Pearson Edexcel

INTERNATIONAL ADVANCED LEVEL

MATHEMATICS/ FURTHER MATHEMATICS/ PURE MATHEMATICS

SCHEME OF WORK

PURE MATHEMATICS 3

Pearson Edexcel International Advanced Subsidiary in Mathematics (XMA01)
Pearson Edexcel International Advanced Subsidiary in Further Mathematics (XFM01)
Pearson Edexcel International Advanced Subsidiary in Pure Mathematics (XPM01)
Pearson Edexcel International Advanced Level in Mathematics (YMA01)
Pearson Edexcel International Advanced Level in Further Mathematics (YFM01)
Pearson Edexcel International Advanced Level in Pure Mathematics (YPM01)

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Pure Mathematics 3

Unit		Title	Estimated hours
1		**Algebra and functions**	
	a	Simplifying algebraic fractions	**2**
	b	Composite and inverse functions	**4**
	c	Modulus function	**4**
	d	Transformations	**3**
2		**Trigonometry**	
	a	Secant, cosecant and cotangent (definitions, identities and graphs) & inverse trigonometrical functions	**4**
	b	Compound and double (and half) angle formulae	**6**
	c	$r\cos(x \pm \alpha)$ or $r\sin(x \pm \alpha)$	**3**
3		**Exponentials and logarithms:** Exponential functions and natural logarithms	**10**
4		**Differentiation**	
	a	Differentiating exponentials, logarithms and the trigonometric functions $\sin x$ and $\cos x$, and their sums, differences and multiples	**5**
	b	Differentiating products, quotients and using the chain rule	**6**
5		**Integration**	
	a	Integrating x^n (including when $n = -1$), exponentials and trigonometric functions	**4**
	b	Integration by recognition of known derivatives and using trigonometric identities	**4**
6		**Numerical methods**	
	a	Location of roots	**1**
	b	Solving by iterative methods	**4**
			60 hours

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UNIT 1: Algebra and Functions

Return to overview

SPECIFICATION REFERENCES

1.1 Simplification of rational expressions including factorising and cancelling, and algebraic division

1.2 Definition of a function. Domain and range of functions. Composition of functions. Inverse functions and their graphs

1.3 The modulus function

1.4 Combinations of the transformations $y = \mathrm{f}(x)$ as represented by $y = a\mathrm{f}(x)$, $y = \mathrm{f}(x) + a$, $y = \mathrm{f}(x + a)$ and $y = \mathrm{f}(ax)$

PRIOR KNOWLEDGE

International GCSE/GCSE (9-1) in Mathematics at Higher Tier

- Algebraic fractions
- Vocabulary and $\mathrm{f}(x)$ notation for functions
- Composite, inverse and transformations of polynomial functions

IAL Mathematics – Pure Mathematics content

Pure 2 (2.1) Algebraic division, factor theorem (Unit 2 of the P2 SoW)

Pure 1 (1.12) Transforming graphs (Unit 1f of the P1 SoW)

KEYWORDS

Polynomial, factor, quadratic, power, index, coefficient, degree, squared, coefficients, identity, algebraic fraction, rational, function, mapping, domain, range, modulus, transformation, composite, inverse, one to one, many to one, mappings, $\mathrm{f}(x)$, $\mathrm{fg}(x)$, $\mathrm{f}^{-1}x$, reflect, translate, stretch.

NOTES

For algebraic fractions, denominators of rational expressions will be linear or quadratic,

e.g. $\frac{1}{ax+b}$, $\frac{ax+b}{px^2+qx+r}$, $\frac{x^3+a^3}{x^2-a^2}$.

Partial fractions to include denominators such as:

$(ax + b)(cx + d)(ex + f)$ and $(ax + b)(cx + d)^2$.

This work has applications in IAL Pure Mathematics topics such as series expansions (Pure 4 Unit 4), differentiation (Pure 3 Unit 4) and integration (Pure 4 Unit 6).

1a. Simplifying algebraic fractions (1.1)

Teaching time
2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to add, subtract, multiply and divide algebraic fractions;
- be able to simplify algebraic fractions by fully factorising and cancelling polynomials up to cubic;
- be able to simplify rational expressions by algebraic division.

TEACHING POINTS

Revise the basic rules of numerical fractions and start with simplifying some IGCSE/GCSE (9-1) Mathematics algebraic fractions.

Exam questions tend to focus on factorising polynomials and then cancelling common factors to simplify algebraic fractions. For example:

Simplify $\frac{x^2-5x-6}{x^2-10x+24} \div \frac{x^2-x-2}{x^2-4x}$

When factorising cubics students can use the factor theorem (link to IAL Pure 2, Unit 2) to find a linear factor of the form $(ax+b)$, or algebraic division where the polynomial has a quadratic factor. Students can also be encouraged to factorise a polynomial by inspection when they have found one factor.

You can use function notation when referring to fractions. (This has been covered in IGCSE/GCSE (9–1) Mathematics.) For example:

The function f is defined by

$$\text{f}: x \rightarrow \frac{3(x+1)}{2x^2+7x-4} - \frac{1}{x+4}, \quad x \in \mathbb{R}, \ x > \frac{1}{2}$$

Show that $\text{f}(x) = \frac{1}{2x-1}$

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

End this section by showing the reverse process where a simplified rational function is split into two (or more) partial factions.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students need to practise factorising quadratics as this is often done incorrectly.

The most common errors include failing to include all necessary brackets, casual miswriting of signs partway through calculations and not dealing correctly with factors. Particular care with signs needs to be taken when a fraction follows a minus sign.

NOTES

Students must be able to divide polynomials for use in the partial fractions in Pure 4.

1b. Composite and inverse functions (1.2)

Teaching time
4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to work out the domain and range of functions;
- know the definition of a one-one and a many-one mapping;
- be able to work out the composition of two functions;
- be able to work out the inverse of a function and sketch its graph;
- understand the condition for an inverse function to exist.

TEACHING POINTS

The notation $f: x \mapsto \ldots$ and f(x) will be used as in IGCSE/GCSE (9-1) Mathematics.

Students will need to understand exactly what functions are and the notation associated with them.

Domain and range from $\mathbb{R}$ (or a subset of $\mathbb{R}$) to $\mathbb{R}$ are important terms for students to understand and should be used regularly. Link this to function machines and graphs (where the domain is the set of x-values and the range is the set of corresponding y-values).

Students should be aware of one-one and many-one mappings and know that a function cannot be one-many.

Students need to know how to find the inverse of a function and it is worth stressing the notation here as lots of students still differentiate when they see this in an exam.

Students should know that if f^{-1} exists, then $ff^{-1}(x) = f^{-1}f(x) = x$. It follows from this that the inverse of a many-one function can only exist if its domain is restricted to make it a one-one function.

Composite functions are also introduced here and it is worth spending some time going over why the order is very important. Students must know that fg means 'do g first and then f". It may be helpful to use an an additional set of brackets in the notation for composite functions, e.g. f[g(x)].

Draw lots of examples of the above using graphing packages and relate the mappings to the graphs. Give an example of a quadratic in which the range is determined by the minimum or maximum point.

Students must also know that the graph of $f^{-1}(x)$ is the image of the graph of $y = f(x)$ after reflection in the line $y = x$. You could relate this to the reverse function machine and the algebraic approach for finding an inverse function (when you change the subject of the formula and rewrite it in terms of x as the final step).

Ask questions such as:

When does the function machine fail to find an inverse?

Do any functions have a self-inverse?

Is an inverse function always possible?

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

The following activities are good for building familiarity and fluency with functions and notation.

1. Play a game where the teacher gives clues about a function and the student have to work out what the function is.

   Begin with numerical examples, e.g. f(3) = 10, f(5) = 26, f(–2) = 5 … until the correct f(x) is given.

   Then to make it more relevant the clues should become algebraic, e.g. f(4) = 14, $f(2a) = 4a^2 - 2$, $f(x^2) = x^4 - 2$ etc.

2. Put all these ideas together by giving students two functions to explore. For example $f(x) = \frac{1}{x}$ and $g(x) = x + 1$ or $f(x) = |x|$ and $g(x) = x - 2$ or $f(x) = x^2 + 1$ and $g(x) = 2x - 3$. Students should explore the following using graphs etc.
   a) compare and contrast the graphs of $fg(x)$ and $gf(x)$
   b) work out if there are any one-one functions here
   c) find the inverses of any one-one functions (relating the inverses to the originals by sketching).

3. Students should investigate whether the following properties of functions are sometimes true, never true or always true.
   $fg(x) = gf(x)$ $\quad$ $g(x) = g^{-1}(x)$ $\quad$ $(fg)^{-1}(x) = g^{-1}f^{-1}(x)$ $\quad$ $(fg)^{-1}(x) = f^{-1}g^{-1}(x)$

An extension activity could be to find as many functions as possible such that $fg(x) = gf(x)$.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students can often successfully find the range in exam questions, but some give their answer in terms of x rather than $f(x)$.

When finding inverse functions, students need to remember to swap x and y. When describing why a function does not have an inverse, students should be advised to answer this question as "because it is not one to one" or "because it is many to one".

NOTES

Relate and link this work on functions to exponentials and logarithms covered in IAL Pure 2 Unit 5.

1c. Modulus function (1.3)	**Teaching time** 4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand what is meant by a modulus of a linear function;
- be able to sketch graphs of functions involving modulus functions;
- be able to solve equations and inequalities involving modulus functions.

TEACHING POINTS

Define the modulus of a set of numbers as being the positive values only. e.g. $|-2| = 2$ and $|5| = 5$.

Begin by using an ICT graph-drawing package (either using the whiteboard or students' individual devices) to sketch some linear graphs using both '$y =$' and '$f(x) =$' notation, e.g. $y = 2x - 1$ or $f(x) = 2x - 1$.

Display the graph of $y = |2x - 1|$ and discuss this with students, drawing comparisons with the 'non-modulus' graph and making sure everyone recognises that $y = |2x - 1|$ does not have any negative values of y (the graph 'bounces up' with the x-axis acting like a mirror).

Define the term modulus function and use the general notation $y = |f(x)|$.

Ask students to predict what the graph of $y = 2|x| - 1$ will look like and then plot it. This time the values of x that are substituted into the function cannot be negative. In other words the graph on the left of the y-axis is a reflection of the graph on the right (where the x-values are positive) with the y-axis being the line of symmetry.

The general notation for this type of function is $y = f|x|$.

Students should be able to sketch the graphs of $y =|ax + b|$ and use their graphs to solve modulus equations and inequalities.

Use the graph-drawing package to sketch the graph of $y = |2x - 1|$ and $y = x$ and use these to solve $|2x - 1| = x$ by considering the points of intersection. Ask students to think about how they might solve this equation algebraically without using a graph. Solving $2x - 1 = x$ gives one solution, but how would the 'modulus' part be represented algebraically? What is the equation of the straight-line graph that represents the 'bounced' part which is now above the y-axis?

Extend this idea to looking at inequalities, for example how to solve $|2x - 1| > x$.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

What happens if we square a modulus? $|-2|^2 = 4$, so a modulus squared is always positive.

Apply this to the modulus equation above $|2x - 1|^2 = x^2$; leading to $3x^2 - 4x + 1 = 0$.

This quadratic gives the two solutions to the equation $y = |2x - 1|$ above. Does this always work? Does this work for inequalities?

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students may find it difficult to sketch graphs involving modulus functions particularly if they are combined with other functions, for example logarithms.

In exam situations, often only the highest scoring students are able to solve modulus equations with x on both sides, or inequalities which involve the modulus function.

1d. Transformations (1.4)

Teaching time
3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand the effect of simple transformations on the graph of $y = \mathrm{f}(x)$ including sketching associated graphs and *combinations* of the transformations:
  $y = a\mathrm{f}(x)$, $y = \mathrm{f}(x) + a$, $y = \mathrm{f}(x + a)$, $y = \mathrm{f}(ax)$;
- be able to transform graphs to produce other graphs;
- understand the effect of composite transformations on equations of curves and be able to describe them geometrically.

TEACHING POINTS

Students should have some understanding of graph transformations from IGCSE/GCSE (9-1) Mathematics and IAL Mathematics (Pure 1 Unit 1), but this will not necessarily include combinations of transformations.

Students need to be able to sketch the transformations $y = a\mathrm{f}(x) + b$, $a\mathrm{f}(x + b)$ and $\mathrm{f}(ax) + b$, but will not be required to sketch $\mathrm{f}(ax + b)$

Use graph drawing packages to investigate the properties of familiar functions (such as trigonometric and reciprocal functions) when you apply the above transformations. Relate the geometry of the transformation to the algebra. For example, $\mathrm{f}(x) + a$ adds a to all the y-coordinates, hence the graph moves ‘up’ by a units (translation vector).

Pose the question, “Does the order in which transformations are applied matter?” Ask students to explore this and present their findings to the class.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Students can explore the difference between transforming x *before* it goes through the function and transforming it afterwards.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students often score well on questions which involve describing geometrical transformations, but incorrect use of terminology will lose marks. Students must use the correct terms: stretch, scale factor and translation.

Students also need to be aware that the order of transformations is often important.

NOTES

Link with the work on transformations in IAL Mathematics Pure 1 Unit 1.

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UNIT 2: Trigonometry

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SPECIFICATION REFERENCES

2.1 Knowledge of secant, cosecant and cotangent and of arcsin, arccos and arctan. Their relationships to sine, cosine and tangent. Understanding of their graphs and appropriate restricted domains

2.2 Knowledge and use of $\sec^2\theta = 1 + \tan^2\theta$ and $\text{cosec}^2\theta = 1 + \cot^2\theta$

2.3a Knowledge and use of double angle formulae; use of formulae for $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$

2.3b Knowledge and use of expressions for $a\cos\theta + b\sin\theta$ in the equivalent forms of $r\cos(\theta \pm \alpha)$ or $r\sin(\theta \pm \alpha)$

PRIOR KNOWLEDGE

International GCSE/GCSE (9-1) in Mathematics at Higher Tier

- Sine and cosine function

IAS Mathematics – Pure Mathematics content

Pure 2 (6.1) $\sin^2 x + \cos^2 x = 1$ and $\frac{\sin x}{\cos x} = \tan x$ (Unit 6 of the P2 SoW)

Pure 2 (6.2) Solving trigonometric equations (Unit 6 of the P2 SoW)

Pure 1 (3.3) Properties of graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$ (Unit 2 of the P1 SoW)

KEYWORDS

Pythagoras, Pythagorean triple, right-angled triangle, opposite, adjacent, hypotenuse, trigonometry, sine, cosine, tangent, secant, cosecant, cotangent, SOHCAHTOA, exact, symmetry, periodicity, identity, equation, interval, quadrant, degree, radian, circular measure, infinity, asymptote, small angles, approximation, identity, proof.

NOTES

This unit is fundamental to future study of hyperbolic functions and integration in Further Maths (FP3), integration in Pure 4 and also links to mechanics. For example, the path of a projectile requires the identity $1 + \tan^2 x = \sec^2 x$.

2a. Secant, cosecant and cotangent (definitions, identities and graphs) & inverse trigonometrical functions (2.1) (2.2)

Teaching time
4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand the secant, cosecant and cotangent functions, and their relationships to sine, cosine and tangent;
- be able to sketch the graphs of secant, cosecant and cotangent;
- be able to simplify expressions and solve involving sec, cosec and cot;
- be able to solve identities involving sec, cosec and cot;
- know and be able to use the identities $1 + \tan^2 x = \sec^2 x$ and $1 + \cot^2 x = \operatorname{cosec}^2 x$ to prove other identities and solve equations in degrees and/or radians;
- be able to work with the inverse trig functions arcsin, arccos and arctan;
- be able to sketch the graphs of arcsin, arccos and arctan.

TEACHING POINTS

Introduce students to the reciprocal trigonometric functions secant θ, cosecant θ and cotangent θ.

A good way to introduce these as reciprocal trig functions is to start by asking whether there is another way of writing x^{-1}. This should lead to the answer $\frac{1}{x}$. If we try this with $\sin^{-1}\theta$ it is not the same meaning as $\frac{1}{\sin\theta}$, so we need to name a different function cosec θ. (Contrast this with inverse trig functions looked at later in this section)

To help students remember which reciprocal function goes with sin, cos and tan, point out that the **third** letter of these new functions, gives the name of the trig function in the denominator, i.e.

se**c** $\theta = \frac{1}{\cos\theta}$ co**s**ec $\theta = \frac{1}{\sin\theta}$ co**t** $\theta = \frac{1}{\tan\theta}$

You should also point out that cot θ can be written as the reciprocal of tan θ to give $\frac{\cos\theta}{\sin\theta}$.

Students will be expected to know what the graphs of each of the reciprocal and inverse functions look like and their key features, including domains and ranges. The relationships between the graphs and their originals can be explored on graphical calculators or graphing Apps.

Show students how to work out new trigonometric identities by dividing $\sin^2\theta + \cos^2\theta = 1$ (IAL Mathematics Pure 1) by $\cos^2\theta$ or by $\sin^2\theta$ to give the two new identities: $1 + \tan^2\theta = \sec^2\theta$ and $1 + \cot^2\theta = \operatorname{cosec}^2\theta$.

This is a good alternative to simply remembering the identities and lessens the chance of mixing them up.

It is a good idea to use the new identities to solve trigonometric equations (which are often quadratic look-a-likes) before proving identities.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

To contrast **reciprocal** trig functions students will also need to be familiar with the **inverse** functions of $\sin\theta$,$\cos\theta$ and $\tan\theta$. They will again need an understanding of the graphs of $\arcsin\theta$, $\arccos\theta$ and $\arctan\theta$. Refer back to the work on functions and emphasise that for arcsin, arccos and arctan to be true functions there must be a one-one relationship between domain and range and so the domains must be restricted to $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ in the case of $\arcsin\theta$ and $\arctan\theta$, and $0 \leq \theta \leq \pi$ in the case of $\arccos\theta$.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

The most common errors in these questions involve using wrong notation or making algebraic mistakes. Students sometimes struggle to deal with more complicated functions such as $\operatorname{cosec}(3x + 1)$ and do not always recognise where trigonometric identities can be used.

NOTES

These trigonometric functions will be useful tools for the calculus units that follow later in the course.

2b. Compound and double (and half) angle formulae (2.3a)

Teaching time
6 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to prove geometrically the compound angle formulae for $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$;
- be able to use compound angle identities to rearrange expressions or prove other identities;
- be able to use compound angle identities to rearrange equations into a different form and then solve;
- be able to recall or work out double angle identities;
- be able to use double angle identities to rearrange expressions or prove other identities;
- be able to use double angle identities to rearrange equations into a different form and then solve.

TEACHING POINTS

A good introduction is to ask the class to work out $\sin(30 + 60)°$. It is equal to $\sin(90)° = 1$. Go on to ask whether $\sin 30° + \sin 60°$ gives the same value (either using a calculator or using surds). They should discover that the values are different. Explain that the reason for this is that you can't simply multiply out functions in this way.

This leads in to explaining why compound angle formulae are needed to calculate $\sin(A + B)$.

Care needs to be taken when using the result to extend to $\sin(A - B)$ for negative values. Students will need to remember that $\cos(-B) = \cos B$ and that $\sin(-B) = -\sin(B)$.

Extend these formulae by substituting $A = B$ to derive the double angle formulae.

Show that there is only one version of $\sin 2x = 2\sin x\cos x$, but the basic version of $\cos 2x = \cos^2 x - \sin^2 x$, can be re-written by substituting $\cos^2 x + \sin^2 x = 1$ (IAL Mathematics Pure 2, Unit 6) into two different versions (exclusively in $\sin x$ or $\cos x$).

A critical part of future questions and proofs involves choosing the correct version of the compound and/or double angle formulae.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Derive and cover examples using half angle formulae by adapting the double angle versions.

The next sub-unit will look at how to solve equations of the type $a\cos\theta + b\sin\theta = C$, using compound angles to rewrite and simplify the expression on the left hand side.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

The most common errors are sign errors when using the compound and double angle formulae.

NOTES

$t\,(\tan\frac{1}{2}\theta)$ formulae will *not* be required.

You should cover reading off obtuse and reflex values by considering a right-angled triangle and assigning a negative or positive sign depending on which quadrant the angle lies in.

Double angle formulae will be a vital substitution when presented in calculus later in the course

2c. $r \cos (x \pm \alpha)$ or $r \sin (x \pm \alpha)$ (2.3b)

Teaching time
3 hours

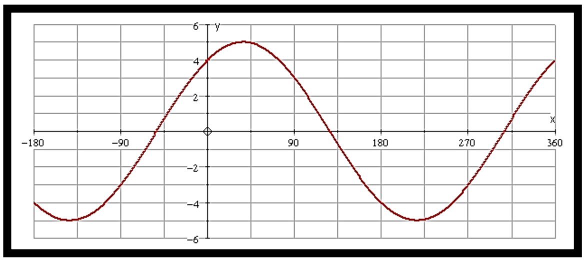
OBJECTIVES

By the end of the sub-unit, students should:

- be able to express $a \cos \theta + b \sin \theta$ as a single sine or cosine function;
- be able to solve equations of the form $a \cos \theta + b \sin \theta = c$ in a given interval.

TEACHING POINTS

Start by drawing a graph of, say, $4 \cos x + 3 \sin x$ to show that is has the basic sin–cos shape. Where are the coordinates of the maximum or minimum points? It approximately fits $5 \cos (x – 40°)$.



Equating $4 \cos x + 3 \sin x$ to an expanded form of $r \cos (x – \alpha)$ gives:

$$4 \cos x + 3 \sin x \equiv r \cos x \cos \alpha + r \sin x \sin \alpha$$

Equating coefficients leads to:

$$r \sin \alpha = 3 \text{ and } r \cos \alpha = 4.$$

By squaring and adding we obtain $r = 5$, and by dividing we obtain $\alpha = 36.9°$. (This confirms the approximate fit above.)

Move on to solving equations of the type $a \cos \theta + b \sin \theta = c$ using $r \cos (x \pm \alpha)$ or $r \sin (x \pm \alpha)$ as the first step. Effectively, the question reduces to a trigonometry equation like those done in Pure 2, but at this level the angles could be in radians.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Ask students whether they can relate the r and α to the basic properties of the curve.
Think about the maximum/minimum value and where it occurs.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Examiner comments suggest that the part of the calculation which causes most problems is working out the angle α:

When writing $a \cos \theta + b \sin \theta$ into the form $r \sin(\theta – \alpha)$ most students found the value of r correctly, the same was not true of the angle α. Some students seemingly failed to notice that α was given as an acute angle.

When solving an equation of the form $a \cos \theta + b \sin \theta = c$ many students seemingly could not cope with the result of –39.23° that their calculator gave them and could not get the first solution. In addition some students found the third quadrant solution only, whereas some found more than two solutions. However many students did give a fully correct solution, often by using a sketch graph to help them decide where the solutions lay.

NOTES

Encourage students to choose which form of the expression to use. It is better to choose the version which, when expanded, gives the same signs for the corresponding terms as the original expression.

UNIT 3: Exponentials and logarithms
Exponential functions and natural logarithms
(3.1) (3.2) (3.3) (4.4)

Teaching time
10 hours

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SPECIFICATION REFERENCES

3.1 The function e^x and its graph

3.2 The function $\ln x$ and its graph; $\ln x$ as the inverse function of e^x

3.3 Use logarithmic graphs to estimate parameters in relationships of the form $y = ax^n$ and $y = kb^x$

4.4 Understand and use exponential growth and decay

PRIOR KNOWLEDGE

IAL Mathematics – Pure Mathematics content

Pure 2 (5.1) Exponentials (Unit 5 of the P2 SoW)

Pure 2 (5.2) Logarithms (Unit 5 of the P2 SoW)

Pure 2 (5.3) Solving exponential equations (Unit 5 of the P2 SoW)

KEYWORDS

Exponential, exponent, power, logarithm, base.

OBJECTIVES

By the end of the unit, students should:

- know and be able to use the function e^x and its graph;
- know and be able to use the function $\ln x$ and its graph;
- know and be able to use $\ln x$ as the inverse function of e^x ;
- be able to use logarithmic graphs to estimate parameters in relationships of the form $y = ax^n$ and $y = kb^x$.

TEACHING POINTS

Explain to students that e^x is a special case of a^x. Graphs of the function e^x should include those in the form $y = e^{ax+b} + c$.

An ability to solve equations of the form $e^{ax+b} = p$ and $\ln(ax + b) = q$ is expected.

Students should be able to plot $\log y$ against $\log x$ and obtain a straight line where the intercept is $\log_a$ and the gradient is n and plot $\log y$ against x to obtain a straight line where the intercept is $\log k$ and the gradient is $\log b$. There should be discussion about why this is an appropriate model and why it is only an estimate.

Contexts for modelling should could include the use of e in continuous compound interest, radioactive decay, drug concentration decay, exponential growth as a model for population growth. Students should be familiar with terms such as initial, meaning when $t = 0$. They may need to explore the behaviour for large values of t or to consider whether the range of values predicted is appropriate. Consideration of a second improved model may be required.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Students can look at different models for population growth using the exponential function.
Use graphing software to investigate varying the parameters of a population model.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Errors seen in exam questions where students have to sketch exponential curves include: stopping the curve at $x = 0$; getting the wrong y-intercept; and believing the curve levels off to $y = 1$ for $x < 0$.

NOTES

Knowledge and use of the result $\frac{\mathrm{d}}{\mathrm{d}x}(a^x) = a^x \ln a$ is expected. Differentiation of exponential functions is covered in the next unit.

UNIT 4: Differentiation

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SPECIFICATION REFERENCES

4.1 Differentiation of e^{kx}, $\ln x$, $\sin kx$, $\cos kx$, $\tan kx$ and their sums and differences

4.2 Differentiate using the product rule, the quotient rule and the chain rule

4.3 The use of $\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$

4.4 Understand and use exponential growth and decay

PRIOR KNOWLEDGE

<u>Covered so far</u>

- Functional notation including $f'(x)$

<u>International GCSE/GCSE (9-1) Mathematics at Higher Tier</u>

- Coordinate geometry
- Changing the subject of the formula, and substitution
- Graphs of linear, quadratic and trigonometric functions

<u>IAS Mathematics – Pure Mathematics content</u>

Pure 1 (2.1, 2.2)	Coordinate geometry (Unit 3 of the P1 SoW)
Pure 2 (6.1)	Trigonometric identities (Unit 6 of the P2 SoW)
Pure 1 (4.2, 4.3)	Differentiation (Unit 4 of the P1 SoW)

KEYWORDS

Derivative, tangent, normal, differential equation, rate of change, product, quotient, first derivative, second derivative.

4a. Differentiating exponentials, logarithms and the trigonometric functions sin x and cos x, and their sums, differences and multiples (4.1)

Teaching time
5 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to differentiate functions involving e^x, ln x, sin x, cos x and related functions such as $6e^{4x}$, 5 ln 3x and $\cos(x - \frac{\pi}{6})$, and sketch the graphs of these functions;
- be able to differentiate to find equations of tangents and normals to the curve.

TEACHING POINTS

It is vital that students understand the functions e^x and ln x and do not just learn how to differentiate them. Use a graphing tool to show that the gradient of a special curve $y = a^x$ has a gradient which is exactly a^x. In other words its rate of growth is exactly the same as its value at that point. This models biological growth in nature (and decay if we consider a^{-x}). The curve sits between 2^x and 3^x and has a value of 2.718… We call this exponential e.

Therefore if $y = e^x$, $\frac{dy}{dx} = e^x$.

Explain that if $y = 2e^x$ then $\frac{dy}{dx} = 2e^x$.

Students could verify this on the graphs below as Fig. 1 is effectively a stretch parallel to the y-axis.

Fig. 2 shows that the graph of $y = e^{2x}$ is twice as steep as e^x, hence if $y = e^{2x}$ then $\frac{dy}{dx} = 2e^{2x}$.

These results will be deduced more formally in the next section, 4b

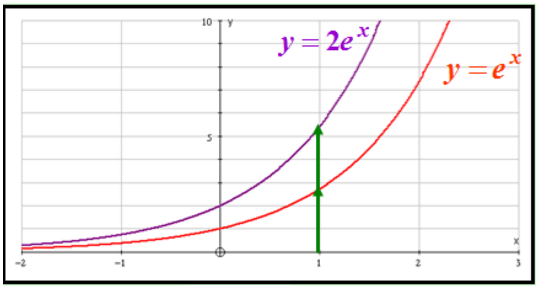


Fig. 1

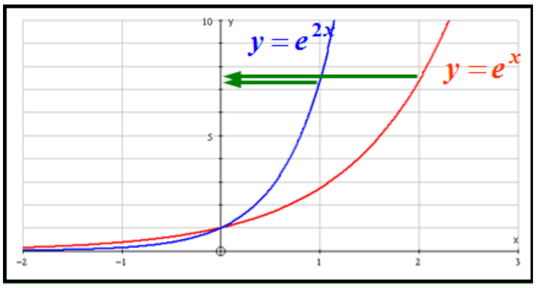


Fig. 2

For natural logarithms, recap the basic definition and graphs (from Pure 2).

By looking at the graph we can see that the gradient of $y = \ln x$ at any particular point is the reciprocal of the x-coordinate of that point where the tangent is drawn. Therefore for $y = \ln x$, $\frac{dy}{dx} = \frac{1}{x}$.

This can be derived in the following way:

If $y = \ln x$, then, from our definition of logs, $x = e^y$. [Write $2 = \log_{10}100$ and $100 = 10^2$ to illustrate this.]

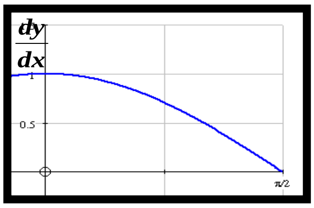
We can differentiate $x = e^y$ by finding $\frac{dx}{dy}$ instead of the usual $\frac{dy}{dx}$.

$\frac{dx}{dy} = e^y$, and taking the reciprocal of both sides gives $\frac{dy}{dx} = \frac{1}{e^y}$.

We know that $e^y = x$ from above, so this gives $\frac{dy}{dx} = \frac{1}{x}$ as the derivative of $y = \ln x$.

The graphical approach could then be used to investigate why, for example, $y = \ln(3x)$ also has a derivative of $\frac{1}{x}$.

The derivatives of $\sin x$ and $\cos x$ are used later to find the derivatives of other trigonometrical expressions. Sketch $y = \sin x$ and consider the gradient at key points by looking at slopes of tangents. If we plot the gradients then we get a shape which looks like the start of a cos graph:



This suggests that if $y = \sin x$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = \cos x$, but this is not a proof or derivation!

Approach the differentiation from first principles in the same way as in IAL Mathematics Pure 1, Unit 4.

Let's take a chord for $y = \sin x$ at $(x, \sin x)$ and $(x + \delta x, \sin(x + \delta x))$, the gradient of the chord is $\frac{\sin(x+\delta x)-\sin x}{\delta x}$

Using compound angle identity for sin $(A + B)$ we find that $\frac{\sin x \cos \delta x+\cos x \sin \delta x-\sin x}{\delta x}$

By manipulation we obtain $\frac{\sin x(\cos \delta x-1)}{\delta x} + \cos x \frac{\sin \delta x}{\delta x}$

Since $\delta x \to 0$, $\frac{\sin \delta x}{\delta x} \to 1$ and $\frac{\cos \delta x-1}{\delta x} \to 0$ we conclude that $\lim_{\delta x\to 0} \frac{\sin(x+\delta x)-\sin x}{\delta x} = \cos x$

Therefore the gradient of the chord $\to$ gradient of the curve and we conclude that $\frac{\mathrm{d}y}{\mathrm{d}x} = \cos x$.

A similar argument with $y = \cos x$ as a starting point leads to:

$$\frac{\cos(x+\delta x)-\cos x}{\delta x} = \frac{\cos x \cos \delta x-\sin x \sin \delta x-\cos x}{\delta x}$$

and therefore finding the derivative to be $-\sin x$.

The alternative notations $h \to 0$ rather than $\delta x \to 0$ are acceptable.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Find gradients and normals for exponential, log and trigonometric functions, using graphs to check and enhance the solutions.

Ask the students to experiment with a graph-drawing package to verify that the gradient functions of $\sin x$ and $\cos x$ match the result found using first principles. Students must understand that the differentiation of $\sin x$ and $\cos x$ can only be used when x is in radians and that they must use radians whether stated in the question or not.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students often miss out minus signs or add an extra x into the answer when differentiating expressions like $\mathrm{e}^{-\frac{1}{4}x}$.

Some students mix up $\frac{\mathrm{d}x}{\mathrm{d}y}$ and $\frac{\mathrm{d}y}{\mathrm{d}x}$ and others struggle to differentiate functions involving ln. For example when differentiating $y = \ln 6x$ they write $\frac{1}{6x}$ rather than $\frac{1}{x}$.

NOTES

Increasingly, exam questions focus on the ability to rearrange and solve equations involving e^x and $\ln x$.

The rest of this unit covers differentiation of more complicated functions in which the derivatives of $\sin x$ and $\cos x$ are building blocks.

4b. Differentiating products and quotients, and using the chain rule (4.2) (4.3) (4.4)

Teaching time
6 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to differentiate composite functions using the chain rule;
- be able to differentiate using the product rule;
- be able to differentiate using the quotient rule;
- use the product and quotient rules to derive the derivatives of different trigonometric functions.

TEACHING POINTS

Most students will be able to differentiate simple instances of e^{3x}, sin 3x and ln 3x without needing formal methods such as $\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$.

Many will also be able to differentiate expressions such as $(3x + 7)^5$ without using the formal method $\frac{d}{dx}(f(x))^n = n(f(x)^{n-1})f'(x)$.

When using the chain rule and the formula $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$, initially u can be given to students, but they must be able to choose their own u and should move onto this quickly. Encourage students to lay out their work carefully, using correct notation and $\frac{dy}{du}$ and $\frac{du}{dx}$, not always $\frac{dy}{dx}$.

Teaching should focus on *how* students know a function needs to be differentiated using the chain rule (or function of a function) and *why* a particular u is selected.

As an introduction for the product rule, ask the students to differentiate x^4. If you rewrite this as the product $(x^2)(x^2)$ and differentiate each part separately, it does not match $4x^3$. Using the product rule will give that match.

In a similar way, writing x^4 as $\frac{x^5}{x}$ can lead into the quotient rule.

Work involving the product and quotient rule often breaks down because of weak algebraic skills and this needs plenty of practice. Students should practise fully simplifying their answers as they may be asked to give a solution in a particular form. Encourage students to lay out their work carefully. Good notation is vital to achieve success.

Show that the product rule and the quotient rule give the same answers on functions that can be written in two ways, for example, $y = \frac{x+1}{x+2}$ and $y = (x + 1)(x + 2)^{-1}$.

Also show that the chain rule and the product rule give the same derivative for $\cos^2 x$ and $\sin^2 x$.

Use the product and quotient rules to derive the differentials of some key trigonometric expressions. For example $\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$ using the quotient rule giving $\sec^2 x$.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Although repeated chain rule questions rarely appear in the exam they provide good extension material and provide an excellent test of good method and correct mathematical notation. Extend the students further by asking them to look up a proof or derivation of the product and quotient rules.

Use the methods above to work out the derivative of a general exponential function, i.e. $\frac{\mathrm{d}}{\mathrm{d}x}(a^{kx}) = ka^{kx} \ln a$

Give the students lots of mixed questions which will enable them to select the correct method. Discussion should focus on why they have selected a particular method and quick ways of identifying the correct method.

Students must be able to use all methods as a particular method is sometime specified in the exam.

Some questions require a trigonometric identity in order to simplify the solution.

Cover questions involving finding tangents, turning points and normals (this links with Pure 1, Unit 4).

Students can look at different models for exponential growth, or decay, and consider rates of change.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Common errors involve: not using the method specified; algebraic errors when manipulating expressions; and being unable to identify the need of the product rule and instead simply differentiating the separate parts and multiplying.

NOTES

Check which differentials are in the formula book and which must be learnt.

UNIT 5: Integration

Return to overview

SPECIFICATION REFERENCES

5.1 Integration of e^{kx}, $\frac{1}{x^n}$, $\sin kx$, $\cos kx$ and their sums and differences

5.2 Integration by recognition of known derivatives to include integrals of the form

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c \quad \text{and} \quad \int f'(x)\left[f(x)\right]^n dx = \frac{\left[f(x)\right]^{n+1}}{n+1} + c$$

PRIOR KNOWLEDGE

Covered so far

- Knowledge of e^x and $\ln x$
- Laws of logarithms
- Trigonometry
- Differentiation

IAL Mathematics – Pure Mathematics content

Pure 2 (5.2) Laws of logarithms (Unit 5 of the P2 SoW)

Pure 2 (6.1) Trigonometry (Unit 6 of the P2 SoW)

Pure 1 (4.2, 5.2) Differentiation and integration (Units 4 &5 of the P1 SoW)

KEYWORDS

Integral, inverse, differential, coefficient, index, power, negative, reciprocal, natural logarithm, $\ln |x|$, coefficient, exponential, identity, sin, cos, tan, sec, cosec, cot, e^x

5a. Integrating x^n (including when $n = -1$), exponentials, logarithms and trigonometric functions. (5.1)

Teaching time
4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to integrate expressions by inspection using the reverse of differentiation;
- be able to integrate x^n for all values of n and understand that the integral of $\frac{1}{x}$ is $\ln|x|$;
- be able to integrate expressions by inspection using the reverse of the chain rule (or function of a function);
- be able to integrate trigonometric expressions;
- be able to integrate expressions involving e^x.

TEACHING POINTS

Recap all the methods of differentiation covered earlier in the course. This can also be used as a starting point for introducing the different rules for integration.

Consider the integral of $x^{-1} = \frac{1}{x}$. Using the rule from IAL Mathematics Pure 1 gives. $\frac{1}{0}$. However, if we recall that the differential of $\ln|x|$ is $\frac{1}{x}$, then the reverse operation tells us that the integral of $\frac{1}{x}$ is $\ln|x| + c$.

Similarly, the differential of e^x is e^x, so the integral will also give the same result.

Finally, the differential of trig expressions should be recapped as this also leads to some standard results for trigonometric integrals.

Take care to show how the integral of $\sin x$ is $-\cos x + c$ (as the differential of $\cos x$ leads to $-\sin x$).

The integral of $\sec^2 x$ looks difficult but is only the reverse of the differential of $\tan x$.

Students must end all indefinite integrations with $+ c$ and use correct notation when integrating, and must include dx.

Encourage students to develop their own technique for integrating problems which require the reverse chain rule. If good examples are used, most students will be able to work out their own method and soon be able to write down the answers directly for integrals like $3e^{2x}$ and $4\sin(3x)$.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

It is always a good idea to advise students to differentiate their answer to see if it goes back to the original (pre-integration) expression. This is a good way to check for sign errors, particularly with the trigonometric questions.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

In exam situations, many students incorrectly integrate functions involving e^x by dividing by the x.

Algebraic errors are also fairly common; clear working and good notation can help here.

NOTES

Make sure students are fluent in these basic integrals, as this will increase the likelihood of success in later integration work.

Pearson

5b. Integration by recognition of known derivatives and using trigonometric identities (5.2) | **Teaching time** 4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- recognise integrals of the form $\int \frac{f'(x)}{f(x)} dx$ = ln |f(x)| + c and $\int f'(x)[f(x)]^n \, dx = \frac{[f(x)]^{n+1}}{n+1} + c$
- be able to use trigonometric identities to manipulate and simplify expressions to a form which can be integrated directly.

TEACHING POINTS

Consider the rule for differentiating ln |f(x)|. This was $\frac{f'(x)}{f(x)}$. A special case of this is the integral of $\frac{1}{x}$, which is ln |x| (+ c).

So, if we have to integrate an expression in which the top of the fraction is the exact differential of the denominator (or a multiple of it), then the answer is the natural log of the denominator (+ c).

Make sure students can adjust questions like the integral of $\frac{4x^2}{x^3}$.

Consider examples like the integral of $\tan x$ by rewriting it as $\frac{\sin x}{\cos x}$, leading to a natural log answer (be careful with the minus!)

One of the most common integrals is $\cos^2 x$. The standard method for integrating this is to rearrange the appropriate double angle formula to create an integral involving not x^2 but $2x$ which is much easier to directly integrate (as shown in the previous section).

Students will need lots of practice in selecting the correct version of $\cos 2x$, which involves only $\cos^2 x$ terms and then rearranging it.

The specification states: 'students are expected to be able to use trigonometric identities to integrate, for example, $\sin^2 x$, $\tan^2 x$, $\cos^2 3x$'.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Students must have lots of practice at working with logarithms and exponentials when integrating, and should leave their answers in exact form. They also need to be fluent in knowing the key trig identities and how to manipulate them from the ones in the formula book.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

The most common errors seen include: mistakes when arranging and substituting identities into integrals; and incorrectly applying laws of logarithms.

UNIT 6: Numerical methods

SPECIFICATION REFERENCES

6.1 Location of roots of $\mathrm{f}(x) = 0$ by considering changes of sign of $\mathrm{f}(x)$ in an interval of x in which $\mathrm{f}(x)$ is continuous

6.2 Approximate solutions using simple iterative methods, including recurrence relations of the form $x_{n+1} = \mathrm{f}(x_n)$

PRIOR KNOWLEDGE

Covered so far

- Series, sequences and recurrence relations
- Graphs, roots and functions
- Differentiation

International GCSE/GCSE (9-1) in Mathematics at Higher Tier

- Iterations and approximate areas under curves (GCSE only)
- Kinematics (velocity–time graphs)

IAL Mathematics – Pure Mathematics content

Pure 1 (1.11), Pure 2 (2.1)	Graphs, roots and functions (Unit 1 of P1 SoW and Unit 2 of P2 SoW)
Pure 1 (4.2, 5.2)	Differentiation and integration (Units 4 &5 of the P1 SoW)

IAL Mathematics – Mechanics content

Mechanics 1 (3.1)	Kinematics (velocity–time graphs) (Unit 3 of the M1 SoW)

KEYWORDS

Roots, continuous, function, positive, negative, converge, diverge, interval, derivative, tangent, chord, iteration.

NOTES

This topic extends the work done on iterations at IGCSE/GCSE (9-1) Mathematics and also links with graphs and functions from Pure 1 and Pure 2.

6a. Location of roots (6.1)

Teaching time
1 hour

OBJECTIVES

By the end of the sub-unit, students should:

- be able to locate roots of $f(x) = 0$ by considering changes of sign of $f(x)$;
- be able to use numerical methods to find solutions of equations.

TEACHING POINTS

Students should be able to recognise that a root exists when there is a change of sign of $f(x)$. Students should recognise this and remember it. There is often an easy mark missed on the exam for this because it is phrased slightly differently.

Students should know that sign change is appropriate for continuous functions in a small interval.

When the interval is too large the sign may not change as there may be an even number of roots.

If the function is not continuous, the sign may change but there may be an asymptote (not a root) so the method will fail.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Look at continuous functions and then contrast this with say $y = \frac{1}{x}$ and $y = \tan x$, which will not have any roots in some intervals despite a change of sign. Use graph drawing packages to investigate similar behaviour in other functions.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students must define $f(x)$ before substituting x-values to find a root.

Most students can successfully identify the root of equations. However there are still many students who then write “change of sign therefore a root” without clarification of where the root lies and hence loose a mark.

Marks are sometimes lost unnecessarily if students do not give their answers to the specified number of significant figures or decimal places.

NOTES

Iterations may be suggested for solving equations which cannot be solved by analytic means (see the next section).

6b. Solving by iterative methods (6.2)

Teaching time
4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand the principle of iteration;
- appreciate the need for convergence in iteration;
- be able to use iteration to find terms in a sequence.

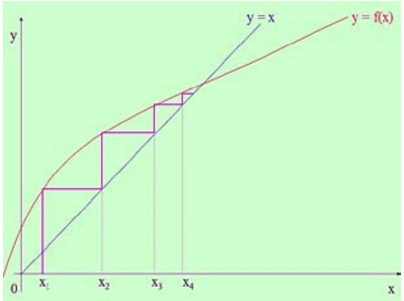
TEACHING POINTS

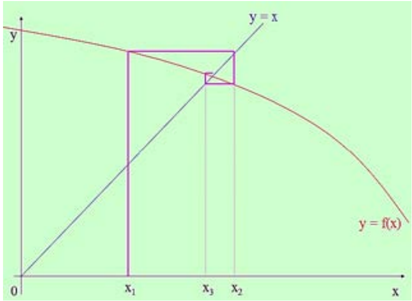
Students may have met iterations at GCSE (9–1) Mathematics, but will need to be introduced to some of the conditions for convergence and understand how the process works (and sometimes does not work).

Revise the method to make one of the x's the subject of the formula, leading to $x = \mathrm{f}(x)$. Use graph-drawing packages to look at the function and decide where would be appropriate for the first iteration value (i.e. x_0).

The method at A level is to consider the roots of the function $y = \mathrm{f}(x)$ as the intersection of the two functions $y = x$ and $y = \mathrm{f}(x)$ (hence $x = \mathrm{f}(x)$).

Use an iteration of the form $x_{n+1} = \mathrm{f}(x_n)$ to find a root of the equation $x = \mathrm{f}(x)$ and show how the convergence can be understood in geometrical terms by drawing cobweb and staircase diagrams like those shown here.



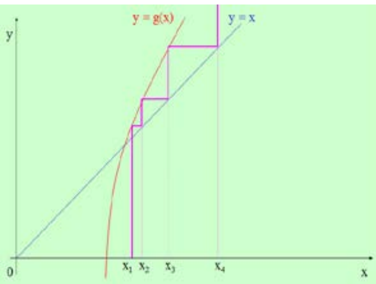


OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Which iterations converge or diverge?

Are there any values which cannot be substituted into certain iterations?

Why does this staircase diagram fail?



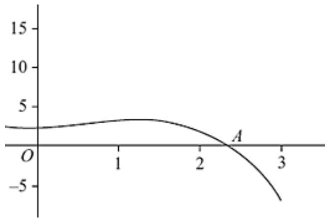
Recurrence relations and iteration methods can be used obtain approximate solution(s) to an equation set in a context. The important point to make is that the original equation is too difficult to solve algebraically (e.g. the roots are decimal and/or the functions will not factorise or contain terms which are non-polynomials).

The choice of degree of accuracy is dependent upon the context of the problem, e.g. nearest minute or number of years.

An example of a possible question is as follows.

The equation $P = -t^3 + 2t^2 + 2$ $(t > 0)$ represents a share price P, at time t months after the money was invested.

The iteration $t_{n+1} = \frac{2}{(t_n)^2} + 2$ represents the solution to the above equation.



Taking $t_0 = 2.5$ months, show that the root gives an approximation to when the share price has zero value. Use the iteration to find the (converged) time at which the shares lose their value before going negative. When were the shares at their highest value?

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Marks will be lost due to using degrees (instead of radians) if functions involve trigonometric terms.

Choosing an unsuitable interval will also prevent progress in these questions.

NOTES

Students should understand that many mathematical problems cannot be solved analytically, but that numerical methods permit a solution to be found to a required level of accuracy.