Pearson Edexcel

INTERNATIONAL ADVANCED LEVEL

MATHEMATICS/ FURTHER MATHEMATICS/ PURE MATHEMATICS

SCHEME OF WORK

PURE MATHEMATICS 2

Pearson Edexcel International Advanced Subsidiary in Mathematics (XMA01)
Pearson Edexcel International Advanced Subsidiary in Further Mathematics (XFM01)
Pearson Edexcel International Advanced Subsidiary in Pure Mathematics (XPM01)
Pearson Edexcel International Advanced Level in Mathematics (YMA01)
Pearson Edexcel International Advanced Level in Further Mathematics (YFM01)
Pearson Edexcel International Advanced Level in Pure Mathematics (YPM01)

First teaching September 2018

First examination from January 2019

First certification from August 2019 (International Advanced Subsidiary) and August 2020 (International Advanced Level)

Introduction

This scheme of work can be used directly as a scheme of work for Pure Mathematics 1 of International Advanced Subsidiary/ Advanced Level in Mathematics/ Further Mathematics/ Pure Mathematics specification (XMA01/XFM01/XPM01/YMA01/YFM01/YPM01).

The scheme of work is broken up into units and sub-units, so that there is greater flexibility for moving topics around to meet planning needs.

Each unit contains:

- Specification references
- Prior knowledge
- Keywords
- Notes.

Each sub-unit contains:

- Recommended teaching time, though of course this is adaptable according to individual teaching needs
- Objectives for students at the end of the sub-unit
- Teaching points
- Opportunities for problem-solving and modelling
- Common misconceptions and examiner report quotes (from legacy Specifications)
- Notes

Teachers should be aware that the estimated teaching hours are approximate and should be used as a guideline only.

Our free support for the International AS and A level Mathematics/ Further Mathematics/ Pure Mathematics specifications can be found on the Pearson Edexcel Mathematics website and on the Emporium.

Pure Mathematics 2

Unit		Title	Estimated hours
1		**Proof:** Examples including proof by deduction, proof by exhaustion and disproof by counter-example	**4**
2		**Algebra and functions:** Algebraic division and the factor and the remainder theorems	**4**
3		**Coordinate geometry in the (x, y) plane:** Circles: equation of a circle, geometric problems on a grid	**7**
4		**Sequences and series**	
	a	Recurrence and iterations	**3**
	b	Arithmetic and geometric sequences and series (proofs of 'sum formulae')	**4**
	c	Sigma notation	**2**
	d	The binomial expansion	**7**
5		**Exponentials and logarithms:** Exponential functions and the laws of logarithms	**8**
6		**Trigonometry:** Trigonometric identities and equations	**10**
7		**Differentiation:** Maxima and minima	**4**
8		**Integration**	
	a	Definite integrals and areas under curves	**5**
	b	The trapezium rule	**2**
			60 hours

UNIT 1: Proof

Examples including proof by deduction, exhaustion and disproof by counter example (1.1), (1.2), (1.3)

Teaching time
4 hours

Return to overview

SPECIFICATION REFERENCES

1.1 Understand and use the structure of mathematical proof, proceeding from given assumptions through a series of logical steps to a conclusion; use methods of proof stated below

1.2 Proof by exhaustion

1.3 Disproof by counter example

PRIOR KNOWLEDGE

International GSCE/GCSE (9-1) in Mathematics at Higher Tier

- Proof

KEYWORDS

Proof, assumptions, exhaustion, disproof, counter-example, conjecture, prediction, implies, necessary, sufficient, converse, therefore, conclusion.

OBJECTIVES

By the end of the unit, students should:

- understand and be able to use the structure of mathematical proof, proceeding from given assumptions through a series of logical steps to a conclusion;
- be able to use methods of proof, including proof by exhaustion and disproof by counter-example.

TEACHING POINTS

Students should be familiar with basic proofs from IGCSE/GCSE (9-1) Mathematics. This knowledge can be built upon to look at the different types of proof. Students will need to understand how to set out each type of proof; the correct conventions in language and layout should be encouraged.

Illustrate proof by exhaustion e.g. Prove that $1^3 + 2^3 + 3^3 + \ldots + n^3 = (1 + 2 + 3 + \ldots + n)^2$ for the positive integers from 1 to 5 inclusive.

This can be proved if you substitute (exhaust) all the possible values of n from 1 to 5. Note that this type of proof can only be used for proving something for a set of given values.

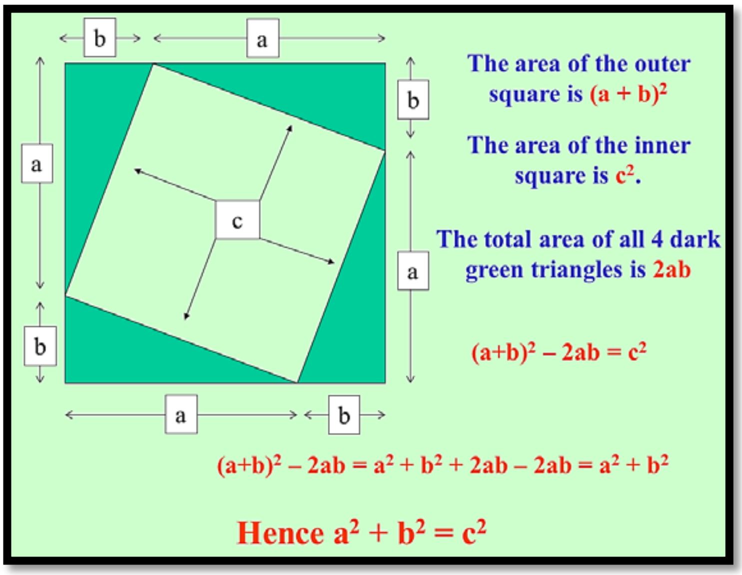
You should also talk about disproof by counter-example.

Explain that all we have to do is find *one* example where the statement does not hold and this is enough to show that it is not always true. This method can be used to disprove trigonometric identities as well as statements such as $a > b \Rightarrow a^2 > b^2$:

Choose any pair of negative numbers with $a > b$ e.g. $a = -2$ and $b = -3$.

Hence $a > b$, but if we square the numbers $a^2 < b^2$ (as $4 < 9$) and so this disproves the statement.

Use areas and the expansion of $(a + b)^2$ to prove Pythagoras' theorem as an example of using a logical sequence of steps in order to deduce a familiar result.



Explain how verification for a set number of values in *not* a proof of a general result (for all values of *n*).

Show how different methods can be used to prove a statement, including:

- Manipulating the LHS of a result and using logical steps (normally algebraic) to make it match the RHS or vice versa (or, sometimes, manipulating both sides to reach the same expression).
- Manipulating an expression to show it holds true for all values. For example, an inequality can always be ≥ 0 if we manipulate the LHS to be in the form of $[\text{something}]^2$ since anything squared will always be bigger or equal to zero. This argument can be used on a gradient function to prove a function is increasing.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Proof gives the opportunity to review previous concepts in a different way for example coordinate geometry. Proof will also be included in later topics.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Some students mistakenly think that substituting several values into an expression is sufficient to prove the statement for all values.

UNIT 2: Algebra and functions | **Teaching time**

Algebraic division and the factor and remainder theorems (2.1) | 4 hours

Return to overview

SPECIFICATION REFERENCES

2.1 Simple algebraic division; use of the factor theorem and remainder theorem

PRIOR KNOWLEDGE

Algebraic manipulation covered so far

- Factorising quadratics
- Notation

International GCSE/GCSE (9-1) in Mathematics at Higher Tier

- Expanding brackets
- Substitution

KEYWORDS

Binomial, coefficient, probability, polynomials, factorisation, quadratic, cubic, rational number, fully factorise, factor, expand.

OBJECTIVES

By the end of the unit, students should:

- be able to use algebraic division;
- know and be able to apply the factor theorem;
- know and be able to apply the remainder theorem;
- be able to fully factorise a cubic expression.

TEACHING POINTS

When using algebraic division, only division by $(ax + b)$ or $(ax – b)$ will be required.

Different methods for algebraic division should be considered depending on students' prior experience and preferred ways of working. Whichever method is used, clear working out should be shown.

Equations in which the coefficient of x or x^2 is 0 for example $x^3 + 3x^2 - 4$ or $2x^3 + 5x - 20$ will need additional explanation and practice.

Students should know that if $f(x) = 0$ when $x = a$, then $(x – a)$ is a factor of $f(x)$. If $f(x) = r$ when $x = a$, then r is the remainder when $f(x)$ is divided by $(x – a)$. Questions in the form $(ax + b)$ should be covered.

Where a negative is being substituted into the equation the distinction between $(-2)^2$ and -2^2 will be important especially when students are using a calculator as examiners often comment on the fact that students will sometimes evaluate $(-2)^2$ as -4.

The factor theorem can be used to find an unknown constant. For example: Find a given that $(x - 2)$ is a factor of $x^3 + ax^2 - 4x + 6$. Two conditions can also be given in order to form simultaneous equations to solve.

When fully factorising a cubic, emphasis should be placed on choosing appropriate values. The final answer may need to be written as a factorised cubic or, alternatively, as the solutions to an equation which can then be used to sketch the curve. Students sometimes use the roots of a polynomial equation to help them factorise but this method must be used with care. Questions sometimes use the word ‘hence’ and so students must be careful which method they chose in these cases.

This is an excellent opportunity to review curve sketching by asking students to give a sketch following factorisation.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Both the factor theorem and the remainder theorem can be introduced through investigation by substituting different values and checking against division to look for patterns. By considering $f(x) = (x - a)g(x)$ and $f(x) = (x - b)h(x) + r$, students can see that $f(a) = 0$ and $f(b) = r$, and that $(x - a)$ is a factor of $f(x)$ and r is the remainder when $f(x)$ is divided by $(x - b)$.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

The majority of errors seen in exam questions are not due to misunderstanding the method, but instead arithmetic and algebraic mistakes. For example, incorrect simplification of terms – especially those involving fractions; mistakes with negative numbers; and writing expressions rather than equations.

Students should be aware that long division is not always the best or quickest method to use and sometimes results in some complicated algebra.

When using the factor theorem, stress the importance of checking the value that is substituted; a common error is to use, for example, f(1) rather than f(–1).

You should also emphasise the importance of fully factorising expressions, as a fairly significant number of students stop when they have reached one linear factor and a quadratic factor.

UNIT 3: Coordinate geometry in the (x, y) plane	Teaching time
Circles: equation of a circle, geometric problems on a grid (3.1)	7 hours

Return to overview

SPECIFICATION REFERENCES

3.1 Coordinate geometry of the circle using the equation of a circle in the form $(x-a)^2+(y-b)^2=r^2$ and including use of the following properties:

(i) the angle in a semicircle is a right angle

(ii) the perpendicular from the centre to a chord bisects the chord

(iii) the perpendicularity of radius and tangent

PRIOR KNOWLEDGE

Algebraic manipulation covered so far

- Simultaneous equations
- Completing the square

International GCSE/GCSE (9-1) in Mathematics at Higher Tier

Circle theorems

KEYWORDS

Equation, bisect, centre, chord, circle, circumcircle, coefficient, constant, diameter, gradient, hypotenuse, intercept, Pythagoras, radius, right angle, segment, semicircle, simultaneous, tangent.

OBJECTIVES

By the end of the unit, students should:

- understand and use the equation of a circle;
- be able to find points of intersection between a circle and a line;
- know and be able to use the properties of chords and tangents.

TEACHING POINTS

Drawing sketches or annotating given diagrams will help students to understand the question in many cases and so should be encouraged.

The equation of the circle $(x-a)^2+(y-b)^2=r^2$ can be derived from Pythagoras' theorem, giving students the opportunity to look at proof.

Students should be able to find the radius and the coordinates of the centre of the circle given the equation of the circle, and vice versa. Students should be familiar with the equations $x^2 + y^2 + 2fx + 2gy + c = 0$ and 'complete the square' method should be used to factorise the equation into the more useful form. Students will need practice within this context to ensure that they are confident with the algebraic manipulation needed, in particular mistakes are often made with the signs and forgetting the constant term.

Circle theorems from IGCSE/GCSE (9-1) Mathematics can be used in questions so a quick recap could be useful and then they should be incorporated into questions. Examples of this include: finding the equation of the circumcircle of a triangle with given vertices; or finding the equation of a tangent using the perpendicular property of tangent and radius.

Simultaneous equations can be used to find the points of intersection between a circle and a straight line. Students can also be asked to show that a line and circle do not intersect, for which the discriminant can be used. Finding intersections with the axes should also be covered.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

The conditions in which a circle and a line intersect can be investigated, with students justifying which will and will not intersect.

Investigate finding the equation of a circle given three points on its circumference.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Most errors when completing the square to find the equation of a circle involve the constant term. Students may forget to subtract it or perhaps add it instead. Having found the equation, when giving the coordinates of the centre students must take care to get the signs the right way round as marks are easily lost by getting this wrong.

When substituting into equations to find the intersections with axes, students sometimes substitute for the wrong variable, for example substituting $y = 0$ when trying to find the intersection with the y-axis. Another error is substituting the entire bracket $(x – a)$ for 0 rather than just x.

When finding the equation of a tangent to a point on the circle, typical errors are: finding the gradient of the radius; finding a line parallel to the radius; and finding a line through the centre of the circle.

UNIT 4: Sequences and series

Return to overview

SPECIFICATION REFERENCES

4.1 Sequences including those given by a formula for the *n*th term and those generated by a simple relation of the form $x_{n+1} = f(x_n)$

4.2 Understand and work with arithmetic sequences and series, including the formulae for the *n*th term and the sum of a finite arithmetic series; the sum of the first *n* natural numbers

4.3 Increasing sequences, decreasing sequences and periodic sequences

4.4 Understand and work with geometric sequences and series including the formulae for the *n*th term and the sum of a finite geometric series; the sum to infinity of a convergent geometric series, including the use of $|r| < 1$

4.5 Binomial expansion of $(a + bx)^n$ for positive integer n

PRIOR KNOWLEDGE

International GCSE/GCSE (9-1) in Mathematics at Higher Tier

- Use simple arithmetic sequences and series
- Generate terms of a sequence from either a term-to-term or a position-to-term rule (GCSE only)
- Use simple geometric progression (GCSE only)
- Finding expressions for the *n*th term of linear and quadratic sequences (GCSE only)

KEYWORDS

Sequence, series, finite, infinite, summation notation, Σ(sigma), periodicity, convergent, divergent, natural numbers, arithmetic series, arithmetic progression (AP), common difference, geometric series, geometric progression (GP), common ratio, *n*th term, sum to *n* terms, sum to infinity (S_∞), limit.

4a. Recurrence and iterations (4.1) (4.3)

Teaching time
3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- know that a sequence can be generated using a formula for the *n*th term or a recurrence relation of the form $x_{n+1} = f(x_n)$;
- understand how a recurrence relation of the form $U_n = f(U_{n-1})$ can generate a sequence;
- know the difference between increasing, decreasing and periodic sequences;
- be able to describe increasing, decreasing and periodic sequences.

TEACHING POINTS

Work with sequences including those given by a formula for the *n*th term and those generated by a simple relation of the form $x_{n+1} = f(x_n)$ and link this with the work done on iterations in GCSE (9-1) Mathematics.

Explore $x_{n+1} = f(x_n)$ type series using graphics calculators or spreadsheets.

Move on to general recurrence relations of the form $U_n = f(U_{n-1})$ and investigate which sequences are increasing, decreasing and periodic. Spend some time looking at the different forms of notation for recurrence relations, making sure you cover examples of increasing, decreasing and periodic sequences. For example:

$u_n = \frac{1}{3}u_n + 1$ describes a decreasing sequence as $u_{n+1} < u_n$ for all integers n

$u_n = 2u_n$ is an increasing sequence as $u_{n+1} > u_n$ for all integers n

$u_{n+1} = \frac{1}{u_n}$ for $n > 1$ and $u_1 = 3$ describes a periodic sequence of order 2.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Cover questions in which sequences can be used to model a variety of different situations. For example finance, growth models, decay, periodic (tide height for example) etc.

Can you tell from the structure of a recurrence relation how it will behave, and the type of sequence it will generate?

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

When asked to find the limit of u_n some candidates use the sum to infinity of a geometric series.

NOTES

Encourage the use of the ANS button on a calculator to obtain the terms for a recurrence relation.

4b. Arithmetic and geometric sequences and series (proofs of 'sum formulae') (4.2) (4.4)	Teaching time 4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- know what a sequence of numbers is and the meaning of finite and infinite sequences;
- know what a series is;
- know the difference between convergent and divergent sequences;
- know what is meant by arithmetic series and sequences;
- be able to use the standard formulae associated with arithmetic series and sequences;
- know what is meant by geometric series and sequences;
- be able to use the standard formulae associated with geometric series and sequences;
- know the condition for a geometric series to be convergent and be able to find its sum to infinity;
- be able to solve problems involving arithmetic and geometric series and sequences;
- know the proofs and derivations of the sum formulae (for both arithmetic and geometric series).

TEACHING POINTS

Start by recapping the work students did on sequences at IGCSE/GCSE (9-1) Mathematics before moving on to the new IAL content, paving the way for the sigma notation in the following section.

Use practical situations, for example involving money, to illustrate arithmetic and geometric sequences and contrast the different ways they grow.

Find the *n*th term of a given arithmetic sequence and also use the rule to find the next two terms.

The Gauss problem (1 + 2 + … + 1000) is a good numerical way to lead into the full proof of the sum of an arithmetic series. Students will need to know the proof and derivation of the formula for the sum of an arithmetic series.

Illustrate how arithmetic sequences are different to geometric sequences, and explain that the common difference (a) becomes the common ratio (r). Students need to be aware that not all geometric sequences converge.

Cover problems where the n in the nth term formula (ar^{n-1}) is to be found using logarithms.

Illustrate when to use $\frac{a(1-r^n)}{(1-r)}$ and when to use $\frac{a(r^n-1)}{(r-1)}$ (depending on the value of r).

Show that $\frac{a}{(1-r)}$ can be derived if we illustrate on a calculator that r^n tends to zero when $-1 < r < 1$.

A way of illustrating the sum to infinity is to imagine hammering a nail into a piece of wood, where each strike makes the nail sink in exactly half its remaining distance. There will be a limit to how many times it will need to be hit, as it surely will end up being 'flush' to the surface of the wood and have a distance of zero above the wood. (You can link this to Zeno's paradox.)

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

This topic can be linked to mechanics by investigating, for example, a ball which is dropped from 2 m and bounces to $\frac{3}{4}$ of its height after each bounce.

Challenge students to come up with a rule to determine which series will have a sum to infinity and which won't.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

When working with formulae for sequences and series, it is important that students state the relevant formula before substituting so that method marks can be awarded even if there is a numerical slip.

NOTES

Move onto general notation of series by using the sigma notation in the next session.

Note that the sum to infinity may be expressed as S_∞.

Pearson

4c. Sigma notation (4.2)

Teaching time
2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be familiar with $\sum$ notation and how it can be used to generate a sequence and series;
- know how this notation will lead to an arithmetic or geometric series and its sum;
- know that $\sum_{1}^{n} 1 = n$.

TEACHING POINTS

The key to understanding the concept of $\sum$ is to look at the limit values and substitute them into the nth term formula to generate the terms of the sequence.

Emphasise to students that they must take care when finding the starting point and never assume it starts with $n = 1$.

Students may initially find the $\sum$ notation tricky, particularly if they are not asked to find the sum of first n terms, but instead asked to find, e.g. the 7th to the 20th.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Challenge students to try to work out whether a sequence is arithmetic, geometric or neither from just looking at the structure of the sigma version of a series.

Ask students to write a series in sigma notation.

Show that $\Sigma n = \frac{1}{2}n(n + 1)$ is the sum of n natural numbers and relate this to the sum formula derived in the previous section.

Think about what to do if the upper limit is infinity.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

A fairly common error is to mix up the formulae for sums and terms, for example finding S_n rather than U_n and vice-versa.

NOTES

Students will need to be clear on the meanings and the usage of the various notations covered in this unit.

4d. The binomial expansion (4.5)	Teaching time 7 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand and be able to use the binomial expansion of $(a + bx)^n$ for positive integer n;
- be able to find an unknown coefficient of a binomial expansion.

TEACHING POINTS

Students should initially be introduced to Pascal's triangle, which can be used to expand simple brackets. Students will need to be familiar with factorials and the ${}_nC_r$ notation.

Introduce the formal binomial expansion in the same way as the formula booklet and discuss the various terms to ensure all students understand.

Setting out work clearly and logically will be invaluable in helping students to achieve the final answer and also to spot mistakes if necessary.

Where there is a coefficient of x (other than 1) students will need to be reminded that the power applies to the whole term, not just the x, and that answers must be simplified appropriately. Negative and fractional coefficients will also need practice.

The limitations of the binomial expansion should be discussed.

Students should practice finding the coefficient of a single term, they should also be able to deal with setting up simple algebraic equations to find unknown constants.

Use of the binomial expansion can be linked to basic probability and approximations.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Encourage students to discover the link between Pascal's triangle and the expansion of simple brackets.

Students could find the general term of a particular expansion.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Marks are most commonly lost because of errors in expanding terms. For example not including the coefficient when calculating, say, $(ax)^2$; not simplifying terms fully; sign errors; and omitting brackets. Good notation will help to avoid many of these mistakes.

When writing expansions with unknown constants, some students don't include the x's in their expansion.

When using their expansions to work out the value of a constant, a significant number of students do not understand that the coefficient does not include the x or x^2 part and so are often unable to form an equation in the unknown alone.

Questions often go on to ask students to use their binomial expansion to evaluate a number raised to a power. For example, evaluating $(1.025)^8$ by substituting $x = 0.025$ into an expansion for $(1 + x)^8$. Students should be advised that simply using their calculator to evaluate $(1.025)^8$ will gain no marks as it is not answering the question.

NOTES

Be aware of alternative notations such as $\binom{n}{r}$ and ${}^{n}C_{r}$.

UNIT 5: Exponentials and logarithms
Exponential functions and the laws of logarithms
(5.1) (5.2) (5.3)

Teaching time
8 hours

Return to overview

SPECIFICATION REFERENCES

5.1 $y = a^x$ and its graph

5.2 Laws of logarithms:

5.3 The solution of equations of the form $a^x = b$

PRIOR KNOWLEDGE

Covered so far

- Indices

International GCSE/GCSE (9-1) in Mathematics at Higher Tier

- Compound interest

KEYWORDS

Exponential, exponent, power, logarithm, base, initial, rate of change, compound interest.

OBJECTIVES

By the end of the unit, students should:

- know and be able to use the function a^x and its graph, where *a* is positive;
- know and be able to use the definition of $\log_a x$ as the inverse of a^x, where a is positive and $x \geq 0$;
- understand and use the laws of logarithms:

  $\log_a(xy) = \log_a x + \log_a y$

  $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

  $\log_a x^k = k\log_a x$

  $\log_a\left(\frac{1}{x}\right) = -\log_a x$

  $\log_a a = 1$

  where $a, x, y > 0, a \neq 1$
- be able to solve equations of the form $a^x = b$.

TEACHING POINTS

When sketching the graph of a^x students should understand the difference in shape between $a < 1$ and $a > 1$.
Students can use the laws of indices to prove the laws of logarithms and show that $\log_a a = 1$.
In solving equations students may use the change of base formula. Equations that require solving may be in the form $2^{3x-1} = 3$.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Students can look at different models for population growth using the exponential function.

Use graphing software to investigate varying the parameters of a population model.

Link to proof: students can be encouraged to derive the laws of logarithms.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Errors seen in exam questions where students have to sketch exponential curves include: stopping the curve at $x = 0$; getting the wrong y-intercept; and believing the curve levels off to $y = 1$ for $x < 0$.

When using laws of logs to answer proof or 'show that' questions, students must show all the steps clearly and not have jumps in their working out.

UNIT 6: Trigonometry	Teaching time
Trigonometric identities and equations (6.1) (6.2)	10 hours

Return to overview

SPECIFICATION REFERENCES

6.1 Knowledge and use of $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta = 1$

6.2 Solution of simple trigonometric equations in a given interval

PRIOR KNOWLEDGE

Algebra covered so far

- Basic algebraic manipulation
- Quadratics

International GCSE/GCSE (9-1) in Mathematics at Higher Tier

- Pythagoras' theorem
- Trigonometry in right-angled triangles

KEYWORDS

Sine, cosine, tangent, interval, period, amplitude, function, inverse, angle of elevation, angle of depression, bearing, degree, identity, special angles, unit circle, symmetry, hypotenuse, opposite, adjacent, intercept.

OBJECTIVES

By the end of the unit, students should:

- be able to solve trigonometric equations within a given interval;
- understand and be able to use $\tan\theta = \frac{\cos\theta}{\sin\theta}$;
- understand and be able to use $\sin^2\theta + \cos^2\theta = 1$.

TEACHING POINTS

When solving trigonometric equations, finding multiple values within a range can initially be illustrated using the graphs of the functions. The decision can then be made whether to move on to using CAST diagrams or continue using graphs. Whichever method is used students will need plenty of practice in identifying all values within the limits correctly.

Intervals with negative solutions as well as positive solutions should be used.

Students should be able to solve equations such as $\sin(x + 70°) = 0.5$ for $0 < x < 360°$; $3 + 5\cos 2x = 1$ for $–180° < x < 180°$; and $6\cos^2 x + \sin x - 5 = 0$ for $0 < x < 360°$, giving their answers in degrees.

Students should be comfortable factorising quadratic trigonometric equations and finding all possible solutions. It should be noted that in some cases only one of the factorisations will give solutions, but in most cases there will be two sets of solutions. Situations where one answer is equal to zero can cause some confusion with students then not looking for further solutions. This sort of example should be covered in class. For example, the equation, $\sin\theta(3\sin\theta + 1) = 0$ will often be simplified to just $3\sin\theta + 1 = 0$, resulting in the loss of solutions to the original equation.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

If trigonometric graphs are used to model situations (such as yearly temperatures, wave lengths and tidal patterns) then the equations can be used to find values at given points.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Common errors include: not finding values in the given range; finding extra, incorrect, solutions; not going on to find the values of x and instead leaving the values for, say $2x$ or $x + 30$; algebraic slips when rearranging the equation; and not giving answers to the correct degree of accuracy. The loss of accuracy in the final answers to trigonometric equations is common and often results in the unnecessary loss of marks. Sketches of the trigonometric functions are often helpful to check all solutions have been found.

UNIT 7: Differentiation | **Teaching time**
Maxima and minima (7.1) | 4 hours

Return to overview

SPECIFICATION REFERENCES

7.1 Applications of differentiation to maxima and minima and stationary points, increasing and decreasing functions

PRIOR KNOWLEDGE

Covered so far

- Differentiation

KEYWORDS

Differentiation, derivative, rate of change, increasing, decreasing, stationary point, maximum, minimum, calculus, function.

OBJECTIVES

By the end of the unit, students should:

- be able to apply differentiation to find gradients, maxima, minima and stationary points;
- be able to identify where functions are increasing or decreasing.

TEACHING POINTS

Maxima, minima and stationary points can be used in curve sketching. Problems may be set in the context of a practical problem. This could bring in area and volume from IGCSE/GCSE (9-1) Mathematics as well as using trigonometry.

Students need to know how to identify when functions are increasing or decreasing. For example, given that $f'(x) = x^2 - 2 + \frac{1}{x^2}$, prove that f($x$) is an increasing function.

Use graph plotting software that allows the derivative to be plotted so that students can see the relationship between a function and its derivative graphically.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Differentiation can be linked to many real-world applications, there can be discussion with students about contexts and the validity of solutions. Use of the second derivative to determine the nature of a stationary point could be considered. Consider for example the curve $y = x^3 + 3x^2 + 1$, which has a local maximum at the point $(-2, 5)$ and a local minimum at the point $(0, 1)$. Students should investigate the gradient of the curve close to $x = 0$. They should see that the gradient, $\frac{dy}{dx}$, goes from negative to positive as x increases, which means that $\frac{dy}{dx}$ is an increasing function. Hence $\frac{d^2y}{dx^2} > 0$ at $x = 0$. A similar argument can be used at the local maximum point $(-2, 5)$ to conclude that $\frac{d^2y}{dx^2} < 0$ at $x = -2$. By considering the functions $y = x^4$ and $y = -x^4$ demonstrate that if $\frac{d^2y}{dx^2} = 0$ at a stationary point it is not possible to determine its nature.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Questions involving finding a maximum or minimum point do require the use of calculus and attempts using trial and improvement will receive no marks.

When finding a stationary point, some students use inequalities as their condition rather than equating their derivative to zero. Another error is to differentiate twice and solve $f''(x) = 0$.

When applying differentiation in context, students should ensure they give full answers and not just a partial solution. For example if asked to find the volume of a box they must not stop after finding the side length.

UNIT 8: Integration

Return to overview

SPECIFICATION REFERENCES

8.1 Evaluation of definite integrals

8.2 Interpretation of the definite integral as the area under a curve

8.3 Approximation of area under a curve using the trapezium rule

PRIOR KNOWLEDGE

Covered so far

- Algebraic manipulation
- Differentiation

KEYWORDS

Calculus, differentiate, integrate, reverse, indefinite, definite, constant, evaluate, intersection.

8a. Definite integrals and areas under curves (8.1), (8.2)

Teaching time
5 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to evaluate definite integrals;
- be able to use a definite integral to find the area under a curve.

TEACHING POINTS

It is important that students show their working out clearly as mistakes are easily made when putting values into a calculator. Students should also be encouraged to check their answers. Calculators that perform numerical integration can be used as a check, but a full method will be needed.

Students will be expected to understand the implication of a negative answer from indefinite integration.

Links can be made with curve sketching in questions where students need to find the points of intersection with the x-axis for a curve in order to find the limits of integration.

Areas can be made up of a combination of a curve and a line so further links can be made to coordinate geometry.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Discuss the implication of a negative answer to encourage students' reasoning skills.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Lack of algebraic fluency can cause problems for some students, particularly when negative/fractional indices are involved or when a negative number is raised to a power. Arithmetic slips are also a common cause of lost marks, often when negative numbers are substituted and subtracted after integration.

Students are generally more successful if they expand any brackets before attempting to integrate the function.

8b. The trapezium rule (8.3)	**Teaching time** 2 hours

OBJECTIVES

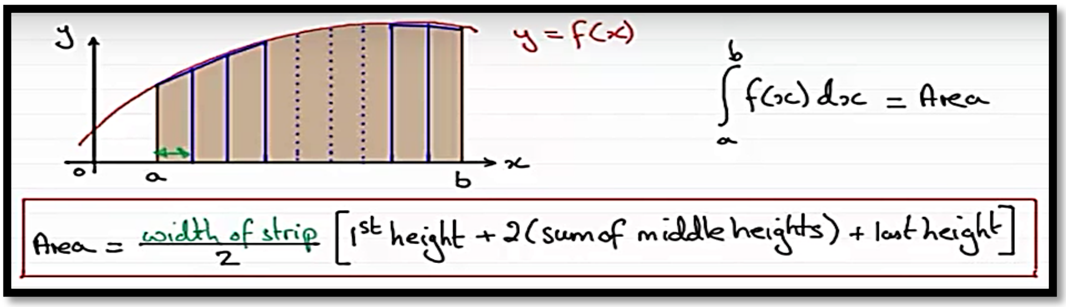
By the end of the sub-unit, students should:

- be able to use the trapezium rule to find an approximation to the area under a curve;
- appreciate the trapezium rule is an approximation and realise when it gives an overestimate or underestimate.

TEACHING POINTS

Make a direct link with the previous section and how to find an estimate for the area under a curve by dividing it into a finite number of strips. Sometimes an estimate is all that we need, and sometimes the integral is very complicated (or sometimes impossible) to integrate and so we have to estimate the area numerically.

The trapezium rule is given in the formula book (and may have also been covered in IGCSE/GCSE (9-1)). Students who struggle with algebra sometimes prefer to use the word version below:



Some students may be able to derive the rule by adding all the individual strip areas (i.e. $\frac{1}{2}h(y_0 + y_1) + \frac{1}{2}h(y_1 + y_2) + \dots$) and then factorising to give the trapezium rule as in the formula book.

Ask students to calculate $\int_1^2 \frac{1}{\sqrt{x}}\,dx$ by integration and also by completing the table and using the trapezium rule. They should compare the answers they get using the different methods.

x	1	1.2	1.4	1.6	1.8	2
$y = \frac{1}{\sqrt{x}}$	1	0.9129		0.7906		0.7071

Another example of the type of question that may be asked is:

Evaluate $\int_0^1 \sqrt{2x+1}\,dx$ using the values of $\sqrt{2x+1}$ at x = 0, 0.25, 0.5, 0.75 and 1.

Use graphing software to sketch of the graph to determine whether the trapezium rule gives an over-estimate or an under-estimate of the exact value of the integral.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

The following exam question shows a modelling example:

A river, running between parallel banks, is 20 m wide. The depth, *y* metres, of the river, measured at a point *x* metres from one bank, is given by the formula:

$y = \frac{1}{10}x\sqrt{20 - x}\,, \quad 0 \le x \le 20$

(*a*) Complete the table below, giving values of *y* to 3 decimal places.

x	0	4	8	12	16	20
y	0		2.771			0

(*b*) Use the trapezium rule with all the values in the table to estimate the cross-sectional area of the river.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

When using the trapezium rule students sometimes mix up the number of strips and the number of *x* or *y* values.

The other main place marks are lost is not giving the final answer to three significant figures.

NOTES

Make sure that you use the same form for the trapezium rule as that given in the formula book.