

INTERNATIONAL ADVANCED LEVEL

**MATHEMATICS/
FURTHER MATHEMATICS/
PURE MATHEMATICS**
SCHEME OF WORK
PURE MATHEMATICS 4

Pearson Edexcel International Advanced Subsidiary in Mathematics (XMA01)

Pearson Edexcel International Advanced Subsidiary in Further Mathematics (XFM01)

Pearson Edexcel International Advanced Subsidiary in Pure Mathematics (XPM01)

Pearson Edexcel International Advanced Level in Mathematics (YMA01)

Pearson Edexcel International Advanced Level in Further Mathematics (YFM01)

Pearson Edexcel International Advanced Level in Pure Mathematics (YPM01)

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A level Mathematics: Pure Mathematics 4

Pure Mathematics 4

Unit	Title	Estimated hours
1	Proof: Proof by contradiction	3
2	Algebra and functions: Partial fractions	3
3	Coordinate geometry in the (x, y) plane: Definition and converting between parametric and Cartesian forms	3
4	Binomial expansion	
<u>a</u>	Expanding $(a + bx)^n$ for rational n ; knowledge of range of validity	4
<u>b</u>	Expansion of functions by first using partial fractions	3
5	Differentiation	
<u>a</u>	Differentiating implicit and parametric functions	5
<u>b</u>	Rates of change problems (including growth and decay)	3
6	Integration	
<u>a</u>	Volumes of revolution	5
<u>b</u>	Integration by substitution	4
<u>c</u>	Integration by parts	3
<u>d</u>	Use of partial fractions	2
<u>e</u>	Differential equations	4
7	Vectors	
<u>a</u>	Definitions, magnitude/direction, addition and scalar multiplication	5
<u>b</u>	Position vectors, distance between two points, geometric problems	3
<u>c</u>	Vector equation of a line	5
<u>d</u>	Scalar product	5
		60 hours

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UNIT 1: Proof: Proof by contradiction (1.1)

Teaching time
3 hours

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SPECIFICATION REFERENCES

1.1 Proof by contradiction

PRIOR KNOWLEDGE

International GCSE/GCSE (9-1) in Mathematics at Higher Tier

- Pythagoras theorem
- Trigonometry
- Algebraic manipulation including completing the square
- Surds, prime and irrational numbers

IAL Mathematics – Pure Mathematics content

Pure 2 (1.1, 1.2, 1.3) Proof (Unit 1 of the P2 SoW)

KEYWORDS

Proof, verify, contradict, rational, irrational, square, root, prime, infinity, square number, quadratic, expansion, trigonometry, Pythagoras.

NOTES

Proof may also be tested throughout the specification through other topics e.g. trigonometry, series, differentiation, etc.

OBJECTIVES

By the end of the unit, students should:

- be able to understand and use proof by contradiction.

TEACHING POINTS

Introduce the topic by reviewing the methods of proof seen in Pure Mathematics 2.

Provide standard examples of proof by contradiction, e.g., $\sqrt{2}$ is irrational:

Assuming it can be written as a rational number $\frac{a}{b}$ which has been written in its lowest terms.

It follows that $\frac{a^2}{b^2} = 2$ and $a^2 = 2b^2$. Therefore, a^2 is even because it is equal to $2b^2$.

It follows that a must be even (as squares of odd integers are never even).

Because a is even, there exists an integer k that fulfills: $a = 2k$.

Substituting $2k$ for a above gives $2b^2 = (2k)^2 = 4k^2$, so $b^2 = 2k^2$.

Because $2k^2 = b^2$, it follows that b^2 is even and b is also even.

Hence a and b are both even, which contradicts that $\frac{a}{b}$ is in its simplest form.

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Another example of proof by contradiction is the proof that there is an infinite number of primes:

Assume there is an integer p , such that p is the largest prime number.

Now $p! + 1 > p$ and is not divisible by p or any other number less than p *

*If 2 is a factor of n , then 2 is not a factor of $n + 1$. Similarly if 3 is a factor of n , 3 is not a factor of $n + 1$.

Now 2, 3, ... p are all factors of $p!$, so none are factors of $p! + 1$.

So, either $p! + 1$ is not divisible by an integer other than 1 or $p! + 1$ which means $p! + 1$ is prime, or $p! + 1$ is divisible by some number between p and $p! + 1$ which implies there is a prime number larger than p .

These both contradict our initial assumption, which proves there are an infinite number of primes.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Students should be given the opportunity to apply the method to a variety of number problems. They could also be asked to identify errors in incorrect or incomplete proofs.

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UNIT 2: Algebra and functions:

Partial fractions (2.1)

Teaching time

3 hours

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SPECIFICATION REFERENCES

- 2.1 Decompose rational functions into partial fractions (denominators not more complicated than repeated linear terms).

PRIOR KNOWLEDGE

International GCSE/GCSE (9-1) in Mathematics at Higher Tier

- Algebraic fractions

IAL Mathematics – Pure Mathematics content

Pure 2 (2.1) Algebraic division, factor theorem (Unit 2 of the P2 SoW)

KEYWORDS

Polynomial, numerator, denominator, factor, quadratic, power, index, coefficient, degree, squared, coefficients, improper, identity, partial fraction, rational.

OBJECTIVES

By the end of the unit, students should:

- be able to split a proper fraction into partial fractions;
- be able to split an improper fraction into partial fractions, dividing the numerator by the denominator (by polynomial long division or by inspection).

TEACHING POINTS

Stress the fact that when we break up a fraction into two or more partial fractions, we use an identity (\equiv) sign, and not an equals sign, as the expressions are equivalent for all values of x .

Start with a pair of algebraic fractions that need to be added together. Stress that the single fraction answer may be simplified, but that it can often be difficult to work with. For example in order to integrate the fraction it may be necessary to split it back up into two (or more) partial fractions. In other words, the reverse process needs to be carried out.

The number of partial fractions and the format of the individual terms, is dependent on two factors.

- The maximum power (or degree) of the polynomials of the numerator and denominator.
The degree of the denominator must be *greater* than that of the numerator.
If the degree is equal or the degree of the numerator is greater (i.e. the fraction is improper), then algebraic division must be carried out first, and then the partial fractions formed.
- The type and power of denominator.
If the denominator is, e.g. $(x + 2)^2$, then we call this a *repeated* factor. In order to cover all possibilities of factors this has to be set up as two partial fractions with denominators $(x + 2)$ and $(x + 2)^2$.

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Show a numerical example with a denominator of 25, and hence the denominators of the partial fractions are 5 and 25.)

Examples of each of the following types need to be covered.

Linear: $\frac{5x-5}{(x+3)(x-2)}$ $\frac{2}{x^2-1}$ $\frac{7x+3}{x(x+1)}$

Repeated: $\frac{4x^2-3x+5}{(x-1)^2(x+2)} \equiv \frac{A}{(x-1)^2} + \frac{B}{(x-1)} + \frac{C}{(x+2)}$

Improper: $\frac{2x^2+5x-6}{(2x-1)(1+x)} \equiv A + \frac{B}{2x-1} + \frac{C}{1+x}$

As students work through examples, encourage them to experiment with the choice of values they substitute. If necessary remind them that $x = 0$ is an option. Also show that equating coefficients can sometimes be a more efficient alternative, sometimes avoiding the necessity for simultaneous equations.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Are there any values which make the denominators zero? Make links with the graphs of the functions and talk about how these values will correspond to exceptions and special cases in future topics where partial fractions need to be found as a simplifying step.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Some students will set up and solve simultaneous equations rather than using values of x to work out missing constants.

Ensure students are aware of the most efficient methods for solving different types of problem so they do not waste time in exam situations.

NOTES

Partial fractions to include denominators such as:

$(ax + b)(cx + d)(ex + f)$ and $(ax + b)(cx + d)^2$.

The specification notes state, 'Quadratic factors in the denominator such as $(x^2 + a)$, $a > 0$ are *not* required.

This work has applications in other topics such as series expansions, differentiation and integration.

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UNIT 3: Coordinate geometry in the (x, y) plane:

Definition and converting between parametric and Cartesian forms (3.1)

Teaching time
3 hours

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SPECIFICATION REFERENCES

3.3 Parametric equations of curves and conversion between Cartesian and parametric forms

PRIOR KNOWLEDGE

Covered so far

- Trigonometric identities
- Knowledge of a variety of functions involving powers, roots, trigonometric functions, exponentials and logarithms

International GCSE/GCSE (9-1) in Mathematics at Higher Tier

- Coordinate geometry
- Changing the subject of the formula, and substitution
- Graphs of linear, quadratic and trigonometric functions

IAL Mathematics – Pure Mathematics content

Pure 1 (2.1, 2.2), Pure 2 (3.1) Coordinate geometry (Unit 3 of P1 and P2 SoW)

Pure 2 (6.1) Trigonometric identities (Unit 6 of the P2 SoW)

KEYWORDS

Parametric, Cartesian, convert, parameter t , identity, eliminate, substitute, circle, hyperbola, parabola, ellipse, domain, modelling.

OBJECTIVES

By the end of the unit, students should:

- understand the difference between the Cartesian and parametric system of expressing coordinates;
- be able to convert between parametric and Cartesian forms.

TEACHING POINTS

Begin by explaining the difference between the Cartesian system, when a graph is described using $y = f(x)$, and the parametric system, which uses $x = f(t)$ and $y = g(t)$ for some parameter t .

Illustrate this by asking the class to consider $x = 5t$ and $y = 3t^2$ and to try to eliminate t from the two equations. This will give $y = \frac{3}{25}x^2$ or $25y = 3x^2$. (This is a quadratic equation – a parabola.)

Repeat for $x = 5t$ and $y = \frac{5}{t}$. This becomes $y = \frac{25}{x}$ (a hyperbola).

Sometimes we need to eliminate the parameter, t , by using identities rather than substitution.

Consider $x = 3 \cos t$ and $y = 3 \sin t$. Squaring both equations and adding means we can use $\cos^2 t + \sin^2 t = 1$ to give $x^2 + y^2 = 9$. (This is a circle, centre $(0, 0)$ of radius 3.)

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Ask students to use similar methods to show that $x = 2 + 5 \cos t$, $y = -4 + 5 \sin t$ describes a circle centre $(2, -4)$ with radius 5.

How do we convert from Cartesian to parametric? For example, what are the pair of parametric equations for a circle, centre $(3, 5)$ radius 10? (Remember we need to be in radians.)

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

What shape is given by $x = 4 \cos t$, $y = 2 \sin t$?

Name and properties of curve?

The trigonometric identities (such as $\sec^2 x = 1 + \tan^2 x$) can be used to convert from parametric to Cartesian form.

A shape may be modelled using parametric equations (e.g. an object moves with constant velocity from $(1, 8)$ at $t = 0$ to $(6, 20)$ at $t = 5$), or students may be asked to find parametric equations for a motion.

Make links to kinematics.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students may have difficulties making any progress with these sorts of questions if they cannot work out which trigonometric identity to apply when eliminating the parameter t .

The examiner comments for these questions illustrate how difficult students find this topic:

The final part proved very demanding and only a minority of students were able to use one of the trigonometric forms of Pythagoras to eliminate t and manipulate the resulting equation to obtain an answer in the required form.

Few even attempted the domain and the fully correct answer $0^\circ \leq t \leq 2\pi$, was very rarely seen.

NOTES

Later in the course, students will need to be able to differentiate (using the chain rule) parametric equations to find tangents, normals, turning points etc.

Also, we will be integrating parametric equations and finding areas under curves (see Units 5 & 6).

Parametric equations is assumed knowledge for the calculus work in section 6, where students must find the volume of revolution for a solid formed by a pair of parametric equations

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UNIT 4: Binomial expansion

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SPECIFICATION REFERENCES

4.1 Binomial series for any rational n

PRIOR KNOWLEDGE

International GCSE/GCSE (9-1) in Mathematics at Higher Tier

- Algebraic fractions

IAL Mathematics – Pure Mathematics content

Pure 2 (2.1) Algebraic division, factor theorem (Unit 2 of the P2 SoW)

Pure 2 (4.5) Binomial expansion of the form $(a + bx)^n$, where n is a positive integer (Unit 4d of the P2 SoW)

KEYWORDS

Binomial, expansion, theorem, integer, rational, power, index, coefficient, validity, modulus, factorial, nC_r , combinations, Pascal's triangle, partial fractions, approximation, converges, diverges, root.

NOTES

The formula book includes formulae for the binomial expansion:

$$(a + b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{r}a^{n-r}b^r + \dots + b^n \quad (n \in \mathbb{N})$$

$$\text{where } \binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1 \times 2}x^2 + \dots + \frac{n(n-1)\dots(n-r+1)}{1 \times 2 \times \dots \times r}x^r + \dots \quad (|x| < 1, n \in \mathbb{R})$$

This unit links with the binomial distribution in the Statistics 1 content.

A level Mathematics: Pure Mathematics 4

4a. Expanding $(a + bx)^n$ for rational n ; knowledge of range of validity (4.1)

Teaching time
4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to find the binomial expansion of $(1 - x)^{-1}$ for rational values of n and $|x| < 1$;
- be able to find the binomial expansion of $(1 + x)^n$ for rational values of n and $|x| < 1$;
- be able to find the binomial expansion of $(1 + bx)^n$ for rational values of n and $|x| < \frac{1}{|b|}$;
- be able to find the binomial expansion of $(a + x)^n$ for rational values of n and $|x| < a$;
- be able to find the binomial expansion of $(a + bx)^n$ for rational values of n and $\left|\frac{bx}{a}\right| < 1$;
- know how to use the binomial theorem to find approximations (including roots).

TEACHING POINTS

Begin by reviewing the expansion of $(a + b)^n$ when n is a positive integer.

Ask students to expand $(1 + x)^4$ and then try $(1 + x)^{-2}$. Why does it fail to work? Which coefficient calculation breaks down?

Explain how the binomial theorem allows us to expand *any* power. (Explain the reasoning behind the factorial notation using the explanation in the *Reasoning and problem solving* section below.)

Consider why the expansions are infinite when the power is *not* a positive integer. How far do we need to expand and to which term? (For example, up to and including coefficients of x^3 .)

Take care to show the precision needed when dealing with negative calculations by demonstrating examples such as $(1 - 2x)^{-\frac{1}{2}}$.

If we expanded $(1 + x)^{\frac{1}{2}}$ then substituted $x = -0.1$, we would effectively be finding the square root of 0.9.

Ask students to use a calculator to find an accurate value for $\sqrt{9}$. How many terms of the expansion would we need to substitute into in order to get a 4 decimal place version of the accurate value?

What happens when we substitute $x = 3$? Does this find the square root of 4?

Explain that if we raise a number > 1 to a positive power, it ‘grows’ and diverges out of control. This means that the value of x must be such that $-1 < x < 1$ or $|x| < 1$ in order to use the expansion of $(1 + x)^n$. The validity of the expansion is dependent upon the value of x we substitute into the terms.

Cover examples that build-up the expansions listed in the objectives above, ending with $(a + bx)^n$ for rational values of n and valid for $\left|\frac{bx}{a}\right| < 1$.

Introduce the concept of expansions of expressions which start with a rather than 1. Begin by showing that if we have $(2 + x)$ and if we want to make this start with a 1 in the bracket, we must take out the factor of 2, giving $2\left(1 + \frac{x}{2}\right)$.

Now show for example, that $2(1 + 4)$ gives the same result as $(2 + 8)$ if we multiplied this out, but that if the bracket were squared the result would not be the same i.e. $2(1 + 4)^2 \neq (4 + 8)^2$.

However, $2^2(1 + 4)^2 = (4 + 8)^2$, so we need to raise the factor to the same power of the bracket and $(a + bx)^n = a^n\left(1 + \frac{bx}{a}\right)^n$.

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OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Show how ${}^n C_r$ will only work on a calculator for positive integer values of n (as done in Pure 2, Unit 4).

However, we can instead use the definition and formula for selections (shown here for ‘choose 2 from n different objects’).

$$\begin{aligned} {}^n C_2 &= \frac{n!}{2!(n-2)!} \\ &= \frac{n(n-1)\cancel{(n-2)}\cancel{(n-3)}\dots}{2!\cancel{(n-2)}\cancel{(n-3)}\dots} \\ &= \frac{n(n-1)}{2!} \end{aligned}$$

This formula works for *all* values of n and follows the pattern of the binomial theorem as stated in the formula book.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

When expanding $(1 + 4x)^{\frac{1}{2}}$ most students got the first two terms of the expansion correct, but often there was a mistake in the x^2 term, with $4x$ becoming just x being the common error. Some students made arithmetic errors with 4^2 , by failing to actually square the 4, and others failed to simplify the binomial coefficient correctly.

When expanding an expression of the form $(a + x)^n$ a common error is to write this as $a(1 + \frac{x}{a})^n$ rather than $a^n(1 + \frac{x}{a})^n$.

Other errors include algebraic errors when combining two expansions, doing more work than is necessary when, for example, only terms up to x^2 are required, including the equality in the expression for the range of valid values for x and lack of understanding when using the modulus symbol (writing expressions such as $|x| < -4$).

NOTES

Link this section to the next part of this unit: expanding functions by first using partial fractions.

A level Mathematics: Pure Mathematics 4

4b. Expansion of functions by first using partial fractions (4.1)

Teaching time

3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to use partial fractions to write a rational function as a series expansion.

TEACHING POINTS

This sub-unit links with the earlier sub-unit on partial fractions and gives the students a purpose for learning how to break-up a rational function into two or more partial fractions.

If we consider the ‘complicated’ fraction below, it needs to be simplified into two simpler fractions each of which only involve a single algebraic bracket.

$$\frac{2x^2 + 5x - 10}{(x-1)(x+2)} \equiv A + \frac{B}{x-1} + \frac{C}{x+2}$$

We can now rewrite each term as a binomial series. (It is important to demonstrate that the $\frac{B}{x-1}$ term will become $B(x-1)^{-1}$.)

Particular care needs to be taken when working with brackets that don’t start with 1, and also when multiplying out all the terms to arrive at the final simplified series (up to and including the power required).

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

You will need to assess all the separate validities for the individual binomial terms to declare the validity for the final series.

Include examples in which one of the terms is not a binomial and just multiplies without expansion. For example, $(2x-1)(1+3x)^{-2}$.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

(These all relate directly to the example given above.)

Nearly all students were able to make the connection between the parts of the question, but there were many errors in expanding both $(x-1)^{-1}$ and $(2+x)^{-1}$.

Few were able to write $(x-1)^{-1}$ as $-(1-x)^{-1}$ and the resulting expansions were incorrect in the majority of cases, both $1+x-x^2$ and $1-x-x^2$ being common errors.

However, $(2+x)^{-1}$ was handled better, but the constant $\frac{1}{2}$ in $\frac{1}{2}\left(1+\frac{x}{2}\right)^{-1}$ was frequently incorrect.

NOTES

Inform the students that partial fractions are also required to break down rational functions before they are differentiated and integrated.

A level Mathematics: Pure Mathematics 4

UNIT 5: Differentiation

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SPECIFICATION REFERENCES

- 5.1 Differentiation of simple functions defined implicitly or parametrically
- 5.2 Formation of simple differential equations

PRIOR KNOWLEDGE

Covered so far

- Function notation including $f'(x)$

IAL Mathematics – Pure Mathematics content

Pure 1 (4.2) Differentiation (Unit 4 of the P1 SoW)

KEYWORDS

Derivative, parametric, implicit, differential equation, rate of change.

NOTES

This topic builds on the differentiation covered already and leads into integration.

A level Mathematics: Pure Mathematics 4

5a. Differentiating implicit and parametric functions (5.1)

Teaching time

5 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to differentiate parametric equations;
- be able to find the gradient at a given point from parametric equations;
- be able to find the equation of a tangent or normal (parametric);
- be able to use implicit differentiation to differentiate an equation involving two variables;
- be able to find the gradient of a curve using implicit differentiation;
- be able to verify a given point is stationary (implicit).

TEACHING POINTS

For parametric differentiation, make links with the chain rule to give $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

Stress that we often substitute in the value of the parameter t at the point which we need to find the gradient. Many questions will involve trigonometric functions, so students must be fluent at differentiating these.

For implicit differentiation, consider the equation of a circle, $x^2 + y^2 = 16$. To differentiate this function we would have to make y the subject of the formula. Sometimes this can be difficult or even impossible.

Make sure students can confidently differentiate terms like x^2y using implicit differentiation. Finally, stress that we need to substitute in *both* x and y coordinates to find the gradient at a certain point.

Students may have to apply the product or quotient rules in implicit differentiation questions and should be given examples of this. In exam questions students are almost always required to find the gradient through implicit differentiation.

Take a point on a circle or another type of curve and find the gradient using both parametric and implicit differentiation. Then find the equation of tangent and/or normal and see that both methods give the same answer.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Cover questions involving finding tangents, turning points and normals (this links with Pure 1, Unit 4).

For example students could be asked to find the normal at the points on the circle $(x - 1)^2 + (y + 2)^2 = 20$ where $x = 3$. Students could be encouraged to convert the Cartesian form of the circle into parametric form. Graphing calculators or Apps can be used to check solutions.

Some questions require a trigonometric identity in order to simplify the solution, so this is a good opportunity to review the trigonometric identities from Pure 3, Unit 2.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Most students are able to implicitly differentiate an equation correctly. Common errors are failing to apply the product rule correctly, and making arithmetic slips when collecting terms and rearranging.

A level Mathematics: Pure Mathematics 4

Most students are able to differentiate using parametric equations by dividing $\frac{dy}{dt}$ by $\frac{dx}{dt}$, although there is often some confusion in notation. Sign errors are common when differentiating sin and cos, and errors are made when re-arranging or simplifying expressions.

Some candidates prefer to convert the parametric equations in to Cartesian form before differentiating, but this should be discouraged as it often leads to an expression which is difficult to differentiate.

A level Mathematics: Pure Mathematics 4

5b. Rates of change problems (including growth and decay) (5.2)

Teaching time

3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- know how to model the growth or decay of 2D and 3D objects using connected rates of change;
- be able to set up a differential equation using given information which may include direct proportion.

TEACHING POINTS

This content links to kinematics, where velocity is considered as $\frac{ds}{dt}$ and acceleration as $\frac{dv}{dt}$.

An example of rate of change in population might be:

In a lake the amount of a particular weed (P) grows at a rate proportional to the amount of weed already present in the lake. The weed is removed by fish eating it at a constant rate of 10 kg per day.

- Write down a differential equation relating P , the amount of weed in kg, and t , the time which has elapsed since the start of the investigation.
- Given that the weed grows at a rate of 5 kg per day when there is 90 kg of weed present in the lake, find the rate of growth of the weed, in kg/day, when there is 120 kg of weed present.

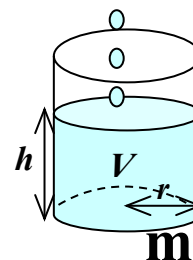
OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Consider water entering this cylinder. To work out the rate at which the height is increasing we need to calculate $\frac{dh}{dt}$.

In exam questions, the rate that the volume of water increases at is often given as $\frac{dV}{dt}$. Therefore, we need to use the chain rule to create $\frac{dh}{dt}$ from $\frac{dV}{dt}$.

$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$ so we need a formula connecting h and V .

$V = \pi r^2 h$ and from this we can work out $\frac{dV}{dh}$ and then $\frac{dh}{dV}$ etc.



COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Most students are able to substitute correctly into a formula for exponential growth and decay.

When required to set up an inequality most students showed that they understood the information given and wrote down a correct opening expression, although there was uncertainty over which way the inequality should go. Some then simplified and solved using logarithms efficiently to get the correct answer. Some resorted to trial and improvement which was accepted for full marks if done correctly, but was worth no marks otherwise.

When solving equations involving exponentials, knowledge of using logarithms varied widely. Many were unable to deal properly with the coefficient and the exponential term and wrote down equations in which t actually should have cancelled out.

Some care needs to be taken when interpreting the answers to exponential growth and decay questions to ensure they are given in the correct form e.g. to the nearest year, second etc.

NOTES

For first order differential equations (which require separating variables) see Unit 6e.

A level Mathematics: Pure Mathematics 4

UNIT 6: Integration

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SPECIFICATION REFERENCES

- 6.1 Evaluation of volume of revolution
- 6.2 Simple cases of integration by substitution and integration by parts. Understand these methods as the reverse processes of the chain and product rules respectively
- 6.3 Simple cases of integration using partial fractions
- 6.4 Analytical solution of simple first order differential equations with separable variables

PRIOR KNOWLEDGE

IAL Mathematics – Pure content

Pure 2 (8.1) Definite integrals (Unit 8 of the P2 SoW)

KEYWORDS

Rotation, solid of revolution, volume of revolution, bounded area, arc, cubic units, parameter, Cartesian equation, partial, fraction, substitution, by parts, area, differential equation, first order, separating variables, initial conditions, general solution, particular solution.

A level Mathematics: Pure Mathematics 4

6a. Volumes of revolution (6.1)

Teaching time

5 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to derive formulae for, and calculate volumes of, revolution about the x -axis;
- be able to find volumes of revolution for functions given in parametric form.

TEACHING POINTS

Volume for rotation about the x -axis $V = \pi \int y^2 dx$ IS required.

Stress the importance of quoting the general formula for volume before substituting values.

Look at using more complex functions for y or x so when squaring will challenge and develop their integration skills.

Revise parametric equations and conversion between parametric and Cartesian forms.

Care should be taken when the equation is given in parametric form as it may be easier to integrate the resulting function in either Cartesian or parametric form.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Finding the volume of a cone.

Finding the volume of a sphere.

Finding the volume of a section rotated about the axis. Subtracting one volume from the other. Torus?

Finding the volume of shapes rotated around a line that is parallel to the x -axis.

Use graphing software to investigate parametric equations which give closed loops.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

‘It was expected that students should at least quote the general formula for volume and then substitute their derivative. It is good practice to quote formulae before substitution. When an error is made on substitution the examiner needs to be sure that the correct formula is being used before the method mark can be awarded’.

Students must be sure of the correct $\pi \int y^2 dx$ formula for volume when the variables are given parametrically. ‘Some rewrote the volume formula as $\pi \int y^2 d\theta$, with dx being replaced by $d\theta$. Those who did not apply $\pi \int y^2 \frac{dx}{d\theta} d\theta$ gained little access to this question, and it was also disappointing to see some students who attempted to apply an incorrect $\pi \int \left(y \frac{dx}{d\theta}\right)^2 d\theta$ ’.

A level Mathematics: Pure Mathematics 4

6b. Integration by substitution (6.2)

Teaching time

4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to integrate expressions using an appropriate substitution;
- be able to select the correct substitution and justify their choices.

TEACHING POINTS

Most students find integration by substitution challenging and will need to complete lots of different styles of questions. It is a good idea to start with an example which can be performed by inspection as the reverse of differentiation.

Students also like to have a step by step process.

1. Use the given substitution or decide on your own. The substitution is usually the contents of a bracket, square root or the 'nasty' bit. i.e. Let $u = \dots$
2. Differentiate the substitution i.e. $\frac{du}{dx} = \dots$
3. Make dx the subject of the formula
4. Replace the dx and make the substitution into the integrand
5. Cancel out any remaining x^*
6. Integrate the resulting (simpler) integral
7. Substitute back to get the answer in terms of x again

*If there are any remaining x , you can re-use the substitution making the x the subject

For expressions including trigonometric functions, the identities involving $\sin^2 x$, $\sec^2 x$ are often useful to simplify the integrand.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Try to encourage students to experiment with different substitutions, particularly types involving expressions such as $\sqrt{3x+4}$. Do we use $u^2 = 3x+4$ or $u = 3x+4$? The former will require implicit differentiation.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Mistakes students make when attempting to integrate by substitution include not changing the dx correctly and simply writing it as du , and failing to substitute back to give an expression in x at the end.

NOTES

Return to this method when covering areas under curves as the limits need to be changed by substituting them into the required substitution.

OBJECTIVES

By the end of the sub-unit, students should:

- be able to integrate an expression using integration by parts;
- be able to select the correct method for integration and justify their choices.

TEACHING POINTS

It is a good idea to show how the product rule for differentiation can be integrated on both sides to derive the ‘by parts’ formula (which is given in the formulae booklet).

Students are usually able to start questions using this method but struggle to get to full solutions and will require lots of practice with algebraic manipulation.

Time should be spent discussing the choice of u and dv . It is usually advisable to select the polynomial to be the u as it simplifies to a lower power after calculating du , thus making the second integral easier than the original question.

Students should recognise that $\ln x$ cannot be integrated simply and should therefore always be chosen as u .

$\ln x$ itself can be integrated using this method taking $u = \ln x$ and $dv = 1$ (as we cannot integrate $\ln x$, but can differentiate it to give $\frac{1}{x}$). The dv becomes more complicated, but then simplifies in the second integral with the $\frac{1}{x}$.

More able students should be able to access questions where it is necessary to use integration by parts twice (e.g. $u = x^2$).

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Consider the integral of $e^x \cos x$ and show that the application of ‘by parts’ loops back to the original question. Refer to the equation $x = 4 - x$ and contrast this with the structure of this example.

Let the original question be I (for integral) and this can lead to $2I = \dots$

[This is a pre-requisite for reduction formulae in Further Pure Mathematics.]

Students should integrate functions such as $\int x(x + 3)^6 dx$ using both ‘by parts’ and ‘substitution’ to show that they give the same answer. This is a good activity for discussion as initially they appear to be different, but after some algebraic manipulation give the same answer.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Common errors when integrating by parts include: choosing u and dv incorrectly (in particular $\ln x$ must always be chosen as u); algebraic errors – especially if they do not remove any common factors to outside the integral sign; incorrect coefficients when integrating dv ; and sign errors where \sin and \cos are involved.

NOTES

The method of integration by parts may be specified in the question.

A level Mathematics: Pure Mathematics 4

6d. Use of partial fractions (6.2)

Teaching time

2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to integrate rational expressions by using partial fractions;
- be able to simplify the expression using laws of logarithms.

TEACHING POINTS

Revise the simplification of rational expressions into partial fractions. We have already seen that this technique is useful in binomial expansions.

Often the first part of an integration question of this sort will ask students to split the fraction into two (or more) partial fractions.

The next part will then ask for the integration to be carried out. For example:

$$\text{Integrate } \int \frac{5}{(x-1)(3x+2)} dx.$$

$$\text{This will lead to } \int \frac{5}{(x-1)(3x+2)} dx = \int \left(\frac{1}{x-1} - \frac{3}{3x+2} \right) dx = \ln(x-1) - \ln(3x+2) (+ c)$$

It is sometimes sufficient to leave the answer in this form, but ‘Show that’ questions will influence the further simplification using laws of logs.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

You should cover repeated factors, which will lead to, for example, $(x-2)^2$ in the denominator, which will not be a log integral.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Partial fractions questions are generally done well though some students attempt to integrate the numerator and denominator separately without using partial fractions.

NOTES

These integrals will sometimes be tested via a differential equation later in the course and laws of logs will form a vital role in finding the general solution. Definite integrals may also need to be calculated and simplified numerically. e.g. $\ln 6 - \ln 2 = \ln \frac{6}{2} = \ln 3$.

The specification states that integration of other rational expressions, such as $\frac{x}{x^2+5}$ and $\frac{x}{(2x-1)^4}$ is also required.

OBJECTIVES

By the end of the sub-unit, students should:

- be able to solve a differential equation by separating variables;
- be able to write a differential equation from a worded problem;
- be able to use a differential equation as a model to solve a problem;
- be able to substitute the initial or otherwise conditions into the equation to find $+c$ and the particular solution;
- be able to use a model to find the value after a given time;
- be able to set up and use logarithms to solve an equation for an exponential growth or decay problem.

TEACHING POINTS

The solution for a differential equation can be found using a method called ‘separating variables’, in which we rearrange and split up the $\frac{dy}{dx}$ as if it is a fraction. It is vital to keep all the y ’s and dy ’s *and* the x ’s and dx ’s together, but also the dx and dy must be in the numerator on each side.

The full solution is shown below.

$$\frac{dy}{dx} = 2y$$

To find the general solution by Calculus, we need to integrate with respect to x , yet the variable on the r.h.s. is y

Separating the variables, we get

$$\frac{dy}{y} = 2dx$$

Notice that the operator is separated. The constant 2 can be on either side as can the $+C$.

We can now integrate: $\int \frac{dy}{y} = \int 2 dx \Rightarrow \ln y = 2x + C$

This can be written as: $y = e^{2x+C}$

which can be simplified to $y = e^{2x}e^C$ or $y = Ae^{2x}$

Index laws can split e^{2x+C}
Also, $e^c = A$ (another constant)

$$\frac{dy}{dx} = ky \Rightarrow y = Ae^{kx} \text{ where } k \text{ is a constant}$$

$y = Ae^{2x}$ is a general solution, but how do we find the value of the constant A ? We need to have some information about the data from which the differential equation originates. Something along the lines of ‘when $x = 0, y = 2$ ’.

Substituting this pair of values into the general solution and finding the value of A , will lead to a particular solution.

Sometimes we may have a choice of pairs to substitute or we may have two pairs of values in order to work out two constants.

A level Mathematics: Pure Mathematics 4

Explain that questions may be set in a context and, in these cases, students need to interpret the solution of the differential equation in the context of the problem. This may include identifying limitations of the solution.

The following example is typical:-

The population of a town was 50 000 in 2010 and had increased to 55 000 by 2015. Assuming that the population is increasing at a rate proportional to its size at any time, estimate the population in 2020 giving your answer to the nearest hundred.

$\frac{dn}{dt} = kn \Rightarrow n = Ae^{kt}$ as above, but now n is the number of people and t is the time in years.

The validity of the solution for large values should be considered. For example, if the question was modelling population growth, would it be realistic for the value to keep increasing forever?

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

For separating variables and finding the particular solution, encourage the more able students to use the initial conditions as the limits of integration, thus avoiding the $+ c$.

Graphing software could be used to investigate solutions to differential equations, and how the solution changes as the initial conditions change.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Examiner comments indicate that this can prove a difficult topic for some students:

When forming a differential equation some students wrote down the correct differential equation apparently fully understanding all the information given and interpreting it correctly. However, all sorts of errors abounded in other attempts, some not even involving a derivative, and some with derivatives in x and y .

Many had a spurious t and/or h , either as a multiple or power, and the k appeared in a variety of places. Some students did not even form an equation, leaving a proportionality sign in their answer.

When solving a differential equation most students knew they were expected to separate the variables and did it correctly, although there were some notation errors in the positioning of dx , at the front rather than the rear of the integrand. Those who failed to separate the variables, just produced nonsense. Many students struggled with the fact that integration by parts or substitution was needed. All students, no matter what their attempt at the integral, could obtain a method mark if they included a constant and tried to find it using the given initial conditions.

NOTES

Link this topic to kinematics. For example solving differential equations of the form $\frac{dv}{dt} = 3t^2$ (when $t = 0$, $v = 4$). Separating variables leads to $v = t^3 + c$ etc.

$\frac{dv}{dt}$ is the acceleration and this shows that if we integrate the acceleration, we get the velocity.

A level Mathematics: Pure Mathematics 4

UNIT 7: Vectors

[Return to overview](#)

SPECIFICATION REFERENCES

- 7.1 Vectors in two and three dimensions
- 7.2 Magnitude of a vector
- 7.3 Algebraic operations of vector addition and multiplication by scalars, and their geometrical interpretations
- 7.4 Position vectors
- 7.5 The distance between two points
- 7.6 Vector equations of lines
- 7.7 The scalar product. Its use for calculating the angle between two lines

PRIOR KNOWLEDGE

Covered so far

- Surds

International GCSE/GCSE (9-1) in Mathematics at Higher Tier

- Vectors

KEYWORDS

Vector, scalar, column, \mathbf{i} , \mathbf{j} , \mathbf{k} , magnitude, direction, component, parallel, perpendicular, modulus, dimension, ratio, collinear, scalar product, position vectors, angle, unit vector, orthogonal, vector addition/subtraction, vector equation, scalar product, dot product, collinear, skew, concurrent.

A level Mathematics: Pure Mathematics 4

7a. Definitions, magnitude/direction, addition and scalar multiplication (7.1) (7.2) (7.3)

Teaching time
5 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to use vectors in two and three dimensions using column vectors and \mathbf{i} , \mathbf{j} and \mathbf{k} unit vectors;
- be able to calculate the magnitude and direction of a vector and convert between component form and magnitude/direction form;
- be able to add vectors diagrammatically and perform the algebraic operations of vector addition and multiplication by scalars, and understand their geometrical interpretations.

TEACHING POINTS

Students need to be familiar with column vectors and with the use of \mathbf{i} , \mathbf{j} and \mathbf{k} vectors. When considering 3D vectors, begin by showing some 3D coordinates on x, y, z axes. (Graph drawing packages are very useful here, especially if you can turn the grid to view from different positions.)

Students should be able to find a unit vector in the direction of \mathbf{a} , and be familiar with the notation $|\mathbf{a}|$.

The triangle and parallelogram laws of addition should be known and students should be able to use them. Students should understand that vectors are commutative.

Where answers are given in surds they should be simplified if possible.

When performing operations on vectors this should also be understood geometrically, diagrams will be helpful here. Students should be able to use given diagrams but also draw their own in order to assist with questions.

Students should understand and be able to use the conditions for parallel vectors.

Use the classroom floor as a 2-dimensional grid to help students visualise vectors. Use the position of students in the room to illustrate concepts. For three dimensions, consider a cuboid (2 by 3 by 4), with one corner at the origin. Ask the class to write down the coordinates of all the vertices.

Consider vectors in the real world, e.g. ask students to think of everyday phenomena that have a magnitude and direction e.g. forces, velocities, displacements.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Students can prove vectors are parallel to demonstrate their reasoning skill.

Given particular vectors, students can investigate places they can or cannot reach, for example the knights problem on a chessboard.

Consider an aircraft landing in a cross-wind – what direction does it need to fly?

Link examples to mechanics (kinematics and forces). For example, consider questions such as:

The velocity of an object is given by vector $\mathbf{v} = 3t\mathbf{i} + t^2\mathbf{j} + 4t\mathbf{k}$. What is its speed after 5 seconds?

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students sometimes make mistakes when manipulating vectors in \mathbf{i} , \mathbf{j} , \mathbf{k} form and should be encouraged to use column vectors when possible.

Encourage students to draw diagrams to help their geometrical thinking when answering vector questions.

Stress the importance of reading the question carefully and giving answers in the correct way, for example coordinates or column vectors may be requested.

A level Mathematics: Pure Mathematics 4

7b. Position vectors, distance between two points, geometric problems (7.4) (7.5)

Teaching time
3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand and be able to use position vectors;
- be able to calculate the distance between two points represented by position vectors;
- be able to use vectors to solve problems in pure mathematics and in context, (including forces).

TEACHING POINTS

Students should know and be able to use $\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.

Students should be able to calculate the distance between two points (x_1, y_1) and (x_2, y_2) using the formula $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$. Extend this idea to calculating the distance d between two points in three dimensions (x_1, y_1, z_1) and (x_2, y_2, z_2) using $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$

Use the ratio theorem to find the position vector of a point C dividing AB in a given ratio.

Use familiar shapes to illustrate the difference between two vectors and vector addition, e.g. parallelogram, rectangle.

Use vectors to solve problems and discuss the 3D geometrical interpretations of solutions.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Finding position vector of the fourth corner of a shape (e.g. parallelogram) $ABCD$ with three given position vectors for the corners A , B and C .

Use regular polygons to find vectors connecting different vertices and to illustrate the ratio theorem.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Examiners comment that students understand the simple basics of vectors but are unable to deal with the complexity of ratios. Students should be given plenty of practice in identifying points that divide line segments in a particular ratio both externally and internally.

Emphasise the importance of good notation. Students do not always understand that AP^2 represents the square of the length AP .

A level Mathematics: Pure Mathematics 4

7c. Vector equation of a line (7.6)

Teaching time

5 hours

OBJECTIVES

By the end of the sub-unit, students should:

- know how to find the vector equation of a line in both two and three dimensions;
- be able to determine whether two lines in three dimensions intersect and find the point of intersection;
- be able to determine whether two lines are parallel.

TEACHING POINTS

Students should be familiar with the vector form ($\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and $\mathbf{r} = \mathbf{c} + t(\mathbf{d} - \mathbf{c})$) of the equation of a straight line. Show similarities with $y = mx + c$. Discuss how we can decide whether two lines are parallel, and discuss whether lines in 3D space will always intersect or not. Students should be able to determine whether two lines intersect, by forming three simultaneous equations, and find the point of intersection.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Can link to mechanics and the intersection of vectors, both in 2D and 3D. Can touch on Euclidean geometry as use of these in the real world context. Problems such as finding the distance between two particles could be considered, and applying intersecting lines to practical situations such as determining whether two ships will collide.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Students must give the equation of a line as an equation and not as an expression, i.e. they will lose marks for not writing $\mathbf{r} = \dots$. They should also remember to consider all three equations when determining whether two lines in 3D intersect.

A level Mathematics: Pure Mathematics 4

7d. Scalar product (7.7)

Teaching time

5 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to find the scalar product of two vectors;
- be able to check whether vectors are perpendicular by using the scalar product;
- be able to use the scalar product to calculate the angle between two lines.

TEACHING POINTS

The formulae for angle between two lines: $\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}$ needs to be learnt. Since $\cos \theta = 0$ when $\theta = \frac{\pi}{2}$, students can deduce that the scalar product = 0 when two vectors are perpendicular. Investigate the scalar product of the vectors \mathbf{i} , \mathbf{j} and \mathbf{k} , and show that the scalar product of the vectors $a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$ and $b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k}$ is $a_1 \times b_1 + a_2 \times b_2 + a_3 \times b_3$. Students should be able to find the angle between two lines using the direction vectors, and they should understand that a negative value for the scalar (or dot) product implies the angle is obtuse.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Proof of the cosine rule using vectors uses the scalar product. Problems such as finding the shortest distance from a point to a line can be considered, as well as problems involving areas of triangles and parallelograms in the 3D plane.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Students must check whether the answers for angles should be given in degrees or radians, and also whether the angle should be acute or obtuse.

When applying the scalar product formula, the direction of the vectors is important. The vectors should be coming 'out' of the angle.