INTERNATIONAL ADVANCED LEVEL

MATHEMATICS/ FURTHER MATHEMATICS/ PURE MATHEMATICS

SCHEME OF WORK

FURTHER PURE MATHEMATICS 3

Pearson Edexcel International Advanced Subsidiary in Mathematics (XMA01)
Pearson Edexcel International Advanced Subsidiary in Further Mathematics (XFM01)
Pearson Edexcel International Advanced Subsidiary in Pure Mathematics (XPM01)
Pearson Edexcel International Advanced Level in Mathematics (YMA01)
Pearson Edexcel International Advanced Level in Further Mathematics (YFM01)
Pearson Edexcel International Advanced Level in Pure Mathematics (YPM01)
First teaching September 2018
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A level Mathematics: Pure Mathematics 3

Further Pure Mathematics 3

Unit		Title	Estimated hours
1		**Hyperbolic functions**	
	a	sinh x, cosh x, tanh x and their inverses	**4**
	b	Logarithmic forms of the inverse hyperbolic functions, solving equations involving hyperbolic functions	**4**
2		**Further coordinate systems**	
	a	Equations of the ellipse and hyperbola and their focus-directrix properties	**3**
	b	Tangents and normals to the ellipse and hyperbola	**3**
	c	Simple loci problems	**4**
3		**Differentiation**	
	a	Differentiate the hyperbolic functions	**3**
	b	Differentiate inverse trigonometric and hyperbolic functions	**4**
4		**Integration**	
	a	Integration of the hyperbolic and inverse hyperbolic functions	**3**
	b	Integrate using hyperbolic and trigonometric substitutions	**3**
	d	Reduction formulae	**4**
	e	The calculation of arc length	**2**
	f	The calculation of the area of a surface of revolution	**2**
5		**Vectors**	
	a	The vector product $\mathbf{a} \times \mathbf{b}$ and the scalar triple product $\mathbf{a} . \mathbf{b} \times \mathbf{c}$, and their applications	**3**
	b	Problems involving points, lines and planes	**5**
	c	Vector and Cartesian equations of a line and a plane	**3**
6		**Further matrix algebra**	
	a	Linear transformations	**2**
	b	Inverse of and 3×3 matrices	**2**
	c	Eigenvalues and eigenvectors of 2×2 and 3×3 matrices	**4**
	d	Reduction of symmetric matrices to diagonal form	**2**
			60 hours

UNIT 1: Hyperbolic functions

Return to Overview

SPECIFICATION REFERENCES

1.1 Definitions of the six hyperbolic functions in terms of exponentials. Graphs and properties of the hyperbolic functions

1.2 Inverse hyperbolic functions, their graphs, properties and logarithmic equivalents

PRIOR KNOWLEDGE

IAS Mathematics – Pure content

- Trigonometry (Unit 6 of the P2 SoW)

IAL Mathematics – Pure content

- Trigonometry (Unit 2 of the P3 SoW)
- Exponential functions (Unit 3 of the P3 SoW)

KEYWORDS

Hyperbolic, sinh, cosh, tanh, sech, cosech, coth, domain, range, exponential, function.

1a. sinh x, cosh x, tanh x and their inverses (1.1)(1.2)

Teaching time
4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- know the definitions of the six hyperbolic functions including their domains and ranges;
- be able to sketch graphs of the hyperbolic functions;
- understand and be able to use the inverse hyperbolic functions including domains and ranges;
- be able to derive and use simple hyperbolic identities.

TEACHING POINTS

Start by considering the graphs of the functions $y = e^x$ and $y = e^{-x}$ and asking what the curves $y = e^x + e^{-x}$ and $y = e^x - e^{-x}$ would look like. Graphing software or online function plotters can be helpful for doing this. Define the hyperbolic functions $\sinh x$ and $\cosh x$ and point out that, although they are exponential in definition, they can behave like trigonometric functions. Using the exponential definitions of $\sinh x$ and $\cosh x$, derive the definitions of $\tanh x$, $\text{sech}\, x$, $\text{cosech}\, x$, and $\coth x$.

Students need to be able to sketch the graphs of the six hyperbolic functions, and know their domains and ranges.

Look at the nature of odd and even functions and point out that they are the same as the trigonometric functions. Consider features such as $\sinh(-x) = -\sinh x$ using graph sketches.

Considering the graphs of each function in turn and reflecting each in the line $y = x$ can help develop the shape, domain and range of the inverse functions.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Start by using the exponential definitions to prove the hyperbolic identities and then consider how these compare with familiar trigonometric identities; for example, $\cosh^2 x - \sinh x = 1$. Develop and use Osborne's rule to compare or to prove some of the hyperbolic identities.

This is a good chance to look at catenary curves and point out their prevalence in the real world.

As an extension, you could consider how the hyperbolic functions work in the complex plane. Explore the nature of x replaced by a complex number and how the domain and range change/adapt. Why they are needed could be a good discussion point.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

A typical comment from the examiner's report is regarding sign errors: 'There was the occasional error in signs for the expression of $\sinh x$ and $\cosh x$ in terms of e^x and e^{-x}.'

1b. Logarithmic forms of the inverse hyperbolic functions. Solving equations involving hyperbolic functions (1.1) (1.2)

Teaching time
4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to derive, use and know the logarithmic forms of the inverse hyperbolic functions;
- be able to solve equations involving hyperbolic functions.

TEACHING POINTS

Start with $y = \sinh x \Rightarrow y = \frac{e^x - e^{-x}}{2}$ and rearrange to give a quadratic in e^x. This can be solved to give $e^x = \ln(y + \sqrt{y^2 + 1})$ which leads quickly to the formula for arsinh x. Take care to consider the domains; for example, cosh x should be restricted to $x \geq 0$.

Review the hyperbolic identities and look at equations involving hyperbolic functions. Consider examples where a hyperbolic identity could be used and examples where the exponential definition is required. Point out that an exact answer in terms of natural logarithms is often required.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Consider the whole function without restricting the domain.

Noting that $x - \sqrt{x^2 - 1} = \frac{(x-\sqrt{x^2-1})(x+\sqrt{x^2-1})}{x+\sqrt{x^2-1}}$ and expanding the numerator can lead to a better understanding of arcosh x and its form.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Students should understand that there are often a number of different approaches to solving problems and the choice of strategy can be crucial. Using the logarithmic form of inverse hyperbolic functions to obtain the final answer is a common approach, along with writing hyperbolic functions in terms of exponentials and proceeding to solve the resulting quadratics in e^x. It should be noted that sometimes this can end up with extra solutions that should be rejected. It should also be noted that attempts to solve a given equation by expressing it in terms of exponentials can result in a dead end when, for example, a quartic in e^x is reached.

UNIT 2: Further coordinate systems

Return to Overview

SPECIFICATION REFERENCES

2.1 Cartesian and parametric equations for the ellipse and hyperbola

2.2 The focus-directrix properties of the ellipse and hyperbola, including the eccentricity

2.3 Tangents and normals to these curves

2.4 Simple loci problems

PRIOR KNOWLEDGE

IAS Mathematics – Pure content

- Coordinate geometry of the circle (Unit 3 of the P2 SoW)

IAL Mathematics – Pure content

- Differentiation in Cartesian (including implicit) and parametric forms (Unit 5 of the P4 SoW)

KEYWORDS

Ellipse, hyperbola, focus, directrix, eccentricity, tangent, normal, loci.

2a. Equations of the ellipse and hyperbola and their focus-directrix properties (1.1) (1.2)

Teaching time
3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- know and be able to use the Cartesian and parametric equations for the ellipse and hyperbola;
- know and be able to use the focus-directrix properties of the ellipse and hyperbola, including the eccentricity.

TEACHING POINTS

Start by reviewing the parametric and Cartesian forms of the parabola and rectangular hyperbola covered in FP1 in IAS Further Mathematics. Move on to developing the properties of the ellipse and the hyperbola, using the graphs to consider key points. Students should know the standard parametric and Cartesian forms.

Make sure students are clear with the terminology used, as it is very specific. They should be encouraged to develop a sheet of key points for each shape that details key aspects (axes intersections, equations of asymptotes etc.) based on standard forms.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Knowledge of the standard form and parameters for each shape will help with problem solving.

Explore the nature of conics in the form $ax^2 + 2bxy + cy^2 + dx + ey + f = 0$ using a graphing package to identify the conics and axes of symmetry, considering whether they are parallel to the coordinate axes, for all integer values of *a* to *f*.

Use graphing software to investigate the effect on the shape of the graph as the constants vary.

As an extension you could consider 3D quadric (conic section) surfaces.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Given the similarity of approach to conic sections, students can use the wrong eccentricity formula and sometimes the foci are not well understood and not given as coordinates. Those who know the focal properties required achieve success with some difficult proofs with minimal effort. Those who opt for an approach using Pythagoras had various degrees of success and often struggle with the algebra or are unable to deal with the square roots. Choosing to show relationships work with specific examples rather than algebraic proof will typically gain no credit.

2b. Tangents and normals to the ellipse and hyperbola (2.3)

Teaching time
3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to derive the equations of tangents and normals to these conic sections and solve related problems.

TEACHING POINTS

A good place to start is with some implicit and parametric differentiation. Students can work with the Cartesian and parametric forms to find the gradient at particular points on an ellipse or hyperbola, and then develop what the tangent equations should be in a general form, working more algebraically. Then encourage the students to develop and prove the solutions for both tangents and normals. Point out that they will need to prove the general forms and should not simply quote them.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Parametric forms are particularly useful for problem solving, and knowing the standard forms can help enormously. However, there are some problem types that require other techniques and these should not be overlooked e.g. sum and product of roots of a quadratic equation.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Emphasise the importance of correct algebra:

In some proofs, simplifying the gradient typically makes simplification of their equation much more successful. Those students who do not do this, typically fail to successfully produce a convincing argument.

Dealing with polynomial expressions underneath a square root often arises and it would be wise to emphasise correct algebraic techniques when this form is seen.

Many errors are made by students with these questions, notably ‘invisible brackets’ (i.e. $6p^2$ instead of $36p^2$).

Problems seen include incorrect differentiation and incorrect substitution. Many students struggle to eliminate one of the variables when this is required in the solution, but of those that do manage it, the correct answer is usually reached.

2c. Simple loci problems (2.4)

Teaching time
4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to describe each conic section in terms of a locus of points and solve related loci problems.

TEACHING POINTS

Ensure that students understand what is meant by the term 'locus'. Consider a range of problems, for example questions involving chords and midpoints, and provide plenty of opportunity for them to practise algebraic manipulation. Encourage graph sketching as the starting point to problem solving. If a sketch is given, encourage students to add missing features given in the question. Also encourage them to factorise and simplify expressions whenever possible.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Online graphing tools or graph sketching packages can be useful to help students get to grips with loci problems. Experiment by changing the values of the constants in the general form of each conic section or by changing the values of the eccentricity.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Latter parts of questions on this topic can prove to be very challenging with many students not knowing where to start or which strategy to adopt. For example, students' choice of approach when finding the locus of the mid-point of a chord of an ellipse.

Substitution of $y = mx + c$ into the ellipse usually results in a dead end. Another common approach is to use the parametric form for two ends of a chord and then apply the factor formulae to find an equation for the locus of the midpoint. Those who adopt this approach often failed to use the factor formulae and could make little progress.

It is wise to consider a number of (challenging) examples, especially those involving chords.

UNIT 3: Differentiation

Return to Overview

SPECIFICATION REFERENCES

3.1 Differentiation of hyperbolic functions and expressions involving them

3.2 Differentiation of inverse functions, including trigonometric and hyperbolic functions

PRIOR KNOWLEDGE

IAS Mathematics – Pure content

- Trigonometry (Unit 6 of the P2 SoW)

IAL Mathematics – Pure content

- Trigonometry (Unit 2 of the P3 SoW)
- Exponential functions (Unit 3 of the P3 SoW)

KEYWORDS

Hyperbolic, sinh, cosh, tanh, domain, range, exponential, function, radical.

3a. Differentiate the hyperbolic functions (3.1)

Teaching time
3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to differentiate the hyperbolic functions and know the standard results.

TEACHING POINTS

A reminder that $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{-x}) = -\mathrm{e}^{-x}$ means that the formulae for the derivatives of the hyperbolic functions can be found as an exercise. These could also be compared to the derivatives of the trigonometric functions. Review the chain, product and quotient rules, which were covered in P3 in IAL Mathematics, and ask students to find the derivatives of a range of expressions involving hyperbolic functions, such as $y = \frac{\cosh 3x}{x^2}$, $y = \ln(\sinh x)$ and $y = \cosh^3 x$.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Find stationary points on curves involving hyperbolic functions and use the second derivative to determine the nature of a stationary point and to identify points of inflection.

As an extension, you could look at catenary curves and related differential equations.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Typically students can correctly differentiate hyperbolic functions, but the final stages of more complex solutions cause problems and a number of errors are seen often in processing terms down to a printed result.

3b. Differentiate inverse hyperbolic and trigonometric functions (3.2)

Teaching time
4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to differentiate inverse hyperbolic and trigonometric functions such as $\frac{1}{2}\arctan x^2$ and $\text{arsinh}\,(3x)$.

TEACHING POINTS

Develop the formulae by using implicit differentiation; for example, $y = \arcsin x \Longrightarrow \sin y = x$ and then $(\cos y)\frac{\mathrm{d}y}{\mathrm{d}x} = 1$.

This leads quickly to the formula $\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$ using $\cos^2 y + \sin^2 y = 1 \Longrightarrow \cos y = \sqrt{1-x^2}$.

The formulae for the other trigonometric functions and for the hyperbolic functions can be shown using a similar approach. After deriving the standard formulae, move on to examples requiring the chain, product or quotient rules such as $\frac{\mathrm{d}}{\mathrm{d}x}\left(\arcsin \frac{x}{a}\right)$ or or $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x \,\text{arsinh}\, x)$

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Developing the formulae for $\frac{\mathrm{d}}{\mathrm{d}x}(\text{arcsec}\, x)$ and $\frac{\mathrm{d}}{\mathrm{d}x}(\text{arccosec}\, x)$ or for expressions involving $|x|$ are more challenging.

Link with the Maclaurin series, which was covered in FP2, IAL Further Mathematics, and the differentiation of arctan θ for finding an approximation for π.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Algebraic techniques involving radicals are very important and are a common source of errors.

UNIT 4: Integration

Return to Overview

SPECIFICATION REFERENCES

4.1. Integration of hyperbolic functions and expressions involving them
4.2 Integration of inverse trigonometric and hyperbolic functions
4.3 Integration using hyperbolic and trigonometric substitutions
4.4 Use of substitution for integrals involving quadratic surds
4.5 The derivation and use of simple reduction formulae
4.6 The calculation of arc length and the area of a surface of revolution

PRIOR KNOWLEDGE

IAL Mathematics – Pure content

- Integration (Unit 6 of the P4 SoW)
- Partial fractions (Unit 2 of the P4 SoW)
- Inverse functions (Unit 1 of the P3 SoW)
- Trigonometric functions (Unit 2 of the P3 SoW)
- Differentiation of trigonometric functions, e^x and $\ln x$ (Unit 4 of the P3 SoW)
- Cartesian and parametric equations (Unit 3 of the P4 SoW)

IAL Further Mathematics – Core Pure content

- Polar coordinates (Unit 7 of the FP2 SoW)

KEYWORDS

Integrate, partial, fraction, radical, inverse, integration by parts, reduction formulae, arc length, surface of revolution, Cartesian, parametric.

4a. Integration of the hyperbolic and inverse hyperbolic functions and functions of the form $(x^2 + a^2)^{-\frac{1}{2}}$ and $(x^2 - a^2)^{-\frac{1}{2}}$ (4.1)(4.2)

Teaching time
3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to integrate expressions involving hyperbolic functions;
- be able to integrate inverse trigonometric and hyperbolic functions using integration by parts.

TEACHING POINTS

Start by reviewing the techniques for integration covered in P4 in IAL Mathematics, and review the derivatives of the hyperbolic functions from the previous unit. Move on to considering integration as the reverse of differentiation and look at examples where students are required to recognise the standard integrals. Extend this to examples which require use of the backwards chain rule or use of the hyperbolic identities such as $\int \sinh(3x-1)\,\mathrm{d}x$ or $\int \cosh^2 3x\,\mathrm{d}x$. You should also cover cases where use of the exponential definition may be the simplest method, such as $\int \mathrm{e}^{2x}\cosh x\,\mathrm{d}x$.

Use integration by parts to integrate the inverse hyperbolic and inverse trigonometric functions. Extend to examples such as $\int x \arctan x\,\mathrm{d}x$ and examples which involve limits.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Provide more challenging problems, such as those which require the use of hyperbolic identities. You could also consider examples where students have to find an area under a curve, or an area enclosed by two curves and examples which require students to prove a result.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

In an exam question to find $\int \text{arsinh}\,2x\,\mathrm{d}x$, whilst most candidates realised they needed to use integration by parts, there were many variations seen for the derivative of arsinh $2x$. These were usually of the right form, but the factor of 2 was often missing in one or both positions, and sometimes the signs were incorrect. There were a large number of candidates who were unable to integrate the $2x\left(1+4x^2\right)^{-\frac{1}{2}}$ term. Strong candidates recognised the form of this integral, and some managed to make an appropriate substitution, but many tried a variety of unsuccessful methods. A few candidates succeeded by using a substitution such as u = arsinh $2x$. Most were able to convert the arsinh term into log form.

4b. Integrate using hyperbolic and trigonometric substitutions (4.3)(4.4)

Teaching time
3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- know how to integrate functions of the form $(a^2 \pm x^2)^{-\frac{1}{2}}$ and $(a^2 + x^2)^{-1}$ and be able to choose trigonometric or hyperbolic substitutions to integrate associated functions;
- know how to integrate functions involving quadratic surds and be able to choose appropriate trigonometric or hyperbolic substitutions.

TEACHING POINTS

This is an opportunity to revise the trigonometric and the hyperbolic identities and the differentiation of sec θ, tan θ etc.

To integrate functions such as $(x^2 + a^2)^{-\frac{1}{2}}$, ask students to consider the trigonometric and hyperbolic identities and to suggest a suitable substitution for x.

Give the functions to integrate and the substitutions and ask the students to match them. This will be a good way to make sure that they can remember and apply the process correctly.

Choosing the appropriate trigonometric substitution is easier if the following prompts are learnt:

$\sqrt{a^2 - b^2x^2} \Rightarrow x = \frac{a}{b}\sin\theta$ or $\frac{a}{b}\tanh u$

$\sqrt{b^2x^2 - a^2} \Rightarrow x = \frac{a}{b}\sec\theta$ or $\frac{a}{b}\cosh u$

$\sqrt{a^2 + b^2x^2} \Rightarrow x = \frac{a}{b}\tan\theta$ or $\frac{a}{b}\sinh u$

$a^2 + b^2x^2 \Rightarrow x = \frac{a}{b}\tan\theta$ or $\frac{a}{b}\sinh u$

Start by looking at examples where $a = 1, b = 1$.

Point out that these integrals are given in the formula booklet and may be stated without proof where use of a suitable substitution is not required.

Point out that in examples of the form $(a^2 - b^2x^2)^{-\frac{1}{2}}$ and $(a^2 + b^2x^2)^{-1}$ it is usually easier to write $a^2 \pm b^2x^2$ as $b^2\left(\frac{a^2}{b^2} \pm x^2\right)$.

Consider other functions involving quadratic surds which require a trigonometric or hyperbolic substitution, such as $\int \sqrt{1 + x^2}\,\mathrm{d}x$. In more complicated examples the substitution may be given, for example: Use the substitution $x = 2\sin\theta - 1$ to find $\int \frac{x}{\sqrt{3 - 2x - x^2}}\,\mathrm{d}x$.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Review partial fractions, covered in P4 in IAL Mathematics, and get students to derive the general result for $\int \frac{1}{a^2 - x^2}\,\mathrm{d}x$ and $\int \frac{1}{x^2 - a^2}\,\mathrm{d}x$. Use the method of completing the square to reduce integrals of the form $\int \frac{1}{\sqrt{px^2 + qx + r}}\,\mathrm{d}x$ and $\int \frac{1}{px^2 + qx + r}\,\mathrm{d}x$ to integrals of the form $(a^2 \pm x^2)^{-\frac{1}{2}}$ or $(a^2 \pm x^2)^{-1}$.

Consider a range of more complicated problems such as $\int \frac{1}{\sqrt{-4x^2 - 12x}}\,\mathrm{d}x$ or $\int \frac{2 + 5x}{\sqrt{2 + x^2}}\,\mathrm{d}x$.

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COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Algebraic techniques involving radicals are very important and are a common source of errors.

An examiner's report on a question involving a 3-term quadratic: 'The majority of candidates knew that completing the square was the required first step and that 'arctan' was the required form. There were some attempts using various substitutions such as $x + 2 = 3 \sinh u$ but in cases like these candidates made little progress. Generally this question was answered well with many fully correct solutions.'

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4c. Reduction formulae (4.5)

Teaching time
4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to derive reduction formulae for a variety of integrals.

TEACHING POINTS

You should begin by giving students the opportunity to revise the work they have done in IAL Mathematics on trigonometrical relationships and integration by parts. It is important they are fluent in these skills before they attempt to derive reduction formulae.

A good starting point is to integrate powers of a function such as $\cos^n x$ for $n = 1, 2, 3, 4$. This will reinforce the ideas already studied as well as giving students an appreciation of the elegance of the reduction formulae when they start to derive these.

You should give students the opportunity to derive sufficient reduction formulae for relatively simple cases such as $\cos^n x$, $\sin^n x$, $x^n e^x$, etc. before looking at cases where trigonometric relationships are also involved.

$$I_n = \int \sin^n x \, dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n} I_{n-2}$$

$$I_n = \int \cos^n x \, dx = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n} I_{n-2}$$

$$I_n = \int \tan^n x \, dx = \frac{1}{n-1}\tan^{n-1} x - I_{n-2}$$

$$I_n = \int x^n e^{ax} \, dx = \frac{1}{a}(x^n e^{ax} - nI_{n-1})$$

Encourage students to include sufficient explanation in their working to show their reasoning as they work through examples.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Student can be asked to develop standard results. Challenge them to show the results for many reduction problems. This is a good test of their integration and algebra skills.

- How to find formulae for cot, sec and cosec.
- Rational functions

  $$I_n = \int \frac{x^n}{\sqrt{ax+b}} dx = \frac{2x^n\sqrt{ax+b}}{a(2n+1)} - \frac{2nb}{a(2n+1)} I_{n-1}$$
- Composite trig functions; e.g. how can the composite trigonometric function shown be developed into the given reduction formulae?

  $$I_n = \int \tan^5 \cos^3 x \, dx = \frac{1}{n+m}\cos^{m-1} u \sin^{n+1} u + \frac{m-1}{n+m}\int \cos^{m-2} u \sin^n u \, du$$

4d. The calculation of arc length (4.6)

Teaching time
2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to calculate the length of an arc where the equation of the curve may be given in Cartesian or parametric form.

TEACHING POINTS

A good idea here is to challenge students to derive the formula in its Cartesian form for themselves with a little guidance, when required. This will help them remember the formula correctly and provide them with a greater understanding than is gained from learning a formula without any understanding of where it comes from. Applying the formula in Cartesian form will give students many opportunities to revise work done on integration by substitution, which was considered in IAL Mathematics.

You can then go on to challenge students to derive the formula needed when the equation of the curve is given in parametric form.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Deriving both of the required forms of the formula will provide students with an opportunity to demonstrate rigour in their algebraic approaches to these.

A good challenge point can be to link this section with the reduction formulae.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Using the formula can sometimes lead to a negative result, Students need to be aware that this is as a result of the limits being considered in incorrect order.

Students should be advised to write down the formula that they are going to use before they use it. The advantage in doing this is that any errors in substitution will only be penalised by loss of accuracy marks; if the general formula is not shown then the method marks are lost as well.

4e. The calculation of the area of a surface of revolution (4.6)

Teaching time
2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to calculate the area of a surface of revolution where the equation of the curve may be given in Cartesian or parametric form.

TEACHING POINTS

To remind the students, start with some challenging volume of revolution work and develop the concept of rotating a shell about the axes. This gives an opportunity to derive both of the required forms of the equation. This should be an opportunity to enhance their understanding of the physical model.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Proof of the surface area of a sphere and a cone.

Significance of volume and surface areas: Torricelli's trumpet. Discuss the nature of finite volume and infinite surface area.

How do the formulae change as the rotation changes? Consider a rotation about the y-axis or about a line of the form $y = mx$. Is it possible to calculate the surface area in these cases?

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students should be advised to write down the formula that they are going to use before they use it. The advantage in doing this is that any errors in substitution will only be penalised by loss of accuracy marks; if the general formula is not shown then the method marks are lost as well.

UNIT 5: Vectors

Return to Overview

SPECIFICATION REFERENCES

5.1 The vector product $\mathbf{a} \times \mathbf{b}$ and the triple scalar product $\mathbf{a}.\mathbf{b} \times \mathbf{c}$

5.2 Use of vectors in problems involving points, lines and planes

The equation of a line in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$

5.3 The equation of a plane in the forms $\mathbf{r}.\mathbf{n} = p$, $\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$

PRIOR KNOWLEDGE

IAL Mathematics – Pure content

- Vectors (Unit 7 of the P4 SoW)

KEYWORDS

Scalar, cross, dot, vector, product, parallelepiped, tetrahedron, Cartesian, line, plane, intersection, equation.

5a. The vector product a × b and the scalar triple product a · b × c and their applications (5.1)

Teaching time
3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to find the vector product and the triple scalar product in 2 and 3 dimensions;
- be able to interpret and use |**a** × **b**| as an area and **a** · **b** × **c** as a volume.

TEACHING POINTS

Start by reviewing the scalar product, covered in P4 in IAL Mathematics. Define the vector product, emphasising the differences with the scalar product to avoid confusion. It is especially important to note that the result of a vector (cross) product is a vector and the result of the scalar (dot) product is a number.

Show that the cross product is the perpendicular vector to the vectors that are crossed. Show the nature of direction is important, so we have to be careful with the result. Graphing software or online apps can be useful in demonstrating this. Also discuss the nature of $\mathbf{u} \times \mathbf{v} = 0$.

For the Cartesian form of the vector product, use the formula $\mathbf{a} \times \mathbf{b} = (ab \sin \theta)\mathbf{n}$. The multiplication of the two vectors gives nine outcomes. Show that the **ii**, **jj** and **kk** parts can be eliminated to leave the **ij**, **ji**, **ik**, **ki** and **jk**, **kj**. As part of the proof you can show that **ij** will give **k** and **ji** will give –**k**. The coefficient of the resulting vector can be linked to the determinant of the 3 × 3 matrix, which will be covered in the next unit.

Using $\frac{1}{2}ab \sin C$, derive the formula for the area of a triangle in vector product form. Derive a similar formula for the area of a parallelogram. Define the triple scalar product and then develop the vector product formulae for the volume of a tetrahedron and a parallelepiped. Note that the triple scalar product can also be linked to the determinant of a 3 × 3 matrix (unit 6).

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

There are modelling opportunities in mechanics contexts, notably rotational motion.

How can torque be represented as a vector system?

Students can investigate the algebraic properties of cross product, make a conjecture and prove it.

- $u \times v = v \times u$?
- $(u \times v) \times w = u \times (v \times w)$?
- $au \times v = a\,(u \times v)$?
- $u \times (v + w) = uv + uw$?

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Note that $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$ and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ for most vectors. This can be a common source of errors.

5b. Vector and Cartesian equations of a plane (5.3)

Teaching time
5 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand and be able to use the vector and Cartesian forms of the equation of a plane.

TEACHING POINTS

Consider a fixed point **a** and a general point **r** on a plane. Using the fact that the scalar product of perpendicular vectors is zero we have $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$. Rearranging, we have $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$, which leads to the scalar product form of a plane, $\mathbf{r} \cdot \mathbf{n} = p$. Link this vector form to the Cartesian equation of the plane, $n_1x + n_2y + n_3z = d$

Review the vector form of a line, $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$, as covered in P4 in IAL Mathematics, and extend this to the vector form of a plane, $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

You can link planes and lines to mechanics and look at the intersection of vectors, both in 2D and 3D. You could touch on Euclidean geometry as a use of these in a real-world context.

Show problems that link vector to Cartesian forms of a plane and how the Cartesian form is used.

Look at the link between vectors and complex numbers in quaternions.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Students must give the equation of a plane as an equation and not as an expression, i.e. they will lose marks for not writing **r** =…

They need to know the difference between giving an answer in vector form or Cartesian form.

5c. Problems involving points, lines and planes (5.2)

Teaching time
3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to find the points of intersection of lines and planes which meet;
- be able to calculate the perpendicular distance between two lines;
- be able to calculate the perpendicular distance from a point to a line or to a plane;
- be able to write the vector equation of a line in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$.

TEACHING POINTS

Start by recapping the vector equation of the line, $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$, then use the fact that the vector product of two parallel vectors is zero to derive the vector equation $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$. Make sure that students are familiar with Cartesian form of a line in 3D, $\frac{x-a_1}{d_1} = \frac{y-a_2}{d_2} = \frac{z-a_3}{d_3}$, and how it relates to the vector forms.

Recap the scalar product form of a plane, $\mathbf{r} \cdot \mathbf{n} = p$. Discuss the fact that p represents to shortest distance from the origin when $\hat{\mathbf{n}}$, the unit vector perpendicular to the plane, is used. Move on to looking at the shortest distance from a point to a plane. Point out that the perpendicular distance from (α, β, γ) to $n_1x + n_2y + n_3z + d = 0$ is $\frac{|n_1\alpha+n_2\beta+n_3\gamma+d|}{\sqrt{n_1{}^2+n_2{}^2+n_3{}^2}}$ and that this is given in the formula booklet.

You should cover a range of applications of lines and planes such as finding the point of intersection of a line and a plane, the line of intersection of two planes and the shortest, or perpendicular, distance between two parallel lines, two skew lines or a point and a line.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

The Nrich website is a good source of problem-solving questions.

Look at the perpendicular distance between two planes with two common points.

Extend the perpendicular distance to look at distance between two planes. You could bring in the cross product here.

Consider the intersection of three planes and the system of three solutions; consider the use of technology to help solve these. Discuss the three possible outcomes: a unique solution, an infinite solution and no solutions.

Finding a solution could also be linked to matrices (in Unit 6)

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

When finding points of intersection, students will sometimes successfully find μ or λ, but then proceed no further and not attempt to find the actual point of intersection.

Take care when using the formula for perpendicular distance from a point to a plane not to miss out the '$+d$' in the numerator.

UNIT 6: Further matrix algebra

Return to Overview

SPECIFICATION REFERENCES

6.1 Linear transformations of column vectors in two and three dimensions and their matrix representation

6.2 Combination of transformations. Products of matrices

6.3 Transpose of a matrix

6.4 Evaluation of 3×3 determinants

6.5 Inverse of 3×3 matrices

6.6 The inverse (when it exists) of a given transformation or combination of transformations

6.7 Eigenvalues and eigenvectors of 2×2 and 3×3 matrices

6.8 Reduction of symmetric matrices to diagonal form

PRIOR KNOWLEDGE

IGCSE (9-1) in Mathematics at Higher Tier

- Transformations (rotations, translations, reflections, enlargement)
- Trigonometric ratios
- Simultaneous equations

IAL Mathematics - Pure content

- Vectors in 3D (Unit 7 of the P4 SoW)
- Matrix algebra and matrix transformations (Units 5 and 6 of the FP1 SoW)

KEYWORDS

Array, dimension, rows, columns, elements, scalar, square matrices, commutative, associative, transformation, rotation, translation, reflection, enlargement, linear transformation, scale factor, vector, position vector, object, image, identity, determinant, inverse, transpose, symmetric, zero matrix, minor, cofactor, singular, non-singular, three-dimensional space, line, plane, parameter, vector equation, Cartesian equation, simultaneous equations, invariant point, invariant line, eigenvector, eigenvalue, normalised vector, orthogonal, diagonal, reduction.

6a Linear transformations (6.1) (6.2)

Teaching time
2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to use matrix products to represent combinations of transformations;
- be able to use matrices to represent linear transformations in three dimensions;

TEACHING POINTS

Begin by reviewing 2D linear transformations and their matrix representation, as covered in FP1 in IAS Further Mathematics. Extend this to three dimensions.

To identify the matrix representing a particular 3D transformation, consider the effect of the matrix or the transformation on three simple vectors; $\begin{pmatrix}1\\0\\0\end{pmatrix}$ (also denoted **i**), $\begin{pmatrix}0\\1\\0\end{pmatrix}$ (also denoted **j**) and $\begin{pmatrix}0\\0\\1\end{pmatrix}$ (also denoted **k**). Given any matrix $\mathbf{M} = \begin{pmatrix}a & b & c\\d & e & f\\g & h & i\end{pmatrix}$, the image of $\begin{pmatrix}1\\0\\0\end{pmatrix}$ is the first column of **M**, the image of $\begin{pmatrix}0\\1\\0\end{pmatrix}$ is the second column of **M** and the image of $\begin{pmatrix}0\\0\\1\end{pmatrix}$ is the third column of **M**.

Remind students that the order matters: **A** followed by **B** followed by **C** is represented by **CBA**. For example, is the transformation *T* is represented by the matrix **T** and the transformation *U* is represented by the matrix **U**, then the matrix **UT** represents the combined transformation of the transformation *T* followed by the transformation *U*.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

You could discuss how 3D matrix transformation are used in computer graphics, and how 3D rotational transformation matrices are used in Mechanics.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

The order of the transformations is important: ‘The majority of candidates used their answer from (b) correctly to multiply two matrices in the correct order, but a significant number multiplied them in the wrong order, gaining no marks’.

‘Many students could not find the matrices securely to represent basic transformations’.

A level Mathematics: Pure Mathematics 3

6b. Inverse of 3 × 3 matrices (6.3) (6.4) (6.5)(6.6)

Teaching Time
2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to calculate determinants of 3 × 3 matrices;
- be able to find the transpose of a matrix;
- understand and use singular and non-singular matrices;
- know the properties of inverse matrices;
- be able to calculate the inverse of non-singular 3 × 3 matrices.

TEACHING POINTS

Whilst understanding the process of finding the inverse of a 3 × 3 matrix is required, students should also be shown how to use a calculator to calculate the inverse of a matrix.

Students should show that they understand the process of inverting a matrix and the change in complexity from inverting a 2 × 2 to inverting a 3 × 3. Adding dimensions to a matrix adds significant complexity to finding the inverse.

This is an opportunity review work on matrix algebra covered in FP1 in IAS Further Mathematics.

Make sure students know that if det(**A**) = 0, then $\mathbf{A}^{-1}$ cannot be found, and **A** is a **singular** matrix.

They should know that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ and that if **A** and **B** are non-singular matrices, then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Show the determinant properties for a $n \times n$ matrix:

- if A has a row or column which is all 0s then |A| = 0
- if A has two identical rows of columns then |A| = 0
- if B is obtained from A due to one row or column multiplying by a scalar $k \in \mathbb{R}$ then $|B| = k|A|$
- for a combination of two matrices. |AB| = |A||B|
- if B is obtained from A by interchanging two rows or columns then |B| = – |A|

Link inverses back to the previous sub-unit and set some problems where students have to find the original points before a transformation was applied using the inverse of the transformation matrix.

Define the transpose of a matrix and make sure students know the that $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Discuss the use of matrices for encryption.

Investigate $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$, $(k\mathbf{A})^{-1} = \frac{1}{k}\mathbf{A}^{-1}$, $(\mathbf{A}^n)^{-1} = (\mathbf{A}^{-1})^n$ where n is a positive integer.

You could investigate the determinant properties and set problems using matrix algebra.

Students should be encouraged to prove the results $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$ and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUO1TES

‘A significant number of candidates thought that the determinant of **A** is $\frac{1}{\det\mathbf{A}}$.’

‘There were some errors in the calculation of the determinant and also some errors in the positions and signs of the elements within the inverse matrix’.

6c. Eigenvalues and eigenvectors of 2 × 2 and 3 × 3 matrices (6.7)

Teaching time
4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to find the eigenvectors and eigenvalues of 2 × 2 and 3 × 3 matrices.

TEACHING POINTS

It may prove useful throughout this sub-unit to work initially with 2 × 2 matrices and then move on to developing students' skills working with 3 × 3 matrices.

Although a formal definition of eigenvalue and eigenvector (i.e. $\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x}$) will be required here as a starting point, it is important that students are introduced at an early stage to the geometrical interpretation of these i.e. that if $\boldsymbol{x}$ is an eigenvalue of a matrix **A** then $\boldsymbol{x}$ is mapped onto a scalar multiple of itself.

Students will then be ready to calculate the eigenvalues first by considering the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. It may prove a suitable challenge for students to derive this equation themselves from the definition $\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x}$.

Having initially found the eigenvalues, students then can find the eigenvectors including the use of normalised vectors.

Where students are given 1 or more eigenvalues of a 3 × 3 matrix, it if often useful to use the result: 'Trace (**A**) = sum of the eigenvalues of **A**'.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

This is an opportunity to link the use of eigenvectors and values with the general equation for a conic. Discuss how the use of these can show a transformation of a conic form.

6d. Reduction of symmetric matrices to diagonal form (6.8)

Teaching time
2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to diagonalise a 2 × 2 or 3 × 3 matrix.

TEACHING POINTS

When starting this sub-unit you should ensure that students are suitably proficient in finding eigenvalues and eigenvectors as without this knowledge little progress can be made here. Begin by discussing the meaning of a normalised vector and the properties of an orthogonal matrix. Get students to develop the relationship$\mathbf{D} = \mathbf{P}^{\mathrm{T}}\mathbf{AP}$, where **D** is a diagonal matrix of **A**, and **P** is an orthogonal matrix formed from the normalised eigenvectors of **A**. It may be useful to first consider the diagonalisation of 2 × 2 symmetric matrices before moving onto 3 × 3 matrices, which is more complex.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

As an application of diagonalising a matrix you could consider the power of a matrix. Begin by asking students to show that if $\mathbf{D} = \mathbf{P}^{\mathrm{T}}\mathbf{AP}$ then $\mathbf{A} = \mathbf{PDP}^{\mathrm{T}}$, where **D** is a diagonal matrix of **A**, and **P** is an orthogonal matrix formed from the normalised eigenvectors of **A.** If matrix **A** is written in the form $\mathbf{PDP}^{\mathrm{T}}$ then $\mathbf{A}^{k} = (\mathbf{PDP}^{\mathrm{T}})^{k} = \mathbf{PDP}^{\mathrm{T}}\mathbf{PDP}^{\mathrm{T}} \ldots \mathbf{PDP}^{\mathrm{T}} = \mathbf{PDIDI} \ldots \mathbf{IDP}^{\mathrm{T}} = \mathbf{PD}^{\mathrm{k}}\mathbf{P}^{\mathrm{T}}$**.** Students can then be asked to show that $\mathbf{D}^{\mathrm{k}} = \begin{pmatrix} (\lambda_1)^k & 0 \\ 0 & (\lambda_2)^k \end{pmatrix}$ (for a 2 × 2 matrix).

Use mathematical software to investigate the geometrical interpretation of repeated eigenvalues.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Some candidates forgot to normalise or possibly did not know what normalised meant.