

INTERNATIONAL ADVANCED LEVEL

**MATHEMATICS/  
FURTHER MATHEMATICS/  
PURE MATHEMATICS**  
SCHEME OF WORK  
**FURTHER PURE MATHEMATICS 2**

Pearson Edexcel International Advanced Subsidiary in Mathematics (XMA01)

Pearson Edexcel International Advanced Subsidiary in Further Mathematics (XFM01)

Pearson Edexcel International Advanced Subsidiary in Pure Mathematics (XPM01)

Pearson Edexcel International Advanced Level in Mathematics (YMA01)

Pearson Edexcel International Advanced Level in Further Mathematics (YFM01)

Pearson Edexcel International Advanced Level in Pure Mathematics (YPM01)

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Further Pure Mathematics 2

Unit	Title	Estimated hours
1	<b>Inequalities:</b> Algebraic inequalities and inequations	5
2	<b>Series:</b> Method of differences	4
3	<b>Further complex numbers</b>	
a	Know and use $z = re^{i\theta} = r(\cos \theta + i \sin \theta)$	3
b	De Moivre's theorem	5
c	Loci	3
d	Elementary transformations from the $z$ -plane to the $w$ -plane	5
4	<b>First order differential equations</b>	
a	Integrating factors to solve first order differential equations	5
b	Differential equations reducible by means of a given substitution	3
5	<b>Second order differential equations</b>	
a	Second order differential equations of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$	6
b	Differential equations reducible by means of a given substitution	3
6	<b>Maclaurin and Taylor series</b>	
a	Maclaurin series	5
b	Derivation and use of Taylor series	4
7	<b>Polar coordinates</b>	
a	Convert between Cartesian and polar and sketch $r(\theta)$	4
b	Area enclosed by a polar curve	5
		<b>60 hours</b>

**UNIT 1: Inequalities**

Teaching time

**Algebraic inequalities and inequations (1.1)**

5 hours

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- 1.1 The manipulation and solution of algebraic inequalities and inequations, including those involving the modulus sign

**PRIOR KNOWLEDGE**IAS Mathematics – Pure content

- The algebra of linear and quadratic inequalities (Unit 1 of P1 SoW)

IAL Mathematics – Pure content

- Modulus function (Unit 1 of P3 SoW)

**KEYWORDS**

Inequality, inequation, greater than, less than, common denominator.

**OBJECTIVES**

By the end of the unit, students should:

- be able to manipulate and solve algebraic inequalities and inequations, including those involving the modulus sign.

**TEACHING POINTS**

Begin by recapping the key features of linear and quadratic inequalities and how they differ from equations as covered in IAS Mathematics (P1 Unit 1).

Move on to inequalities involving algebraic fractions. Include a range of problems for example  $\frac{4x-1}{x+2} < 1$  and  $\frac{1}{x-2} \geq \frac{x}{x+4}$ . Discuss the impact of e.g.  $-4 < -1$  and  $4 > 1$  in the algebraic approaches. This should help with the understanding of the problems. Cover all the variations that are possible to solve inequations of this type, for example:

- multiply both sides by the denominator squared and solve the resulting quadratic or cubic inequality;
- re-arrange so that both terms are on the LHS, combine terms to a single fraction, find the critical values and consider the sign of the LHS between each critical value.

Encourage graph sketching, where appropriate, only in the final stages of solving problems to determine the correct regions of the solution set.

Review modulus functions as covered in Unit 1 of Pure 3, then move onto inequations involving modulus functions. Include examples with linear and quadratic expressions, and examples with modulus functions on one or both sides. Encourage students to use a graphical approach to solving the inequality. For example to solve  $|x^2 - x - 6| < 2 - x$  sketch both  $y = x^2 - x - 6$  and  $y = 2 - x$  on the same axes, find the points of intersection and then use the graph to determine the regions of the solution set.

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### OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

This is a good opportunity to give the students more complex examples and see how they can develop their solutions. Encourage them to explore different ways of approaching problems, including graphical methods.

You could discuss how to sketch curves of the form  $y = \frac{ax+b}{cx+d}$ .

### COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

If a question states that algebra should be used, then purely graphical approaches with little working will be heavily penalised.

The most successful approaches involve the algebraic method being supported with a sketch and these are most likely to achieve full marks. Those who chose to square both sides subsequently ran into difficulty as they had extra values and did not know which ones to use.

Some candidates simply cross-multiplied, sketched a quadratic and missed two of the critical values.

**SPECIFICATION REFERENCES**

2.1 Summation of simple finite series using the method of differences

**PRIOR KNOWLEDGE**IAL Mathematics – Pure content

- Algebra and functions (Unit 2 of P2 SoW)
- Algebraic fraction manipulation (Unit 1 of P3 SoW)
- Partial fractions (Unit 2 of P4 SoW)
- Implicit differentiation (Unit 5 of P4 SoW)
- Differentiating products (Unit 4 of P3 SoW)

IAS Further Mathematics – Core Pure Mathematics

- Sums of Series (Unit 7 of FP1 SoW)

**KEYWORDS**

Series, sigma, differences, sum, term, general term, partial fraction.

**OBJECTIVES**

By the end of the unit, students should:

- be able to use the method of differences to sum simple finite series.

**TEACHING POINTS**

Start this topic with differences that do not involve fractions, to introduce the idea e.g.

$\sum_{r=1}^n \frac{1}{2}(r(r+1) - r(r-1)) = \sum_{r=1}^n r = \frac{n(n+1)}{2}$ . Carefully list the terms at the beginning and the end of the series in pairs. Show a sufficient number of terms to demonstrate how the cancelling process works to help understanding.

Move on to expressing the general term as  $f(r+1) - f(r)$ .

With the move to problems involving partial fractions, revisit how to find the necessary fractions.

Stress the need to set out working carefully, especially with differences that involve fractions.

**OPPORTUNITIES FOR REASONING/PROBLEM SOLVING**

Consider problems with multiples of  $n$  on both the lower and upper limits leading to the difference between series found by the method of differences.

Look at proving the standard formulae, from Unit 7 of FP1, using this method.

It can be shown that there are some standard expressions that they can develop, e.g.

$$(k+1)^2 - k^2 = 2k + 1 \text{ (sum of natural numbers)}$$

$$(k+1)^3 - k^3 = 3k^2 + 3k + 1 \text{ (sum of square positive numbers)}$$

$$(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1 \text{ (sum of cube positive numbers)}$$

## A level Mathematics: Further Pure Mathematics 2

### COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Emphasise the importance of presentation as poor layout can lead to a student miscopying their own work or making other errors and so achieving a lower score.

Students should take care to include only the terms required and not extend beyond the limits given on the summation sign.

Where an answer is given, it is essential to show all the steps required to produce this answer; making leaps in working can result in loss of marks.

## UNIT 3: Further complex numbers

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### SPECIFICATION REFERENCES

- 3.1 Euler's relation  $e^{i\theta} = \cos \theta + i \sin \theta$
- 3.2 De Moivre's theorem and its application to trigonometric identities and to roots of a complex number
- 3.3 loci and regions in the Argand diagram
- 3.4 Elementary transformations from the  $z$ -plane to the  $w$ -plane

### PRIOR KNOWLEDGE

#### IAL Mathematics Pure content

- Compound angle formulae (Unit 2 of P3 SoW)

#### IAS Further Mathematics – Core Pure content

- Complex numbers (Unit 1 of FP1 SoW)

### KEYWORDS

Conjugate, real part, imaginary part, complex conjugate,  $n$ th root, distinct root, discriminant, Argand diagram, Cartesian coordinates, vector, magnitude, modulus, argument, principal argument, radians, modulus-argument form, polynomial, coefficient, quadratic, quartic, cubic, de Moivre, unity, exponential, multiple angle, locus, loci,  $z$ -plane,  $w$ -plane, Argand, modulus, argument, transformation, translation, enlargement.

**3a. Know and use  $z = re^{i\theta} = r(\cos \theta + i \sin \theta)$  (3.1)**

Teaching time

3 hours

**OBJECTIVES**

By the end of the sub-unit, students should:

- be able to multiply and divide complex numbers in modulus–argument and exponential form;
- know and use cosine and sine in terms of the exponential form.

**TEACHING POINTS**

Link exponential form and modulus–argument form and  $z = a + ib$  introduced in IAS Further Mathematics – FP1 Unit 1 with an Argand diagram. Although this is not covered until Unit 6, you could link modulus–argument form and exponential form using series expansions of  $\sin x$ ,  $\cos x$  and  $e^x$ . Emphasise the importance of this relationship and move on to consider cosine and sine in terms of the exponential function. Extend the work done in FP1 to ensure understanding of the relationship between modulus and argument when multiplying or dividing complex numbers; use specific examples and the proof of the general form. Relate back to the Cartesian form to ensure the links to FP1 content are well understood.

**OPPORTUNITIES FOR REASONING/PROBLEM SOLVING**

Prove  $e^{i\pi} = -1$  and consider the history and importance of this result.

**COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES**

Encourage students to use an Argand diagram whenever possible to illustrate problems and facilitate approaches to solutions.

Final answers where the modulus and argument are known should be evaluated and not left in forms such as  $4096(\cos 2\pi + i \sin 2\pi)$ .

## 3b. De Moivre's theorem (3.2)

Teaching time

5 hours

**OBJECTIVES**

By the end of the sub-unit, students should:

- understand, remember and be able to use de Moivre's theorem:  $z^n = r^n e^{in\theta} = r^n(\sin n\theta + i \cos n\theta)$ ;
- be able to derive multiple angle formulae/expressions e.g.  $\cos 3\theta$  in terms of powers of  $\cos \theta$ , and  $\sin^3 \theta$  in terms of multiple angles of  $\sin \theta$ ;
- be able to find the roots of a complex number.

**TEACHING POINTS**

Derive the general statement of de Moivre's theorem by developing expressions for  $z^2$ ,  $z^3$  and  $z^4$ , and spotting the pattern to support the more obvious approach using  $(e^{i\theta})^n = e^{in\theta}$ . Ensure that students are familiar with  $\cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$  and  $\sin \theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$  before going on to look at the application to trigonometric identities. Make a clear distinction between the approach to derive e.g.  $\cos 3\theta$  in terms of powers of  $\cos \theta$  using de Moivre and the approach to derive e.g.  $\sin^3 \theta$  in terms of multiple angles of  $\sin \theta$  using  $(z^n + \frac{1}{z^n}) = 2 \cos n\theta$  etc. This distinction is important, so students do not confuse the two methods.

Look at how De Moivre's theorem can be used to find the  $n$ th roots of a complex number. Start by solving  $z^n = 1$  before moving onto finding the roots of a general complex number.

**OPPORTUNITIES FOR REASONING/PROBLEM SOLVING**

Consider the Argand diagram in more detail whenever possible. Look at the relationship between rectangular and polar forms of a complex number:  $x = r \cos \theta$ ,  $y = r \sin \theta$  etc. and illustrate solutions whenever appropriate. This will support the ideas that are introduced later.

Make links with proof by induction to prove De Moivre's theorem for positive integers. Prove that De Moivre's theorem is also true for negative integers.

Student can look at using the binomial theorem to show  $(z^n + \frac{1}{z^n}) = 2 \cos n\theta$ . This will let them see the distinct difference between the two methods.

The formula for the sum of a geometric series can be used to show that the sum of the  $n$ th roots of 1 is zero.

**COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES**

Reading the question carefully and reflecting on the approach required is a common theme in examiners' reports. If the question asks for the use of de Moivre, then attempting alternative approaches by expanding brackets etc. may not gain any credit.

Encourage students to draw Argand diagrams to reflect on the demands of the question whenever the opportunity arises.

### OBJECTIVES

By the end of the sub-unit, students should:

- be able to construct and interpret simple loci and regions in the Argand diagram such as  $|z - a| > r$  and  $\arg(z - a) = \theta$ ;
- understand and be able to use a wider range of loci in the Argand diagram.

### TEACHING POINTS

Consider sums and differences and how they look in ‘vector’ diagrams. Illustrate how modulus diagrams look. Stress the vital fact that the angle can vary for a fixed distance – hence loci.

The locus of  $z$  (the path which represents all possible values of  $z$ ) will form a curve or a straight line.

Students should be familiar with circles of the form  $|z - a| = r$  and  $|z - a| = c|z - b|$ , half lines and lines of the form  $\arg(z - a) = \theta$  and  $|z - a| = |z - b|$  and arcs of the form  $\arg\left(\frac{z-a}{z-b}\right) = \theta$ .

Encourage students to recognise and be able to construct simple circles, lines, half lines and arcs. Where the shape of the locus is not obvious from the equation, students will need to use algebra to find the Cartesian equation of the locus. An example might be: sketch the locus of  $|z - 2| = 2|z + i|$ .

Let  $z = x + iy$

Substitute into the equation:  $|x - 2 + yi| = 2|x + (1 + y)i|$

Square both sides and use Pythagoras to find the modulus:  $x^2 - 4x + 4 + y^2 = 4(x^2 + y^2 + 2y + 1)$

Simplify:  $\left(x + \frac{2}{3}\right)^2 + \left(y + \frac{4}{3}\right)^2 = \frac{20}{9}$ . Hence the locus is a circle, centre  $\left(\frac{2}{3}, \frac{4}{3}\right)$  and radius  $\frac{2\sqrt{5}}{3}$ .

To work out which region is required for the inequalities, choose a point (for circles, the centre of the circle is usually a good choice) and use the inequality to find whether it is in the region required.

You should then give students the opportunity to extend their knowledge further by considering regions defined by inequalities such as  $|z - a| > 2|z - b|$ . It is vital to encourage students to think carefully about the regions formed in their Argand diagrams and how these relate to the inequalities.

### OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Sketch the locus of  $\arg\left(\frac{z-6}{z-2}\right) = \frac{\pi}{4}$  and find the Cartesian equation of this locus.

For the complex number  $z = x + iy$ , find the minimum and maximum values of  $|z|$  given that  $z$  satisfies an equation such as  $|z - 12 - 5i| = 3$ .

Students should be challenged both to draw the Argand diagram given the inequalities and to form the inequalities from a given Argand diagram.

### COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

‘Many lost a mark by failing to conclude explicitly that their equation represented a circle’.

‘A surprising number of candidates shaded the outside of the circle rather than the inside and few gave any reason for their decision. Few thought to use the coordinates of the centre of the circle to find whether it was in the region required’.

3d. Elementary transformations from the  $z$ -plane to the  $w$ -plane (3.4)

Teaching time

5 hours

**OBJECTIVES**

By the end of the sub-unit, students should:

- be able to carry out elementary transformations from the  $z$ -plane to the  $w$ -plane.

**TEACHING POINTS**

This is a new skill, so students may be unfamiliar with the notation and the language used. It is important that you encourage students to fully consider the locus of  $z$  in the first instance rather than rushing to the transformation without first considering what  $z$  represents (e.g. that  $|z| = 2$  represents a simple circle with Cartesian Equation  $x^2 + y^2 = 4$ ).

This transformation can be a complex process for many students so it is important to reassure them that they are simply changing the subject of a formula from  $w$  to  $z$ . Start with some simple transformations which can be done graphically to give students a good understanding of the processes involved. It is also useful here to allow students to describe the transformations in geometrical terms (e.g. as translations, rotations or enlargements) to further enhance their understanding. Then move on to more complex examples of the form  $w = \frac{az+b}{cz+d}$ . Consider cases where a simple circle, such as  $|z| = 1$ , is mapped to a line or to another circle, and cases where simple lines are mapped to circles. Encourage students to recognise the equations of circles and lines in complex form, so that they do not need to go on unnecessarily to find a Cartesian equation algebraically.

**OPPORTUNITIES FOR REASONING/PROBLEM SOLVING**

Students can develop relationships that can show the different transformations.

E.g.

- $w = z + 2$  Translation
- $w = z + 2 - i$  Translation
- $w = 2z$  Enlargement
- $w = iz$  Rotation

The rotation and enlargement can be developed to incorporate matrices.

**COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES**

Students need to be aware that this type of question commonly contains the instruction ‘hence’; many ignore this. Emphasise that this word should give some guidance as to how to approach the question.

A commonly seen error is for students to substitute  $x + iy$  for  $z$  rather than  $x + ix$  which then gives them complicated algebraic expressions.

## UNIT 4: First order differential equations

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### SPECIFICATION REFERENCES

- 4.1 Further solution of first order differential equations with separable variables
- 4.2 First order linear differential equations of the form  $\frac{dy}{dx} + Py = Q$  where  $P$  and  $Q$  are functions of  $x$
- 4.3 Differential equations reducible to the above types by means of a substitution

### PRIOR KNOWLEDGE

#### IAL Mathematics – Pure content

- Differentiation (Unit 4 of P3 SoW)
- Integration (Unit 5 of P3 SoW and Unit 6 of P4 SoW)

### KEYWORDS

Integrating, factor, function, differential, equation, order, general, particular, substitution, solution, differential, equation, reducible.

**4a. Integrating factors to solve first order differential equations**  
**(4.1) (4.2)****Teaching time**  
5 hours**OBJECTIVES**

By the end of the sub-unit, students should:

- be able to identify the form of first order differential equations that can be solved by an integrating factor or by separating variables and carry out the solution;
- be able to find general and particular solutions of differential equations of this form.

**TEACHING POINTS**

It is good to start by revisiting what a first order differential equation is. Review the method of separating variables as covered in Unit 6 of IAL Pure 4. Extend this by getting students to consider and to sketch the family of solution curves. Graphing software is a useful way to demonstrate this for a variety of differential equations.

A good way to move on from here is to give examples that are the reverse to the product rule. This can be developed into the form of the integrating factor. It is good to show the proof of the integrating factor form as this will help with the development of the solution, as will a firm reminder that the constant of integration should not be left out. It may also help to look at how the constant of integration can be chosen to help with the solution, especially where an answer is given.

Knowing the general form  $\frac{dy}{dx} + P(x)y = Q(x)$  will help identify when this technique is appropriate. It is important that  $e^{\int P(x)dx}$  is known and used, but that's where the use of formulae should stop and each problem worked through once the IF is identified.

A solution process is a useful aide memoir:

- 1 Rearrange the differential equation into the form  $\frac{dy}{dx} + P(x)y = Q(x)$
- 2 Find IF =  $e^{\int P(x)dx}$
- 3 Multiply everything by the IF and verify the left-hand side becomes the product rule.
- 4 Integrate both sides taking care when choosing the constant of integration.
- 5 Rearrange to an acceptable form/given answer.

An interesting extension might be to consider the graphs of the solutions, especially when the problems are more involved e.g.  $\frac{dy}{dx} - \frac{1}{2}y = 2 \sin 5x$ .

**OPPORTUNITIES FOR REASONING/PROBLEM SOLVING**

Considering different ranges for the constant of integration determines whether or not the solution is divergent or remains finite as  $x \rightarrow \infty$ . Online tools can be used to help graph the solution for various values of the constant of integration.

The convergence (or otherwise) of the solution can be further developed by using initial conditions to find the constant of integration and which initial conditions will determine that the solution remains finite.

**COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES**

Interpreting what is required in examinations is crucial here. Most mistakes occur when students do not recognise the form of the equation and do not deploy the correct strategy for the initial method mark. Since the remaining marks typically depended on this mark, students will lose any marks available. It is wise to

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ask students to learn the form so they can recognise it, especially when some initial rearrangement is required.

Also emphasise that the constant of integration is important and should not be forgotten!

**4b. Differential equations reducible by means of a given substitution (4.3)**
**Teaching time**  
 3 hours

**OBJECTIVES**

By the end of the sub-unit, students should:

- be able to solve differential equations by means of a given substitution reducible to the types encountered in the previous sub-unit.

**TEACHING POINTS**

Some differential equations are reducible to the types covered in the previous sub-unit by means of a suitable substitution.

For example:

Use the substitution  $z = \frac{1}{y}$  to transform the differential equation  $\frac{dy}{dx} + xy = xy^2$  into the differential equation  $\frac{dz}{dx} - xz = -x$ .

- Find expressions for  $y$  and  $\frac{dy}{dx}$ :  $y = \frac{1}{z}$  and  $\frac{dy}{dx} = -z^{-2} \frac{dz}{dx}$  (using implicit differentiation).
- Substitute these into the differential equation to obtain a differential equation in terms of  $x$  and  $z$ :  

$$\frac{-1}{z^2} \frac{dz}{dx} + \frac{x}{z} = \frac{x}{z^2}$$
- Multiply through by  $-z^2$  to give the required form.

Use the strategy set out in Sub-Unit 4a to solve the resulting differential equation after the substitution has been made.

**OPPORTUNITIES FOR REASONING/PROBLEM SOLVING**

Set some problems where the substitution is not given and ask students to find their own. Where students produce a solution using a substitution that is different from the one you prepared consider them both and look at the equivalence of the solutions. Graphing software can be useful for comparing equivalent solutions.

**COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES**

When a question requires it, the majority show confidence using the chain rule in this topic. The more successful students tend to use the method that involves finding an expression (for the derivative) then substituting into the original equation.

Given that this was a proof, in order to obtain full marks students needed to fully demonstrate their method with no errors seen.

## UNIT 5: Second order differential equations

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### SPECIFICATION REFERENCES

- 5.1** The linear second order differential equation  $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$ , where  $a$ ,  $b$  and  $c$  are real and the particular integral can be found by inspection or trial
- 5.2** Differential equations reducible to the above types by means of a given substitution

### PRIOR KNOWLEDGE

#### IAL Mathematics – Pure content

- Differentiation (Unit 4 of P3 SoW and Unit 5 of P4 SoW)
- Integration (Unit 6 of P4 SoW)

### KEYWORDS

Complementary function, differential, equation, order, auxiliary, discriminant, general, particular, substitution, solution, differential equation, reducible, particular integral.

**5a. Second order differential equations of the form**

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x) \quad (5.1)$$

**Teaching time**

6 hours

**OBJECTIVES**

By the end of the sub-unit, students should:

- be able to solve second order differential equations of the form  $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$  where  $f(x)$  is a polynomial, exponential or trigonometric function;
- be able to find general and particular solutions of second order differential equations of this form.

**TEACHING POINTS**

A methodical approach is essential to the successful completion of this sub-unit.

Begin with  $f(x) = 0$  and move quickly to the principle of superposition.

A good starting point is to give a solution and have the students simplify and substitute in the  $\frac{d^2y}{dx^2}$ ,  $\frac{dy}{dx}$  and  $y$  forms and develop the auxiliary equation, but then show how it can be written down without the initial working required. Consider each type based on the roots/discriminant of the auxiliary equation. It is useful if the form of the general solution is known for each type. This is a good opportunity to revise De Moivre's theorem as the students can then be asked to develop the complex/imaginary solutions.

It is a good idea once the general solution is found to look at initial conditions/boundary value problems to determine the constants in the particular solution.

Move on to  $f(x) \neq 0$  and consider the particular integral required based on the form of  $f(x)$ . It will help problem solving if each of these particular integrals are known. Take every opportunity to consider particular solutions using boundary value problems.

**OPPORTUNITIES FOR REASONING/PROBLEM SOLVING**

Develop a 'strategy sheet', perhaps in the form of a flowchart, that identifies what to do based on the problem being posed.

Develop the need for the additional  $x$  in a particular integral and work out why it is needed.

Ask students to sketch the graphs of the resulting solutions.

**COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES**

Getting variables mixed up in the complementary function is common, as is the wrong sign on the power of  $e$ . The correct standard form of the particular integral and completing the necessary calculations to find the coefficients is usually well done. However, a common error in some examples is to then ascribe them to the wrong trigonometric term.

When it is required almost all students know that they needed to construct and solve an auxiliary quadratic, and choose the appropriate complementary function. Some students, having incorrectly solved the auxiliary equation and obtained a root, then went on to use an incorrect particular integral. This is a method error and will not be awarded the marks for that part of the solution. Those students who chose an appropriate expression for the PI tend to be successful in differentiating it twice and finding a correct value.

**5b. Differential equations reducible by means of a given substitution (5.2)**
**Teaching time**  
3 hours

**OBJECTIVES**

By the end of the sub-unit, students should:

- be able to solve differential equations by means of a given substitution reducible to the types encountered in the previous sub-unit.

**TEACHING POINTS** As with first order differential equations, some second order differential equations are reducible to the types covered in the previous sub-unit by means of a suitable substitution.

An example might be:

Show that the transformation  $y = xv$  transforms the equation  $4x^2 \frac{d^2y}{dx^2} - 8x \frac{dy}{dx} + (8 + 4x^2)y = x^4$  into the equation  $4 \frac{d^2v}{dx^2} + 4v = x$ .

- Obtain expressions for  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$ .
- Substitute these into the given differential equation.
- Re-arrange to the required form.

Clear layout is very important given the complexity involved, as is the presentation of solutions. Encourage students to derive the relationships carefully using the chain rule and product rule rather than relying on learned results

Use the ‘strategy sheet’ as suggested in the previous sub-unit to solve the resulting differential equation after the substitution has been made.

**OPPORTUNITIES FOR REASONING/PROBLEM SOLVING**

Set some problems where the substitution is not given and ask students to find their own. Where students produce a solution using a substitution that is different from the one you prepared consider them both and look at the equivalence of the solutions.

**COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES**

When a question requires it, the majority showing confidence using the chain rule in this topic. The more successful students tended to use the method of finding an expression for the derivative then substituting these into the original equation. Problems arise in the application of the chain rule within the product rule in order to find the second derivative. It is not uncommon for students to use only the product rule, then on substituting into the given equation realise a factor was missing and introduce it at this stage.

## UNIT 6: Maclaurin and Taylor series

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### SPECIFICATION REFERENCES

- 6.1 Third and higher order derivatives
- 6.2 Derivation and use of Maclaurin series
- 6.3 Derivation and use of Taylor series
- 6.4 Use of Taylor series method for series solution of differential equations

### PRIOR KNOWLEDGE

#### IAL – Pure content

- Algebra and functions (Unit 2 of P2 SoW)
- Algebraic fraction manipulation (Unit 1 of P3 SoW)
- Partial fractions (Unit 2 of P4 SoW)
- Implicit differentiation (Unit 5 of P4 SoW)
- Differentiating products (Unit 4 of P3 SoW)

#### IAS Further Mathematics – Pure content

- Sums of Series (Unit 7 of FP1 SoW)

### KEYWORDS

Series, term, function, Maclaurin, range, power, Taylor, differential equation, solution, factorial.

### OBJECTIVES

By the end of the sub-unit, students should:

- be able to find and use higher derivatives of functions;
- know how to express functions as an infinite series in ascending powers using Maclaurin's expansion;
- be able to find the series expansion of composite functions.

### TEACHING POINTS

Start by reviewing the techniques covered so far for differentiation and discuss the notation used in higher order derivatives. Include as many different types of function as possible to practise techniques including implicit differentiation and differentiation of a product. Finding values of these derivatives at  $x = 0$  will facilitate the introduction of the Maclaurin series.

Look explicitly at some binomial expansions to show how they can be expressed as infinite series, then suggest that other familiar functions can also be expressed as series by considering a simple function, say  $e^x$ . Develop the formula for the Maclaurin expansion from here and stress the factorial denominators of the coefficients as these can be forgotten.

Find the series expansion for  $\sin x$  and  $\cos x$ , but point out that these are standard formulae and can be found in the formulae booklet. This could be a good time to prove the small angle formula; graphing software could be used to investigate how small  $x$  should be.

Find the Maclaurin expansion for other functions such as  $\ln(x + 1)$  and other trigonometrical functions, using graphing software to investigate any restrictions on the range of  $x$  for which the expansion is valid.

Consider composite functions and discuss how, for example, a series expansion for  $\cos(2x^2)$  can be found from the expansion for  $\cos x$ , and a series expansion for  $\ln\left(\frac{1+x}{1-x}\right)$  found from the series expansions for  $\ln(x + 1)$  and  $\ln(1 - x)$ . Emphasise the need for clear layout and proper use of brackets when moving on to composite functions; for example, the expansion of  $\cos(2x^2)$  has a second term of  $-\frac{(2x^2)^2}{2!}$ .

### OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Consider functions such as  $\ln x$  and discuss if they have a Maclaurin expansion. Include the range of validity and consider how many terms are required to give a good degree of accuracy to the actual value of the function.

Use technology to show the accuracy that you can go to with the series expansion by modelling the function and the truncated series expansions on a graph. Sliders can be used to develop this.

A good problem to link here with differentiation of inverse trig functions is Leibniz proof of finding an approximation for  $\pi$  using infinite series and arctan. Great discussion can follow on how and why he did this and how sophisticated it was for the 18th century.

Discuss the usefulness of these series expansions in solving problems. Look at using a series expansion for finding approximations to definite integrals, for example  $\int_{-1}^1 e^{-\frac{x^2}{2}} dx$ .

### COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

## **A level Mathematics: Further Pure Mathematics 2**

The general form of the series should be quoted before an attempt is made to substitute derivatives. Without it, method marks may not be awarded.

**OBJECTIVES**

By the end of the sub-unit, students should:

- be able to derive and use Taylor series;
- be able to use the Taylor series method to find a series solution of differential equations.

**TEACHING POINTS**

Recap the key points of the Maclaurin series, possibly via a problem-solving exercise.

Extend these ideas to Taylor series. Encourage a methodical approach, finding all the required derivatives before evaluating at required points and then substituting into Taylor series form. Emphasise the presence of factorials in the denominators and look at problems that extend to higher powers.

Develop a ‘series expansion’ sheet that details the Taylor series of standard functions expanded around key points.

Link the Taylor series solution back to Maclaurin by considering solutions in powers of  $x$ . Look at first and second order differential equations and how these can be solved using Taylor series.

Encourage students to check solutions by differentiating whenever practicable.

**OPPORTUNITIES FOR REASONING/PROBLEM SOLVING**

Consider differential equation problems where the coefficients are not polynomials or constants and perhaps singular rather than ordinary points.

Use technology to show how closely the truncated series expansions approximate the function. Sliders can be used to develop this.

Discuss when we might use the more general Taylor series rather than a Maclaurin’s series. Point out that approximations will be more accurate close to  $x = a$  in a series expansion in powers of  $(x - a)$ .

**COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES**

Most candidates become familiar with the Taylor expansion, but a large minority cannot differentiate implicitly. The absence of the use of the chain rule can cost several marks.

Differentiation is done accurately in the main, up to the 3rd derivative, with a few variations in the route to this derivative, although there are usually some slips in the use of product and chain rule.

Substitution is almost always correct, and substitution in the general series is also usually well done. As usual, some students do not link the series to the original function given in the question, using an undefined  $f(x) =$  or  $y =$ .

Candidates are usually well prepared for this type of question. There are typically very few poor attempts and in many cases full marks are attained. Most candidates, having worked out the values of the derivatives needed, write the required expansion straightaway without writing down the formula first. Luckily, most give enough working for it to be clear that the correct formula was being used, so it is rare for the method mark for the use of the correct formula to be lost. The most frequently lost mark is that for the final answer where the use of  $f(x) = \dots$  or (nothing at all)  $= \dots$  rather than the required  $y = \dots$  is penalised. Candidates should be careful to check what the question actually asks for in all cases.

## UNIT 7: Polar coordinates

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### SPECIFICATION REFERENCES

- 7.1 Polar coordinates  $(r, \theta)$ ,  $r \geq 0$
- 7.2 Use of the formula  $\frac{1}{2} \int_{\alpha}^{\beta} r^2$  for area

### PRIOR KNOWLEDGE

#### IAL – Pure content

- Coordinate geometry in the  $(x, y)$  plane (Unit 3 of P2 SoW and Unit 3 of P4 SoW)
- Curve sketching (Unit 1 of P1 SoW)
- Integration (Unit 6 of P4 SoW)

### KEYWORDS

Polar, Cartesian, coordinates, convert, parallel, point, enclosed, area.

**7a. Convert between Cartesian and polar and sketch  $r(\theta)$  (7.1)**

Teaching time

4 hours

**OBJECTIVES**

By the end of the sub-unit, students should:

- understand and be able to use polar coordinates and equations and be able to convert between polar and Cartesian coordinates and equations;
- know how to sketch standard polar curves.

**TEACHING POINTS**

Introduce polar coordinates using a simple curve e.g. a circle centred at the origin. Relate to work already covered on complex numbers. Support the development of ideas by sketching points whenever possible. Reminding students that  $x = r \cos \theta$  and  $y = r \sin \theta$  can also be a very useful prompt. Online calculators can be used to check the answers of more demanding questions.

Online polar plotting tools allow you to experiment quickly with the range of values chosen for  $\theta$ . Use sliders to vary the values  $r = p \cos(q\theta)$  or  $a + b \cos(q\theta)$ .

This is a good opportunity to develop some enquiry-based learning with how the constant coefficients affect the function and what it looks like on a polar plot. This will develop the students' understanding of the shape that they are looking for when hand plotting and the necessary turning points. For example, plot  $r = 3 \cos(2\theta)$  for  $0$  to  $\pi$  and then  $0$  to  $2\pi$ . Look at the symmetries of the functions as they are developed. This will help students with their sketching and understanding of what the plots are for various different types. Ensure that students are familiar with simple circles, lines and half-lines, spirals, roses of the form  $r = a \cos(b\theta)$  and  $r = a \sin(b\theta)$  and cardioids, and are able to sketch them.

**OPPORTUNITIES FOR REASONING/PROBLEM SOLVING**

Derive  $r^2 = x^2 + y^2$  and make links to the (unit) circle.

Discuss how you can have a straight line on a polar plot; this is a good question to link Cartesian and polar equations. Using the relationships  $x = r \cos \theta$ ,  $y = r \sin \theta$  and  $r^2 = x^2 + y^2$ , ask students to convert between polar and Cartesian equations.

**COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES**

A sketch is an essential part of any solution strategy and should be used whenever appropriate.

**OBJECTIVES**

By the end of the sub-unit, students should:

- be able to find tangents parallel and perpendicular to the initial line;
- be able to find (compound) areas under polar graphs using the formula  $\frac{1}{2} \int r^2 d\theta$ .

**TEACHING POINTS**

Use and understand  $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$  to derive gradients of tangents parallel and perpendicular to the initial line for each problem rather than using a general formula. Be careful to stress that the product rule typically results in more than one term in the numerator and denominator.

Derive and use the formula  $\frac{1}{2} \int r^2 d\theta$  to find areas.

Emphasise that typically areas to be found are ‘enclosed by’ rather than ‘under’ a curve. Choose examples that have been sketched in the earlier sub-unit and encourage sketching whenever possible. Students should use their sketches to help them determine the required limits.

Integration of trigonometric functions is required here, so it would be good to begin with a reminder of standard integrals and the formulae for double angles to make the resulting integrals more accessible.

**OPPORTUNITIES FOR REASONING/PROBLEM SOLVING**

Consider more complex problems involving areas, such as finding the area between two curves. Encourage students to use a sketch to identify the required area and the limits needed. A further example might be to find the area between inner and outer loops of the curve  $r = 2 + 4 \cos \theta$ . Online graphing/calculating tools can be used to confirm the areas being found, especially when problems are more complex e.g. finding the area enclosed by inner loops. Find the equation of a tangent or normal to a polar curve, giving the answer in both polar and Cartesian form.

You could extend the work done finding tangents to finding tangents not parallel to the initial line e.g.

$$r = 3 + 8 \sin \theta \text{ at } \theta = \frac{\pi}{6}.$$

**COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES**

Students should take care to provide their answers in the required form and to the required accuracy. For example, marks will be lost for a decimal approximation if the question asks for an exact area.

Errors in integration are often caused by incorrect limits or by substituting limits incorrectly. Marks are also lost due to basic errors in expanding expressions and algebraic manipulation.