Pearson Edexcel

INTERNATIONAL ADVANCED LEVEL

MATHEMATICS/ FURTHER MATHEMATICS/ PURE MATHEMATICS

SCHEME OF WORK

FURTHER PURE MATHEMATICS 1

Pearson Edexcel International Advanced Subsidiary in Mathematics (XMA01)
Pearson Edexcel International Advanced Subsidiary in Further Mathematics (XFM01)
Pearson Edexcel International Advanced Subsidiary in Pure Mathematics (XPM01)
Pearson Edexcel International Advanced Level in Mathematics (YMA01)
Pearson Edexcel International Advanced Level in Further Mathematics (YFM01)
Pearson Edexcel International Advanced Level in Pure Mathematics (YPM01)
First teaching September 2018
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First certification from August 2019 (International Advanced Subsidiary) and August 2020 (International Advanced Level)

A level Mathematics: Pure Mathematics 1

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Further Pure Mathematics 1

Unit		Title	Estimated hours
1		**Complex numbers**	
	a	Introduction of complex numbers, basic manipulation	**3**
	b	Complex conjugate, division and solving polynomial equations	**5**
	c	Argand diagrams	**2**
	d	Modulus and argument	**4**
2		**Roots of quadratic equations**	
	a	Roots of polynomial equations	**4**
	b	Formation of polynomial equations	**2**
3		**Numerical solution of equations:**	
	a	Numerical solution of equations	**4**
	b	Newton-Raphson method	**2**
4		**Coordinate systems**	
	a	Equations of parabola and rectangular hyperbola and the focus-directrix properties of the parabola	**6**
	b	Tangents and normals to the parabola and hyperbola	**4**
5		**Matrix algebra integration**	
	a	Matrix addition, subtraction and multiplication	**3**
	b	Inverse of 2×2 matrices	**3**
6		**Transformations using Matrices:** Linear transformations	**8**
7		**Series:** Sums of series	**4**
8		**Proof:** Proof by mathematical induction	**6**
			60 hours

UNIT 1: Complex numbers

Return to Overview

SPECIFICATION REFERENCES

1.1 Definition of complex numbers in the form $a + \mathrm{i}b$ and $r\cos\theta + \mathrm{i}r\sin\theta$

1.2 Sum, product and quotient of complex numbers

1.3 Geometrical representation of complex numbers in the Argand diagram. Geometrical representation of sums, products and quotients of complex numbers

1.4 Complex solutions of quadratic equations with real coefficients

1.5 Finding conjugate complex roots and a real root of a cubic equation with integer coefficients

1.6 Finding conjugate complex roots and/or real roots of a quartic equation with real coefficients

PRIOR KNOWLEDGE

IGCSE/GCSE (9-1) in Mathematics at Higher Tier

- Like terms, expanding brackets and factorising (including difference of two squares)
- Surds
- Solving quadratic equations by factorising, completing the square and using the quadratic formula
- Trigonometric ratios, such as $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$
- Equation of a straight line

IAS Mathematics – Pure content

- Quadratic functions and use of the discriminant (Unit 1 of the P1 SoW)
- Algebraic division and the factor theorem (Unit 2 of the P2 SoW)
- Equation of a circle (Unit 3 of the P2 SoW)
- Binomial Expansion (Unit 4 of the P2 SoW)
- Radians (Unit 2 of the P1 SoW)

KEYWORDS

Conjugate, real part, imaginary part, complex conjugate, root, discriminant, Argand diagram, Cartesian coordinates, vector, magnitude, modulus, argument, principal argument, radians, modulus-argument form, polynomial, coefficient, quadratic, quartic, cubic, complex conjugate pair.

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1a. Introduction of complex numbers, basic manipulation (1.1) (1.2) (1.4)

Teaching time
3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to solve any quadratic equation with real coefficients;
- be able to add, subtract and multiply complex numbers in the form $x + \mathrm{i}y$ with x and y real;
- understand and use the terms 'real part' and 'imaginary part'.

TEACHING POINTS

Begin by looking at previously 'unsolvable' quadratics and relate to the discriminant: $b^2 - 4ac < 0$. Sketch the graph and show that it does not cross the x-axis, hence there are no **real** roots. Put the idea across that we need an **imaginary** axis to cross for some sort of root to exist.

Define i as $\sqrt{-1}$. Hence $\mathrm{i}^2 = -1$.

Emphasise that in a complex number, the real part and the imaginary part cannot be combined to form a single term. Introduce addition and subtraction of complex numbers as adding (or subtracting) the real parts and adding (or subtracting) the imaginary parts. Point out the similarity with adding and subtracting vectors. Similarly, multiplying complex numbers uses the same techniques as for multiplying brackets in algebra followed by simplifying powers of i and using $\mathrm{i}^2 = -1$.

Show students both the quadratic formula method and the completing the square method for finding complex solutions of a quadratic equation. Particular care must be taken with signs when using the quadratic formula.

It is worth noting that if the quadratic formula is quoted incorrectly, no marks will be gained.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Expand brackets such as $(2 + 3\mathrm{i})^8$ using the binomial theorem.

Ask questions such as:

'Can you multiply or add two different complex numbers to get a real number?'

'What happens when you square or cube complex numbers?'

'Can some complex numbers be squared to become imaginary numbers?'

Look at equating complex numbers by comparing the real to real and imaginary to imaginary parts. For example, find the values of a and b if the following complex numbers are equal:

$z_1 = a(2 + \mathrm{i}) + (b - \mathrm{i})$ and $z_2 = a(3 + 2\mathrm{i}) + 4 + b\mathrm{i}$

For further practice at multiplying complex numbers you could use complex numbers in geometric series, for example: Given that $u_{n+1} = (2 + \mathrm{i})u_n$ and $u_1 = 3$, find the first five terms of the series and find the sum to n terms.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

The examiners commented on how handwriting was difficult to read at times. In particular, the number '2' and the letter 'z' were often written badly by many candidates. It is important that candidates write numbers, letters and symbols clearly so that marks are not lost unnecessarily.

Students should be encouraged to write down the quadratic formula before they use it.

Care must be taken when using the quadratic formula; a common error is the loss of the square root sign so that $\frac{1\pm\sqrt{1-4\times1\times2}}{2}$ becomes $\frac{1}{2}\pm\frac{7}{2}\text{i}$.

NOTES

To review work on simplifying surds, practice simplifying terms such as $\sqrt{-24}$.
Encourage students to write complex numbers in the form $a+b\text{i}$, where the real and the imaginary parts are separate.

1b. Complex conjugate, division and solving polynomial equations (1.2) (1.5) (1.6)

Teaching time
5 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand and be able to use the complex conjugate of a complex number;
- be able to divide two complex numbers by using the complex conjugate of the denominator;
- know that non-real roots of polynomial equations with real coefficients occur in conjugate pairs;
- be able to solve cubic or quartic equations with real coefficients.

TEACHING POINTS

Point out that the complex roots of a quadratic equation are always of the form $a \pm b\text{i}$, and that these are called complex conjugates. Use the difference of two squares and rationalising a denominator in surd form to illustrate the method of manipulation of complex conjugates. The process for dividing complex numbers is similar to the process used to divide surds. For surds the denominator is rationalised. For complex numbers the denominator is made real.

Emphasise that zz* and $z + z^*$ are real.

Before moving on to solving cubic and quartic equations, revisit the factor theorem.

Highlight the fundamental theorem of algebra and the nature of polynomials and roots.

For cubic equations:

i. all three roots are real; or
ii. one root is real and the other two roots from a complex conjugate pair.

For quartic equations:

i. all four roots are real; or
ii. two roots are real and the other two roots form a complex conjugate pair; or
iii. two roots form a complex conjugate pair and the other two roots also form a complex conjugate pair.

Take care when finding the complex conjugate of a number given in the form '(imaginary part)i + real part', e.g. 8i – 3. The complex conjugate is – 8i – 3, and not 8i + 3.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

When solving polynomials, relate the roots of cubic and quartic equations to the graphs. Extend the nature of the roots to higher powers of polynomials. You could also ask students to find the equation of the polynomial given the roots.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

'It is worth noting that a significant number of candidates did not know what was meant by z^*'.

Remind students to be careful not to negate the real part instead of the imaginary part.

'Almost all students knew that they were required to multiply through by the conjugate of the denominator, but some lost accuracy because of sign errors or not collecting like terms correctly. There were a number of students who did not present the solution in the required form'.

Common algebraic errors include $(2\text{i})^2 = 2\text{i}^2$ and equivalent.

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‘Candidates need to be reminded that correct answers may not get full marks if insufficient working is shown’.

1c. Argand diagrams (1.3)

Teaching time
2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to use and interpret Argand diagrams.

TEACHING POINTS

Link Argand diagrams with **i**, **j** vectors, if previously covered in IAL Mathematics (M1 or P4).

The position of the complex number on the Argand diagram (the quadrant in which it appears) will determine whether its argument is positive or negative and whether its argument is acute or obtuse.

Students should be encouraged to take care when drawing Argand diagrams, and to use an accurate scale.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Begin by plotting powers of i on the Argand diagram and then think about what the square or cube of other complex numbers would look like. Ask students if they can explain what happens. You could also look at what the Argand diagram looks like when you add or subtract complex numbers. This can be explored graphically with mathematical software such as Autograph or Geogebra.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Students will lose marks due to lack of required labelling. ‘Many Argand diagrams were messy with little regard for scale, but labelling enabled the candidates to earn the marks’.

‘Errors made included plotting the pure imaginary roots on the real axis or not plotting the complex roots as a conjugate pair’.

1d. Modulus and argument (1.1) (1.3)	**Teaching time** 4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to convert between the Cartesian form and the modulus-argument form of a complex number;
- understand and be able to use $|z_1 z_2| = |z_1||z_2|$.

TEACHING POINTS

Link the modulus and argument of a complex number with the magnitude and direction of vectors if previously covered in IAL Mathematics (M1 or P4).

Encourage students to always draw a diagram when calculating $\arg(z)$. Point out that $\tan^{-1}\left(\frac{y}{x}\right)$ only works for the 1st and 4th quadrants. Tell students that the argument should be in radians and that the 'principal argument' is $-\pi < \text{Arg}(z) \leq \pi$.

Make the mod-argument form clear and note that it can also be written as $z = r(\cos(\theta) + \text{i}\sin(\theta))$.

Show that when complex numbers are multiplied the modulus is multiplied. Start by finding the modulus of two complex numbers, for example $z_1 = 4 + 3\text{i}$ and $z_2 = 5 - 12\text{i}$, then find $z_1 z_2$ and $|z_1 z_2|$. You could use mathematical software to look at further examples plotted on an Argand diagram. If the compound angle formulae (Unit 2 of IAL P3) have been covered you could then go on to prove the result $|z_1 z_2| = |z_1||z_2|$ using $z_1 = r_1(\cos(\theta_1) + \text{i}\sin(\theta_1))$, and $z_2 = r_2(\cos(\theta_2) + \text{i}\sin(\theta_2))$ to show that $z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

You could extend the work done so far on complex numbers by finding the square roots of a complex number. Start with, for example, $z = 4\text{i}$. $\sqrt{z} = \sqrt{4\text{i}} = a + b\text{i}$. Square both sides and then equate the real parts and the imaginary parts to find a and b. Point out that a and b are real, so any imaginary solutions should be discarded.

Although knowledge of the result $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ is not required, discussing the result informally using particular examples or using mathematical software to explore the result graphically would be a useful extension, and will lead into the complex number work in FP2. If the compound angle formulae (IAL P3 Unit 2) have been covered you could then go on to prove the result:

If $z = r_1(\cos(\theta_1) + \text{i}\sin(\theta_1))$ and $w = r_2(\cos(\theta_2) + \text{i}\sin(\theta_2))$,

then $zw = r_1 r_2(\cos(\theta_1 + \theta_2) + \text{i}\sin(\theta_1 + \theta_2))$.

Point out that for division the moduli are divided and the arguments subtracted. Emphasise that using the mod-arg form for dividing is often much easier and quicker that dividing the Cartesian form.

You could also discuss what the modulus and argument are for a complex number in the form $z = \cos\theta - \text{i}\sin\theta$, using $\cos\theta = \cos(-\theta)$ and $\sin\theta = -\sin(-\theta)$.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Students are advised to draw a diagram so they can visualise the argument. Some students will use $\tan\left(\frac{2}{3}\right)$ rather than $\arctan\left(\frac{2}{3}\right)$. i should not be included in the calculation of the modulus.

UNIT 2: Roots of quadratic equations

Return to Overview

SPECIFICATION REFERENCES

2.1 Sum of roots and product of roots of a quadratic equation

2.2 Manipulation of expressions involving the sum of roots and product of roots

2.3 Forming quadratic equations with new roots

PRIOR KNOWLEDGE

IGCSE/GCSE (9-1) in Mathematics at Higher Tier

- Expanding and factorising quadratic

IAS Mathematics – Pure 2 content

- Factor theorem (Unit 2 of the P2 SoW)

KEYWORDS

Quadratic, cubic, quartic, polynomial, coefficient, degree, root, complex conjugate, degree, Vieta's formulae.

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2a. Roots of polynomial equations (2.1) (2.2)

Teaching time
4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand and use the relationship between roots and coefficients of quadratic equations;
- be able to manipulate and use expressions involving the sum and product of roots of quadratic equations.

TEACHING POINTS

Although a quadratic may have complicated roots, we can often determine a lot about them using Vieta's formulae, which don't necessarily tell us what the roots are, but enable us to evaluate other expressions.

Go through Vieta's formulae, which tell us that:

- A quadratic equation with roots α and β can be written as $ax^2 + bx + c = a(x - \alpha)(x - \beta)$. Expanding and equating coefficients of x will yield $\alpha + \beta = \frac{-b}{a}$ and $\alpha\beta = \frac{c}{a}$. If $a = 1$, this simplifies to $\alpha + \beta = -b$ and $\alpha\beta = c$.

These formulae need to be learned.

Show that these formulae can be used to evaluate expressions such as $\alpha^2 + \beta^2$, $\alpha^3 + \beta^3$ and $\frac{1}{\alpha} + \frac{1}{\beta}$, without knowing the actual roots, which are most likely complex in many cases. Explain that the expressions should be manipulated to get them in terms of Vieta's formulae. For example, $(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$. This can be rearranged to make $\alpha^2 + \beta^2$ the subject, and the values from Vieta's formulas can be substituted in. Similarly $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta}$.

Knowledge of the following identities will be helpful:

- $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
- $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Given that, for example, $2 + i$ is a root of a quadratic equation, ask students to find a suitable quadratic equation using: $x^2 - (sum\ of\ roots)x + (product\ of\ roots) = 0$. Point out that this can be used as an alternative method for finding solutions to cubic and quartic equations (Unit 1b).

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

It is usually more difficult to actually try to find the roots of the polynomial. 'A surprising number of candidates found the roots of the original equation and proceeded to answer the whole question using them. This often required lengthy manipulations and calculations where mistakes were common'.

Sometimes the question explicitly states 'without solving the equation', and students who answer the question ignoring this instruction will gain no marks.

'A number of candidates thought that $\alpha^3 + \beta^3$ was $(\alpha + \beta)^3$ and so lost two marks. A number of candidates worked with individual roots involving surds and were presumably unaware of the sum and product properties.'

Students must be careful with signs, particularly in the bracketed terms.

2b. Formation of polynomial equations (2.2) (2.3)

Teaching time
2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to form a quadratic equation with new roots.

TEACHING POINTS

The method used could be substitution, but is most likely to be using sums and products of roots.
An example might be: 'if α and β are roots of the equation $3x^2 + 5x - 1 = 0$ find a quadratic equation that has roots $\alpha + 3$ and $\beta + 3$'. Using sums and products of roots, this problem can be solved as follows: Using $\alpha + \beta = \frac{-5}{3}$ and $\alpha\beta = \frac{-1}{3}$, then the sum of new roots is $\alpha + 3 + \beta + 3 = \frac{-5}{3} + 6 = \frac{13}{3}$ and the product of new roots is $(\alpha + 3)(\beta + 3) = \alpha\beta + 3(\alpha + \beta) + 9 = \frac{-1}{3} - 5 + 9 = \frac{11}{3}$. So the quadratic equation with roots $\alpha + 3, \beta + 3$ becomes $z^2 - \frac{13}{3}z + \frac{11}{3} = 0$, ie. $3z^2 - 13z + 11 = 0$.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Students should be given plenty of practice at forming quadratics with new roots and manipulating the algebraic expressions, particularly those involving fractions such as $\frac{1}{\alpha}, \frac{1}{\beta}$ or $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Errors in forming the new quadratic equation include: the omission of '= 0', not giving integer coefficients and the omission of '*x*' in the middle term i.e. giving an answer as $8x^2 - 33 + 8 = 0$.
'The most common error was a failure to perform basic algebraic manipulation on the sum of the roots'.

UNIT 3: Numerical solution of equations

Return to overview

SPECIFICATION REFERENCES

3.1 Equations of the form f(x) = 0 solved numerically by:
(i) interval bisection
(ii) linear interpolation
(iii) the Newton-Raphson process

PRIOR KNOWLEDGE

IGCSE/GCSE (9-1) in Mathematics at Higher Tier

- Iterations (GCSE only)
- Interpret the gradient on a curve as an instantaneous rate of change.

IAS Mathematics – Pure Mathematics content

- Graphs, roots and functions (Unit 1 of the P1 SoW)
- Differentiation (Unit 4 of the P1 SoW, Unit 7 of the P2 SoW)
- Series, sequences and recurrence relations (Unit 4 of the P2 SoW)

KEYWORDS

Roots, continuous, function, positive, negative, converge, diverge, interval, derivative, tangent, chord, iteration, Newton-Raphson, staircase, cobweb, trapezium rule.

NOTES

This topic extends the work done on iterations at GCSE (9-1) Mathematics and also links with graphs and functions.

3a. Numerical solution of equations (3.1)

Teaching time
4 hours

OBJECTIVES

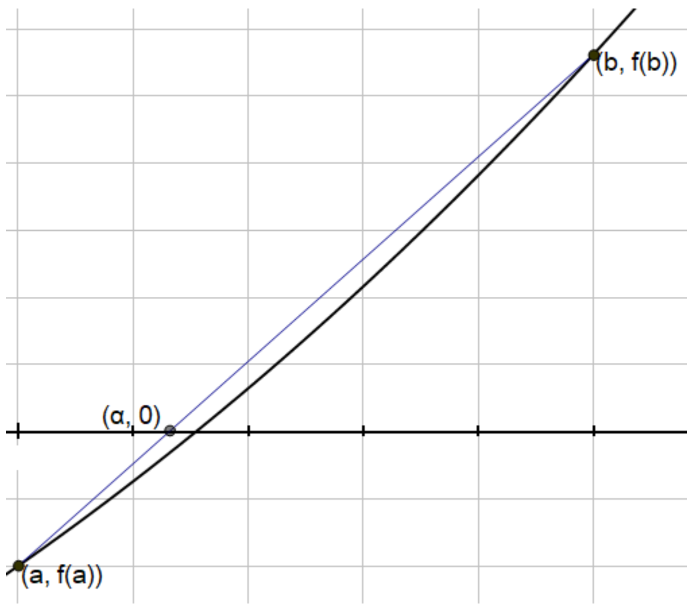
By the end of the sub-unit, students should:

- be able to determine whether an interval contains a root for a continuous function;
- be able to solve equations approximately using linear interpolation;
- understand how linear interpolation works in geometrical terms;
- understand and be able to use the method of interval bisection to find an interval containing a root of a continuous function.

TEACHING POINTS

Begin by reviewing the method of iteration from GCSE (9-1) Mathematics. Discuss the idea that if, for a continuous function, $\mathrm{f}(x)$, $\mathrm{f}(a) < 0$ and $\mathrm{f}(b) > 0$ then a root must lie in the interval (a, b). Use graph-drawing packages to find suitable intervals. Explain that ‘interval bisection’ is the process of reducing the interval within which a root lies by half at each iteration. Encourage students to record the upper limit (b), lower limit (a), midpoint ($\frac{a+b}{2}$) and the values of $\mathrm{f}(a)$, $\mathrm{f}(b)$and $\mathrm{f}(\frac{a+b}{2})$ clearly in a table for each iteration. Students should be reminded that they are finding an interval, and not an approximate root, and they should make sure that their final answer is clear.

Explain that linear interpolation is a process for finding an approximate root in a given interval by finding the x-coordinate of the point where the chord between the points (a, f(a)) and (b, f(b)) intersects the x-axis.



There are several ways to do this including the use of two similar triangles along the line segment, equating the gradient between two pairs of points along the line segment or finding the equation of the chord linking (a, f(a)) and (b, f(b)). Students should be encouraged to show their method clearly, and make use of the store button on their calculators to avoid losing accuracy through premature rounding. Mathematical software can be used to demonstrate graphically how these methods.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Exam questions generally require the method of linear interpolation to be used only once to find an approximate root. However, it is useful to investigate what happens if the process is applied several times. Get students to think about how the shape of the curve determines whether the approximations will approach the actual root from the left or the right, and which limit of the initial interval, (a, f(a)) or (b, f(b)), will always be used in successive iterations. Mathematical software can also be used to demonstrate this.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

When showing that a root lies in a given interval, students should be reminded to comment on the sign change and draw a conclusion. A comment from the examiner's report was: 'A few candidates correctly calculated f(6) and f(7), with negative and positive signs respectively, but then failed to comment on the sign change and/or draw a conclusion, therefore losing the accuracy mark.'

Marks will also be lost due to using degrees (instead of radians) if functions involve trigonometric terms.

Comments on interval bisection: 'this was generally well understood, with most candidates arriving at the correct answer. A few only bisected the interval once, while several continued until the function values (rather than input values) had an interval of less than 0.25.'

Comments on linear interpolation: 'this was generally well attempted. Some candidates used negative signs and many made several attempts as they realised that their answer did not seem reasonable. Some candidates used the line between two points, usually with less success.'

NOTES

Students should understand that many mathematical problems cannot be solved analytically, but that numerical methods permit a solution to be found to a required level of accuracy.

3b. Newton-Raphson method (3.1)

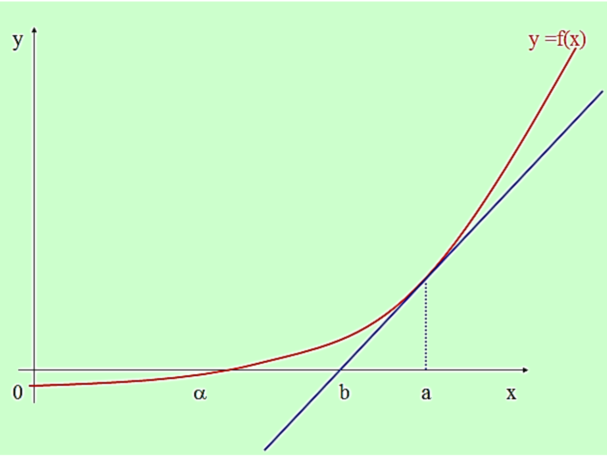
Teaching time
2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to solve equations approximately using the Newton-Raphson method;
- understand how the Newton-Raphson method works in geometrical terms.

TEACHING POINTS

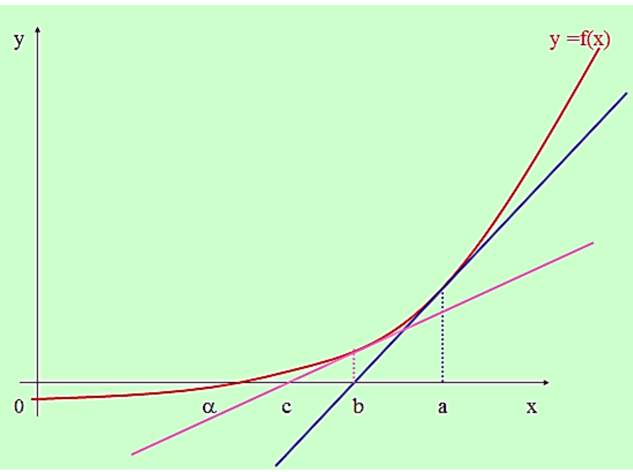


Consider the diagram above. The tangent crosses the x-axis at b (which is quite near the actual root α).

By considering the gradient of the tangent, we get $f'(a) = \frac{f(a)}{a-b}$ which can be rearranged to give $b = a - \frac{f(a)}{f'(a)}$.

We therefore have an expression for an approximation of the root (b), which uses the equation of the curve and its derivative at the point a.

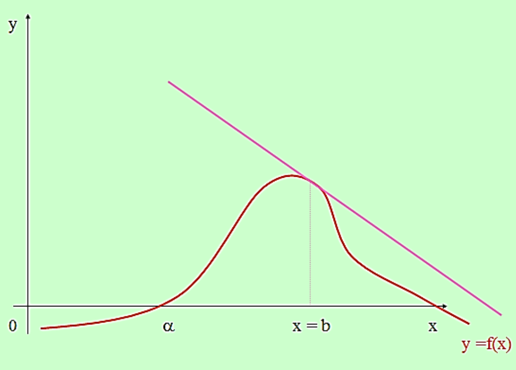
If we now go up from the point b, hit the curve and then construct another tangent (as in the diagram below) then, a similar argument, gives a better approximate root at c (nearer than b). Therefore we would get $c = b - \frac{f(a)}{f'(a)}$.



So if we continued this process we would get $d = c - \frac{f(a)}{f'(a)}$ and generally $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

Sometimes the process fails for some curves or starting points.

What happens to the tangent if we try to apply the process here?



An example of the type of question which may be seen is:

$f(x) = x^3 + 8x - 19$.

Obtain an approximation to the real root of $f(x) = 0$ by performing two applications of the Newton-Raphson procedure to $f(x)$, using $x = 2$ as the first approximation.

Give your answer to 3 decimal places.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Try different methods to find the roots of the same function. Which is the most efficient method or leads to the more accurate approximation? Consider, for example, linear interpolation vs Newton–Raphson.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Marks are often lost for sign errors and other numerical slips.

Students must show full working leading to the correct answer for full marks. Giving a correct answer either without working or following wrong working will result in zero marks.

NOTES

Graph drawing packages are an essential way to 'look' at the curve and the potential position of the roots depending on the first approximation of the root.

The only differentiation required will be as defined in units P1 and P2.

UNIT 4: Coordinate systems

Return to Overview

SPECIFICATION REFERENCES

4.1 Cartesian equations for the parabola and rectangular hyperbola

4.2 Idea of parametric equation for parabola and rectangular hyperbola

4.3 The focus-directrix properties of the parabola

4.4 Tangents and normals to these curves

PRIOR KNOWLEDGE

IAS Mathematics – Pure content

- Coordinate geometry of the circle (Unit 3 of P2 SoW)
- Differentiation (See P1 SoW Unit 4)

KEYWORDS

Parabola, rectangular hyperbola, focus, directrix, tangent, normal, loci.

4a. Equations of parabola and rectangular hyperbola and the focus-directrix properties of the parabola (4.1) (4.2) (4.3)

Teaching time
6 hours

OBJECTIVES

By the end of the sub-unit, students should:

- know and be able to use the Cartesian and parametric equations for the parabola and rectangular hyperbola;
- know and be able to use the focus-directrix property of the parabola.

TEACHING POINTS

A good place to start here is ask the students what curves you can get from a double cone when you cut it. This is a great opportunity to let them consider the four conics (circle, ellipse, parabola and hyperbola) and their properties. To extend this they could consider the degenerate conics (a point, a line or two lines). Encourage curve sketching by hand. This can be also done on circular linear graph paper with the non-degenerate conics and using graphing tools.

Once they have developed the two curves (parabola and rectangular hyperbola), look at each separately and develop the properties of each one. Work with the Cartesian forms first to help with finding the key points. Use the graphs to consider key points required from the constants in each standard form.

It is important to make sure they are clear with the terminology used, as it is very specific. They should be encouraged to develop a sheet of key points for each shape that details key aspects (intersections with axes, equations of asymptotes etc.) based on standard forms.

Once all the key points and forms are understood then move onto the idea of the parametric form.

Discuss the focus-directrix property of the parabola. Graphing software is useful to demonstrate that the locus of points is equidistant from the focus and directrix.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Look at parabolas in the real world and discuss the importance of the focus. Mathematical software can be used to demonstrate that the reflection of any line in the parabola will pass through the focus. Examples might include a parabolic heater, car headlight or a parabolic satellite dish.

Students could be given questions involving proof. Examples might include: A point P(x, y) is such that the distance of P to a point (4, 0) is the same as the distance of P to the straight line $x + 4 = 0$. Prove that the locus of P has an equation of the form $y^2 = 4ax$.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Sometimes the foci are not well understood and not given as coordinates. Those who know the required focal properties achieve success in some difficult proofs with minimal effort. Those who opt for an approach using Pythagoras had various degrees of success and often struggle with the algebra or are unable to deal with the square roots. Choosing to show that relationships work with specific examples rather than giving an algebraic proof will typically gain no credit.

4b. Tangents and normals to the parabola and rectangular hyperbola (4.4)

Teaching time
4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to derive the equations of tangents and normals to these conic sections and solve related problems.

TEACHING POINTS

Using the Cartesian forms of the parabola and rectangular hyperbola, re-arrange these to make y the subject. Differentiate and then find the gradient and equation of the tangent at a given point. Start with particular curves with numerical coefficients and particular points, then move on to the general form, working more algebraically:

Find the equation of the tangent to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

Then encourage students to develop similar equations for the tangents and normals to the rectangular hyperbola.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Although implicit and parametric differentiation is not required, if students have covered these topics they could be encouraged to use these methods to differentiate the equations for the parabola and rectangular hyperbola without re-arranging to make y the subject.

A variety of problems can be given to students involving tangents and normal. Examples might include finding where the normal to a curve at a particular point intersects the curve again or investigating where the tangents or normals to the parabola at particular values of x intersect.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Emphasise the importance of correct algebra:

In some proofs, simplifying the gradient typically makes simplification of their equation much more successful. Those students who do not do this, typically fail to successfully produce a convincing argument.

Many errors are made by students with these questions, notably 'invisible brackets' (i.e. $6p^2$ instead of $36p^2$).

Problems seen include incorrect differentiation and incorrect substitution. Many students struggle to eliminate one of the variables when this is required in the solution, but of those that do manage it, the correct answer is usually reached.

UNIT 5: Matrix algebra integration

Return to Overview

SPECIFICATION REFERENCES

5.1 Addition and subtraction of matrices

5.2 Multiplication of a matrix by a scalar

5.3 Products of matrices

5.4 Evaluation of 2×2 determinants

5.5 Inverse of 2×2 matrices

PRIOR KNOWLEDGE

IGCSE/GCSE (9-1) in Mathematics at Higher Tier

- Transformations (rotations, translations, reflections, enlargement)
- Trigonometric ratios
- Simultaneous equations

KEYWORDS

Array, dimension, rows, columns, elements, scalar, square matrices, commutative, associative, vector, identity, determinant, inverse, zero matrix, singular, non-singular, simultaneous equations.

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5a. Matrix addition, subtraction and multiplication (5.1) (5.2) (5.3)

Teaching time
3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to find the dimension of a matrix;
- be able to add and subtract matrices of the same dimension;
- be able to multiply a matrix by a scalar;
- be able to multiply conformable matrices.

TEACHING POINTS

Stress the importance of thinking it terms of rows and columns. (There are some great videos on-line to embed this!) An $n \times m$ matrix has n rows and m columns.

It is not always possible to find both **AB** and **BA** when multiplying matrices as the dimensions do not allow it. In the case of square matrices though, it will always be possible to find both **AB** and **BA**. However, do not assume **AB** = **BA**. The technical term for this is to say that matrix multiplication is not **commutative**. It is very important to place matrices in the correct order when finding the product of two matrices.

Do not assume $\mathbf{A}^2 - \mathbf{B}^2 = (\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{B})$.

$\mathbf{A}^2$ means $\mathbf{A} \times \mathbf{A}$.

Matrix multiplication is associative. This means that when evaluating the product of three or more matrices, provided the order is kept the same it does not matter which product pair is evaluated first. For example, when evaluating **ABC**, it does not matter is we evaluate **(AB)C** or **A(BC)**, although sometimes forward planning can make the resulting calculations easier.

The 2×2 identity matrix is given by $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and the 3×3 identity matrix is given by $\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Multiplying by the identity matrix is equivalent to multiplying by 1 in arithmetic. So **AI** = **IA** = **A**.

Care must be taken with signs and arithmetic when multiplying two matrices.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Provide students with plenty opportunities to practise multiplying matrices. For example, prove that for all 2×2 matrices **A**, **B** and **C**, (**AB**)**C** = **A**(**BC**), or show that if **A** (1×2), **B** (2×3) and **C** (3×4) then (**AB**)**C** = **A**(**BC**).

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

‘Sums, differences and/or products of matrices often had inaccurate elements which lost at least two marks’.

‘A number of students did not take the importance of the order of matrix multiplication into account’.

‘A few students failed to get the correct dimensional order of the answer’.

‘Explanations for why **AB** and **BA** are not equal were many and varied and often correct. There were lots of references to transformations and matrix multiplication not being commutative, but these attempts were not usually sufficient for this mark. The most common approach was a discussion about the dimensions of the resulting matrices’.

5b. Inverse of 2 × 2 matrices (5.4) (5.5)	**Teaching Time** 3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to calculate determinants of 2 × 2 matrices;
- understand and use singular and non-singular matrices;
- be able to know the properties of inverse matrices;
- be able to calculate the inverse of non-singular 2 × 2 matrices.

TEACHING POINTS

Start by introducing the idea that there is a 2 × 2 matrix $\mathbf{A}^{-1}$ such that $\mathbf{AA}^{-1} = 1$. Using a simple 2 × 2 matrix, **A**, students could set up and solve simultaneous equations to find the matrix $\mathbf{A}^{-1}$. Define the determinant of the matrix **A**, and discuss how the inverse can be found.

Whilst understanding the process of finding the inverse of a matrix is required, students can use a calculator to calculate the inverse of a matrix. However, for a 2 × 2 matrix it is often much quicker to simply apply the method and write down the answer.

Point out that if $\det(\mathbf{A}) = 0$, then $\mathbf{A}^{-1}$ cannot be found, and **A** is a **singular** matrix.

Note that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

Students should know that if **A** and **B** are non-singular matrices, then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

This is an opportunity to introduce some matrix algebra. Ask students to prove the result $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ or investigate $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$, $(k\mathbf{A})^{-1} = \frac{1}{k}\mathbf{A}^{-1}$, $(\mathbf{A}^{n})^{-1} = (\mathbf{A}^{-1})^{n}$, where n is a positive integer. The determinant properties could also be investigated: $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$ and $\frac{1}{\det \mathbf{A}} = \det \mathbf{A}^{-1}$.

Look at how to solve matrix equations such as **AB** = **C** using the inverse. You could also look at Cramer's rule using determinants to find the solution to a system of simultaneous equations.

Look at how matrices are used in the real world, for example the use of matrices for Encryption.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUO1TES

'A significant number of candidates thought that the determinant of **A** is $\frac{1}{\det\mathbf{A}}$.'

'A large group could not find the inverse of a 2 × 2 matrix accurately'.

'There were some errors in the calculation of the determinant and also some errors in the positions and signs of the elements within the inverse matrix'.

'A significant number of students took the time to either factorise the determinant or to multiply the determinant into their matrix, neither of which were necessary to get all the available marks'.

'There were some candidates who attempted to write down the inverse matrix in one step. With errors in their inverse in these cases it was difficult to identify a convincing method'.

NOTES

Students should be encouraged to show how they found their determinant so that they do not lose a method mark for making a sign error in their calculation.

A level Mathematics: Pure Mathematics 1

UNIT 6: Transformations using matrices | **Teaching time**
Linear transformations (6.1) (6.2) (6.3) (6.4) | 8 hours

Return to Overview

SPECIFICATION REFERENCES

6.1 Linear transformations of column vectors in two dimensions and their matrix representation

6.2 Applications of 2 × 2 matrices to represent geometrical transformations

6.3 Combinations of transformations

6.4 The inverse (when it exists) of a given transformation or combination of transformations

PRIOR KNOWLEDGE

IGCSE/GCSE (9-1) in Mathematics at Higher Tier

- Transformations (rotations, translations, reflections, enlargement)
- Vectors

KEYWORDS

Array, dimension, rows, columns, elements, scalar, square matrices, commutative, associative, transformation, rotation, translation, reflection, enlargement, linear transformation, scale factor, vector, position vector, object, image, identity, determinant, inverse, symmetric, zero matrix, singular, non-singular, line, parameter, Cartesian equation, simultaneous equations.

OBJECTIVES

By the end of the unit, students should:

- be able to use matrices to represent 2D rotations, reflections, enlargements and one-way stretches;
- understand and use zero and identity matrices;
- be able to use matrix products to represent combinations of transformations;
- be able to use inverse matrices to reverse the effect of a linear transformation;
- be able to use the determinant of a matrix to determine the area scale factor of a transformation.

TEACHING POINTS

Students will need to know the exact trigonometric ratios for sin, cos and tan of 0°, 30°, 45°, 60° and 90°.

Explain that a linear transformation is one in which the origin does not move, it involves only linear expressions and it can be represented by a matrix.

To identify a 2D transformation from its matrix, consider how the points (1, 0) and (0, 1) in the form of the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are transformed. For example, for the matrix $\mathbf{T} = \begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix}$, the point (1, 0) is transformed to (6, 0), and the point (0, 1) is transformed to (0, 3) so the matrix **T** represents a stretch scale factor 6 parallel to the x-axis and a stretch scale factor 3 parallel to the y-axis.

The order matters: **A** followed by **B** followed by **C** is represented by **CBA**. For example, if the transformation T is represented by the matrix **T** and the transformation U is represented by the matrix **U**, then the matrix **UT** represents the combined transformation of the transformation T followed by the transformation U.

For 2D, identification and use of the matrix representation of single and combined transformations will be confined to:

- **reflection** in coordinate axes or the lines $y = \pm x$
- **rotation** about (0, 0) through any angle
- **stretches** parallel to the x-axis and y-axis
- **enlargement** about centre (0, 0), with scale factor k, $(k \neq 0, k \in \mathbb{R})$

The identity matrix does not carry out any transformation.

When describing an enlargement, the centre will always be (0, 0), and this should be stated, along with the scale factor.

When describing a rotation, the centre will also always be (0, 0), and, again, this should always be stated, along with the angle and direction (anticlockwise is positive). Review the exact trigonometric ratios for sin, cos and tan of 0°, 30°, 45°, 60° and 90°.

When describing a reflection, the mirror line must be stated.

Students will need to be able to describe the transformation represented by a matrix and be able to find a matrix to represent a given transformation. Drawing diagrams to show the images of the original vectors is very useful, so students should be encouraged to do this.

The determinant of a matrix gives the area scale factor of a transformation. If the determinant is negative, the transformation involves a reflection.

Students may be asked to find the original points of an object before a transformation was applied. They need to know that they have to use the inverse of the transformation matrix to transform each point.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

To extend work on areas, you could investigate what happens to an object if it is transformed by a singular matrix. Another investigation would be to take a unit square and use matrix transformations to see how many quadrilaterals you can generate, using rotations and stretches. What is the matrix for each one? Although shears are not required, you could find the form of the matrix which represents a shear with the axes as invariant lines.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

'The most common error was failing to state the centre of rotation, the origin'.

The order of the transformations is important: 'The majority of candidates used their answer from (b) correctly to multiply two matrices in the correct order, but a significant number multiplied them in the wrong order, gaining no marks'.

'The majority knew the determinant property for area but some divided by det **A** instead of multiplying.'

'Many students could not find the matrices securely to represent basic transformations'.

'Some students left the matrix in terms of trigonometric ratios'.

NOTES

Transformation matrices are often used to create 3D computer graphics.

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UNIT 7: Series | **Teaching time**

Sums of series (7.1) | 4 hours

Return to Overview

SPECIFICATION REFERENCES

7.1 Summation of simple finite series

PRIOR KNOWLEDGE

IGCSE/GCSE (9-1) in Mathematics at Higher Tier

- Expanding and factorising quadratic and cubic expressions

IAS Mathematics - Pure content

- Sigma notation (Unit 4 of P2 SoW)

KEYWORDS

Sigma notation, series, sum, arithmetic series, geometric series, binomial series, integer, natural numbers.

OBJECTIVES

By the end of the unit, students should:

- be able to use sigma notation;
- understand and use formulae for the sums of integers, squares and cubes;
- be able to use known formulae to sum more complex series.

TEACHING POINTS

If arithmetic and geometric series in IAS Pure 2 has been covered, this could be reviewed, showing that $\sum r$ is an arithmetic series with $a = 1$ and $d = 1$.

Knowledge that $\sum_1^n 1 = n$ is expected.

Any number of terms can be 'split up' using the addition rule and the multiple rule.

For example, $\sum_{r=1}^{n}(r^2 + r - 2)$should first be 'split up' to $\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r - 2\sum_{r=1}^{n} 1$. Then the standard results for $\sum_{r=1}^{n} r^2$, $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} 1$ can be used. Always multiply brackets before attempting to evaluate summations of series such as $\sum_1^n r(r^2 - 2r)$.

Show the proofs of the standard results as students should have an understanding of how they are generated. However, the proofs for the sum of the squares and cubes may be better done when looking at proof by induction in the next unit.

Look carefully at the limits for the summation. In general: $\sum_{r=k}^{n} \mathrm{f}(r) = \sum_{r=1}^{n} \mathrm{f}(r) - \sum_{r=1}^{k-1} \mathrm{f}(r)$; however a common mistake is to use $\sum_{r=k}^{n} \mathrm{f}(r) = \sum_{r=1}^{n} \mathrm{f}(r) + \sum_{r=1}^{k} \mathrm{f}(r)$.

When asked to find a general result for a sum it is good practice to test it for small values of n. This does not prove it is correct, but if one value of n does not work, then it is obvious that the result is incorrect.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Make sure students have plenty of practice simplifying expressions. They should be encouraged to take out common factors as early as possible, as this tends to make the algebra more manageable. Extend the

work on the standard formulae and include more challenging examples such as:
Find $\sum_{r=1}^{2n} r^2$. To do this use the standard result $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$ and replace n by $2n$ to give $\sum_{r=1}^{2n} r^2 = \frac{2n}{6}(2n+1)(4n+1)$.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

It is a common mistake to write $5\sum_{r=1}^{n} 1$ as 5, instead of $5n$.
As shown above, a common mistake is to use $\sum_{r=k}^{n} \mathrm{f}(r) = \sum_{r=1}^{n} \mathrm{f}(r) + \sum_{r=1}^{k} \mathrm{f}(r)$ instead of $\sum_{r=k}^{n} \mathrm{f}(r) = \sum_{r=1}^{n} \mathrm{f}(r) - \sum_{r=1}^{k-1} \mathrm{f}(r)$.
In expanding $(3r-2)^2$, it is not unusual to see $(3r)^2 = 3r^2$. Students almost always use the correct formulae for $\sum r^2$ and $\sum r$ but evaluating $\sum 4$ as 4 rather than $4n$ is common. Taking out a factor of $\frac{n}{3}$ or $\frac{n}{6}$ in order to achieve the required result usually results in fewer errors than expanding completely to obtain a cubic before attempting a factor. The most common errors result from taking out the factor of $\frac{n}{3}$ or $\frac{n}{6}$ and not compensating for the $\frac{1}{3}$ or the $\frac{1}{6}$ on all the terms inside the bracket.
The approach to take out common factors as early as possible should be encouraged if it is appropriate to the question.

NOTES

$$\sum_{r=1}^{n} r^3 = \left(\sum_{r=1}^{n} r\right)^2$$

The formulae for $\sum_{r=1}^{n} 1 = n$ and $\sum_{r=1}^{n} r = \frac{n}{2}(n+1)$ should be learnt.

UNIT 8: Proof **Teaching time**

Proof by mathematical induction (8.1) 6 hours

Return to Overview

SPECIFICATION REFERENCES

8.1 Construct proofs using mathematical induction

PRIOR KNOWLEDGE

Covered so far

- Series notation
- Matrix multiplication

IGCSE/GCSE (9-1) in Mathematics at Higher Tier

- Index laws

KEYWORDS

Mathematical induction, general statement, basis, assumption, inductive, conclusion, integer, summation, divisible, matrix.

OBJECTIVES

By the end of the unit, students should:

- be able to obtain a proof for the summation of a series, using induction;
- be able to use proof by induction to prove that an expression is divisible by a certain integer;
- be able to use mathematical induction to prove general statements involving matrix multiplication.

TEACHING POINTS

You may need to revisit and highlight the nature of proof in mathematics and that this is one tool that will help but does have limitations.

In proof by induction, all the situations basically follow the same procedure:

1 Prove the statement is true for $n = 1$, i.e. show that when $n = 1$, LHS = RHS.

2 Assume the statement to prove **is** true for $n = k$ (where k is a positive integer). This is just a matter of rewriting the original statement with n replaced by k.

3 Write an expression for the next term i.e. $n = k + 1$.

4 Manipulate this expression and simplify it to look like the original expression with a $(k + 1)$ replacing all the k's.

5 This has proved that the situation works for k +1 if it works for k.

6 Therefore, it can be induced that it would work for $k + 2$, $k + 3$, etc. in a similar way. Hence it works for all positive integers k.

Students need to demonstrate their understanding of the concept of proof by induction, and not just learn the appropriate statements.

Give examples of the four different types of proof covered in this specification, with particular care given to proving that an expression is divisible by a certain integer, as this is the type students typically find the most challenging.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Use mathematical induction to produce a proof for the standard formulae for the sum of squares and the sum of cubes, from Unit 7. As an extension you could discuss Proof by Strong Induction.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Setting out is crucial at each stage. 'Almost all candidates started by attempting to check for $n = 1$, but often neglected to state that they had shown it to be true next to their working at this stage'.

Candidates may lose marks for not including all parts of their proof. An example of a minimum acceptable conclusion, following on from completely correct work, would be 'if the result is true for $n = k$ then it has been shown to be true for $n = k + 1$ and as it was shown true for $n = 1$ then the result is true for all positive integers'.

It is important to point out that for some proofs, students must show that the formula is true for both $n = 1$ and $n = 2$.

Simply trying to use standard results will result in a loss of a number of marks.

For proofs of the summation of a series, it is usually best to expand brackets as little as possible. Common factors of bracketed expressions should be factorised out, as well as fractions. 'Candidates who took $(2k + 1)$ out as a factor at the beginning were generally successful'.

For proofs involving matrix multiplication, students must show sufficient work to justify being awarded full marks for their solution.

The proofs for divisibility tend to be the most problematic, with many students getting stuck after forming $f(k + 1) - f(k)$; this should be practised.

'Statements were often ones that had been learned, rather than being used in the appropriate context. The conclusions were often ill-conceived, particularly when defining the values for which the proof was valid.'

NOTES

Proof by induction is not a method to derive formulae from first principles. It is simply used to check whether or not a general statement is true.