A9.1 Linear programming

Before you start

You should be able to:

- show by shading a region defined by one or more linear inequalities.

Why do this?

Linear programming is an example of optimisation which is very important in manufacturing.

Objectives

- You will be able to find the maximum and minimum value of a linear function within a region in the xy plane.
- You will be able to formulate and solve a linear programming problem in two variables.

Get Ready

1 Draw the lines with equations:
a $y = x + 4$ **b** $2x + 3y = 6$

2 Show by shading on your graph the regions:
a $y \leqslant x + 4$ **b** $2x + 3y \leqslant 6$

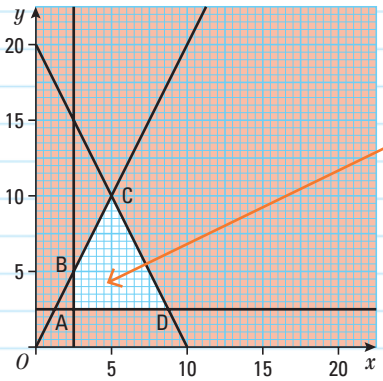
3 Show by shading the region of points which satisfy all of these inequalities:
$x \geqslant 5, y \geqslant 6$ and $x + y \leqslant 13$

Key Points

- A set of linear inequalities of the form $ax + by \leqslant c$ can define an enclosed region, R, known as the feasible region.
- The coordinates of all the points within and on the boundaries of the feasible region satisfy all the inequalities.
- A linear function P is of the form $P = ax + by + c$ where a, b and c are numbers.
- Within an enclosed region R, the maximum and minimum values of any linear function are attained at one of the corners of the region or along an edge of the region.
- Note: it is easier to show the region which satisfies all the inequalities as unshaded.

Example 1

The feasible region ABCD shown (unshaded) below satisfies these linear inequalities:
$2x + y \leqslant 20, x \geqslant 2, y \geqslant 2$ and $y \leqslant 2x$



The feasible region is the interior (and edges) of the quadrilateral ABCD

Find the maximum and minimum values of the following linear functions:
a $P = 4x + 3y$ **b** $P = 4x + 2y$

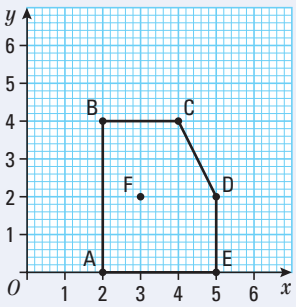
a So the maximum value of P (50) is at D
and the minimum value (14) is at A.

b At each of the corners, P takes the values A 12, B 16, C 40, D 40
So the maximum value occurs (40) at the Points C and D
(and in fact anywhere along the edge CD).
The minimum value (12) occurs at A.

(a) At the corner A (2, 2), P takes the value $4 \times 2 + 3 \times 2 = 14$.
At the corner B (2, 4), P takes the value $4 \times 2 + 3 \times 4 = 20$.
At the other corners P takes the values (C), 35 and (D), 42

Exercise 9A

C

1 

Find the value of each of these linear functions at the point A, B, C, D, E and F as shown in the diagram.

a $2x$ **b** $3y$ **c** $x + y$ **d** $2x + y$
e $3x - y$ **f** $x + 2y + 3$

B

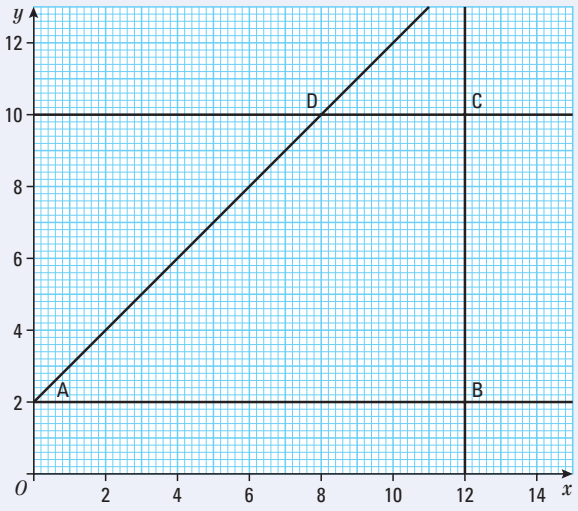
2 Draw the region which satisfies all of the following inequalities:
$x \geqslant 2,\ y \geqslant 3, x + 2y \leqslant 12$
Find the value of each of these linear functions at the vertices of the region and at the point with coordinates (3, 4):

a $2x$ **b** $x + y$ **c** $2y - 3$ **d** $3x + y$ **e** $x - 2y$

A

3 The diagram shows the finite region ABCD.
Write down the equations of each of the boundary lines of ABCD.
Find the inequalities that points within or on the boundary of ABCD must satisfy.
Work out the maximum and minimum values of the following functions at points within or on the boundary of ABCD:

a $x + y$ **b** $2x + 3y$
c $x - y$ **d** $2x - y + 8$



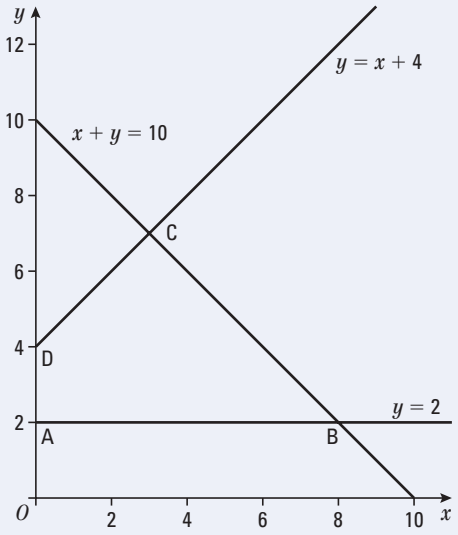
A

4 Here is a sketch.

Find the maximum and minimum values taken by each of these functions within or on the boundary of the finite region ABCD.

a $x + 2y$ **b** $2x + y$

c $4x + 4y$ **d** $2y - 2x$



5 The region ABCD satisfies the following inequalities:

$y + 3x \leqslant 48$ $y \geqslant x + 4$ $y \leqslant 12$ $4y + x \geqslant 16$.

Show, by shading, the region ABCD.

Find the maximum and minimum values of the following functions which satisfy all of the above inequalities.

a $x + 2y$ **b** $2y - x$ **c** $4x + y$ **d** $6x + 2y$

6 Show, by shading, the region which satisfies all of the following inequalities:

$y \leqslant x$ $2y + x \leqslant 150$ $y + x \leqslant 130$ $8y \leqslant 3x + 50$ $y \geqslant 0$

Find the maximum and minimum values of each of the following functions for the set of points which satisfy all of the above inequalities.

a $x + y$ **b** $2x - y$ **c** $3x + 2y + 50$ **d** $2x + 3y + 40$

Example 2 A radio broadcast company has two stations: Hot Hits and Cool Classics.

The company spends daily at least twice as much on Hot Hits as on Cool Classics.
The company spends daily at least £1000 on Cool Classics and at least £4000 on Hot Hits.
The company can afford to spend daily no more than a total of £12 000.
Let £x be the money the company spends daily on Hot Hits.
Let £y be the money the company spends daily on Cool Classics.

a Write down 4 constraints that x and/or y must satisfy.

b Draw a suitable diagram and identify the region that satisfies all of the constraints.

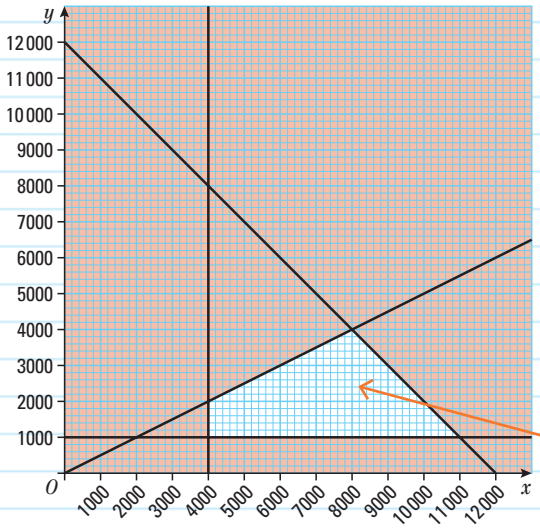
The daily profit on Cool Classics is expected to be £30 per pound spent and the daily profit on Hot Hits is expected to be £15 per pound spent.

c Write down an expression for the total daily profit £P in terms of x and y.

d Use your diagram to find the maximum daily profit and the values of x and y at which it occurs.

a $x \geqslant 2y$ $x \geqslant 4000$ $y \geqslant 1000$ $x + y \leqslant 12000$

b



A constraint is a mathematical condition that must be satisfied. In linear programming the constraints can be written as linear inequalities

ResultsPlus
Examiner's Tip

Candidates often write $x \geqslant 2y$ the wrong way round as $x \geqslant \frac{1}{2}y$.

This is the feasible region – it satisfies all four of the constraints.

c $P = 15x + 30y$

d Maximum profit occurs at (8000, 4000) and is £240 000.

Exercise 9B

A*

1 A farmer puts fertiliser on his fields.
He knows he must put on at least 500 kg of phosphate and at least 800 kg of nitrate.
The maximum total amount of fertiliser he will put on his fields is 3000 kg.
Let x kg be the mass of phosphate.
Let y kg be the mass of nitrate.

a Write down 3 constraints that x and/or y must satisfy.

b Draw a graph and indicate on the graph the region which satisfies all 3 constraints.

The cost of one kg of phosphate is 30p.
The cost of one kg of nitrate is 20p.

c Write down an expression for the cost C pence of x kg of phosphate and y kg of nitrate.

d Find the minimum and maximum cost which satisfies all the constraints.

2 A company makes shirts and vests.
Each day the company must make at least 300 shirts and must make at least 200 vests.
The company makes at least as many shirts as vests each day.
The company can make a maximum of 1000 of these garments each day.
Let x be the number of shirts.
Let y be the number of vests.

a Write down the inequalities that x and/or y must satisfy.

b On graph paper, show by shading, the region which satisfies all of the inequalities.

The cost of making a shirt is £20.
The cost of making a vest is £30.

c Write an expression for the total cost £C of making x shirts and y vests.

d Work out the minimum cost that satisfies all the constraints.

e The profit on a shirt is £10 and the profit on a vest is £5. Assuming that the company sells all the articles it makes, work out the maximum profit.

A*

3 A company makes chairs and settees.
Every day the company can make a maximum of 600 pieces of furniture.
The company makes at most twice as many chairs as settees.
The company makes at least 100 chairs and at most 300 settees each day.
The cost of making a chair is £100 and the cost of making a settee is £160.

Let x be the number of chairs and y be the number of settees.

a Express each of the constraints as inequalities.
b Express the total cost in terms of x and y.
c Draw the feasible region on a grid of squares.
d Find the minimum and maximum costs and the number of chairs and the number of settees at which the minimum cost is attained.

4 A market gardener grows cabbages and carrots.
She has a maximum of 80 hectares for growing.
She grows carrots on at least 50% more land than she grows cabbages.
She must use at least 20 hectares for carrots and at least 10 hectares for cabbages.

Let x hectares be the area used for growing carrots and y hectares be the area used for growing cabbages.

a Write down the inequalities.

The revenue from a hectare of carrots is £300 and the revenue from a hectare of cabbages is £400.

b Write down an expression in terms of x and y for the total revenue, £R.
c Find the maximum value of the revenue £R.

5 A newspaper runs a lottery in which there are £10 prizes and £20 prizes.

There must be at least 12 £20 prizes.
There must be at least 20 £10 prizes.
The number of £20 prizes must be not be more than 16 more than the number of £10 prizes.
The total amount of money available for the prize fund must not be greater than £1000.

Let x be the number of £10 prizes and y be the number of £20 prizes.

a i Explain why $x + 2y \leqslant 100$
ii Write the other constraints as inequalities.
b Draw these inequalities on graph paper and identify the region that satisfies all the inequalities.
c What is the maximum total number of prizes that can be given?

6 Tickets at a concert cost either £20 or £50.

The number of £50 tickets must be no more than 200 more than the number of £20 tickets.
There must be at least 300 £50 tickets and at most 600 £20 tickets.
The total number of tickets must not be more than 1200.

Let x be the number of £50 tickets and let y be the number of £20 tickets.

a Write these constraints as inequalities.
b Draw these inequalities on a suitable grid.

The profit from each £50 ticket is £20 and the profit from each £20 ticket is £10.

c Write down an expression for the total profit, £P.
d Find the maximum profit.

A*

7 Bill takes lots of exercise. Each week he covers between 40 miles and 80 miles by a combination of walking and jogging.
He walks at most half as far as he jogs.
He walks a minimum of 16 miles, and jogs a minimum of 20 miles.
Let x miles be the distance he walks and let y miles be the distance he jogs.

a Write down 5 relevant constraints that x and y must satisfy.

b Draw a graph to show the region of points satisfied by the constraints.

Bill uses up 150 calories per mile when he walks and 250 calories per mile when he jogs.

c Use your graph to find the smallest number and the largest number of calories that Bill can use up each week through this exercise.

8 A company makes two types of phones, A and B.
Each day it must make at least 200 type A and at least 300 type B.
The number of type B must be at most 50% more than type A.
The total number of phones made each day must not be more than 1000 and must not be less than 600.
Let x be the number of type A phones.
Let y be the number of type B phones.

a Write down all the constraints as inequalities.

b Show by shading the region which satisfies all the constraints.

The profit from making a type A is £6. The profit from making a type B is £7.50.

c Assuming that the company sells all the phones it makes, work out the maximum profit from the day.

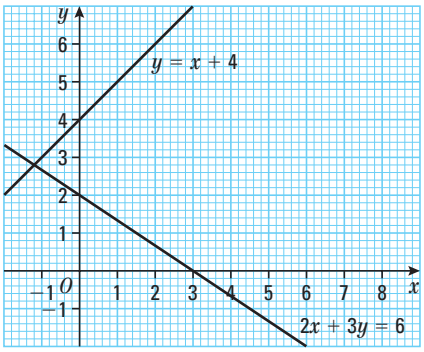
Review

- The solution to a linear programming problem requires the evaluation of a linear function at the corners of the feasible region.
- The maximum (minimum) value of a linear function in an enclosed region defined by a set of linear inequalities occurs at one of the corners of the region or at all the points along the edge of the region.

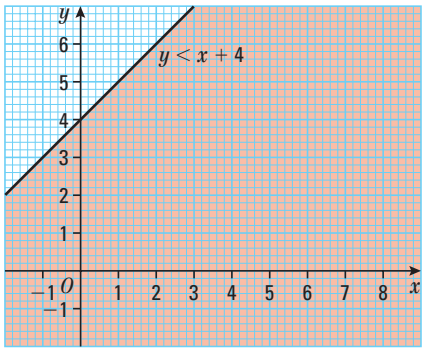
Answers

A9.1 Get Ready answers

1



2 a



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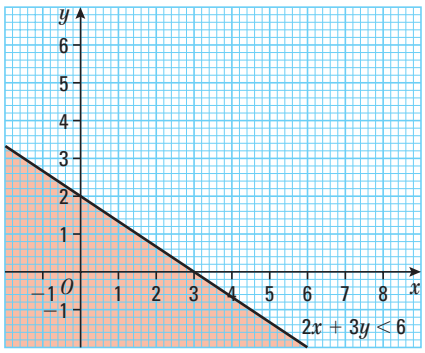

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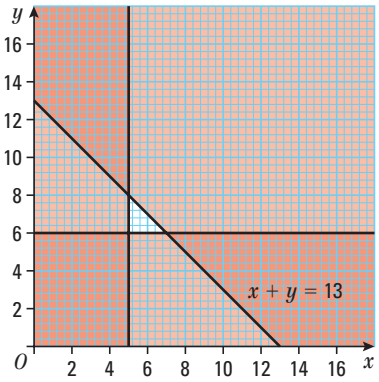
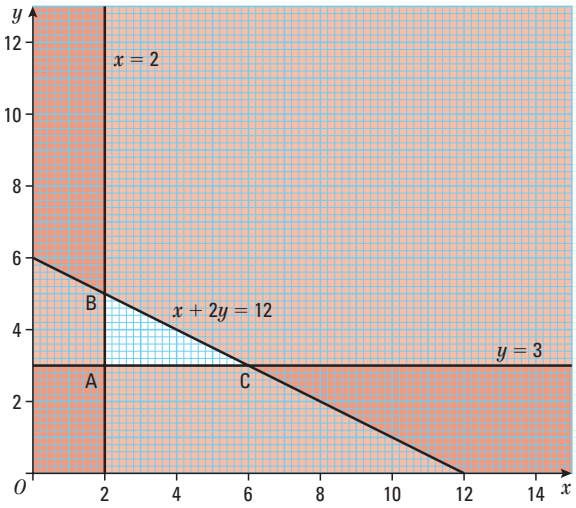


Exercise 9A

1

	A	B	C	D	E	F
$2x$	4	4	8	10	10	6
$3y$	0	12	12	6	0	6
$x + y$	2	6	8	7	5	5
$2x + y$	4	8	12	12	10	8
$3x - y$	6	2	8	13	15	7
$x + 2y + 3$	5	13	15	12	8	10

2



	A	B	C	(3, 4)
$2x$	4	4	12	6
$x + y$	5	7	9	7
$2y - 3$	3	7	3	5
$3x + y$	9	11	21	13
$x - 2y$	−4	−8	0	−5

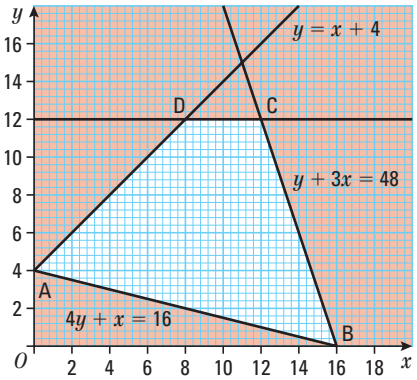
3 Through AB $y = 2$
Through BC $x = 12$
Through AD $y = x + 2$
Through DC $y = 10$
Inequalities satisfied are
$y \geqslant 2$ $x \leqslant 12$ $y \leqslant 10$ $y \leqslant x + 2$

	A	B	C	D
$x + y$	2 (min)	14	22 (max)	18
$2x + 3y$	6 (min)	30	54 (max)	46
$x - y$	−2 (min)	10 (max)	2	−2 (min)
$2x - y + 8$	6 (min)	30 (max)	22	14

4

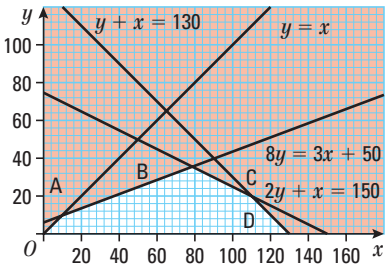
	A	B	C	D
$x + 2y$	4 (min)	12	17 (max)	8
$2x + y$	2 (min)	18 (max)	13	4
$4x + 4y$	8 (min)	40 (max)	40 (max)	16
$2y - 2x$	4	−12 (min)	8 (max)	8 (max)

5



	A	B	C	D
$x + 2y$	8 (min)	16	36 (max)	32
$2y - x$	8	−16 (min)	12	16 (max)
$4x + y$	4 (min)	64 (max)	60	44
$6x + 2y$	8 (min)	96 (max)	96 (max)	72

6

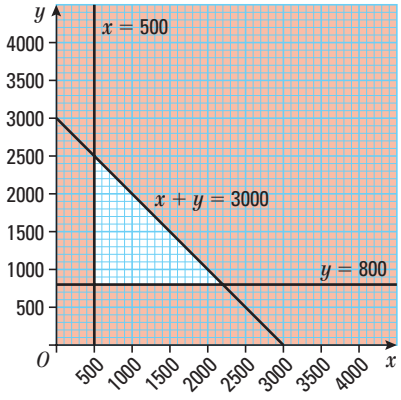


	O	A	B	C	D
$x + y$	0 (min)	20	114	130 (max)	130 (max)
$2x - y$	0 (min)	10	120	200	260 (max)
$3x + 2y + 50$	50 (min)	100	356	420	440 (max)
$2x + 3y + 40$	40 (min)	90	304	320 (max)	300

Exercise 9B answers

1 **a** $x \geqslant 500, y \geqslant 800, x + y \leqslant 3000$

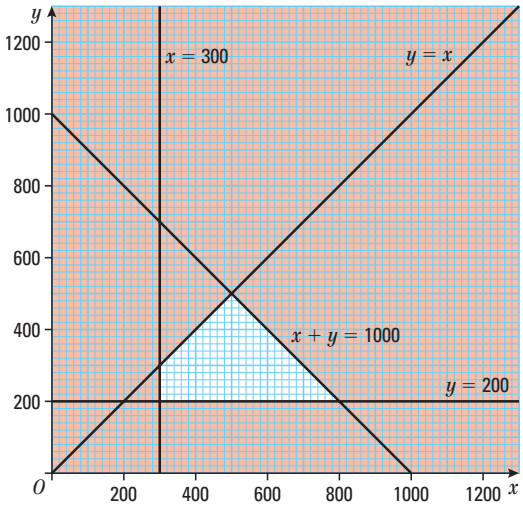
b



c $C = 30x + 20y$

d min = £310, max = £820

2 **a** $x \geqslant 300, y \geqslant 200, y \leqslant x, x + y \leqslant 1000$

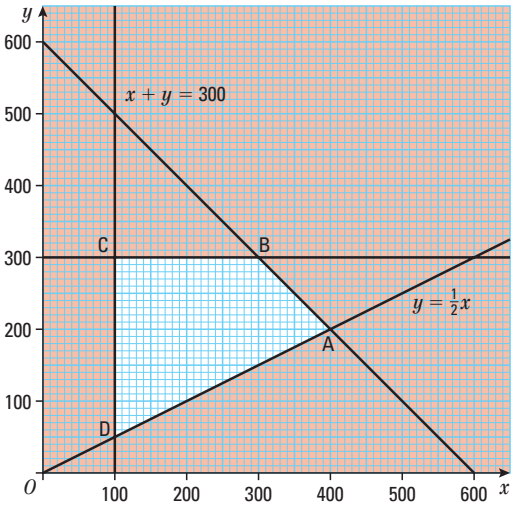


c $C = 20x + 30y$

d £12 000 **e** £9000

3 **a** $x \geqslant 100$, $y \leqslant 300$, $x + y \leqslant 600$, $y \geqslant \frac{1}{2}x$

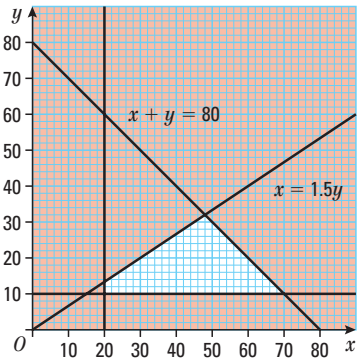
b $C = 100x + 160y$



d min = £18 000 at $x = 100, y = 50$,
max = £78 000 at $x = 300, y = 300$

4 **a** $x + y \leqslant 80$, $x \geqslant 1.5y$, $x \geqslant 20$, $y \geqslant 10$

b $R = 300x + 400y$

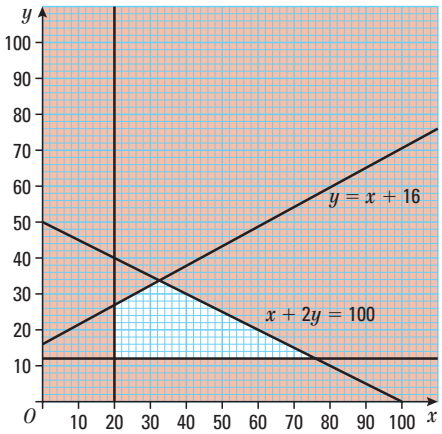


c R max = 27 200 at (48,32)

5 **a** **i** Total prize fund = $10x + 20y$, so $10x + 20y \leqslant 1000$

ii $x \geqslant 20$, $y \geqslant 12$ $y \leqslant x + 16$

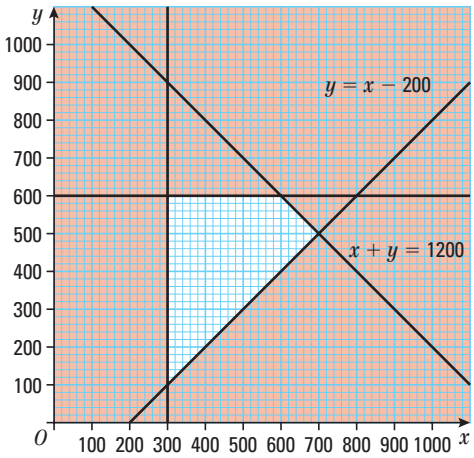
b



c Max value of $x + y$ is 88

6 **a** $x + y \leqslant 1200$, $x \geqslant 300$, $y \leqslant 600$, $y \geqslant x - 200$

b

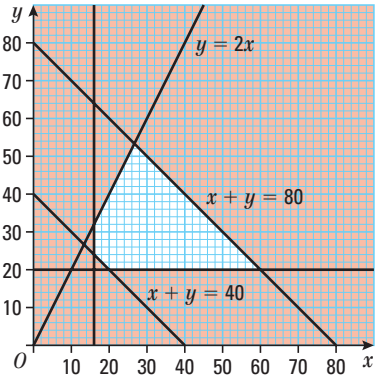


c $P = 20x + 10y$

d P max = 19 000

7 **a** $x \geqslant 16$ $y \geqslant 20$

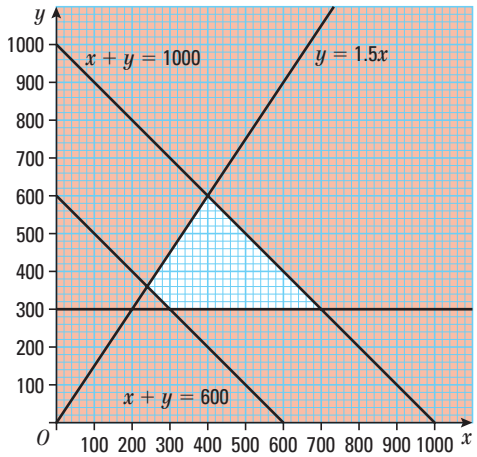
b $x + y \geqslant 40$ $x + y \leqslant 80$ $y \geqslant 2x$



c Maximum and minimum values of $150x + 250y$ are 17 300 and 8000

8 **a** $x \geqslant 200$, $y \geqslant 300$, $x + y \geqslant 600$,
$x + y \leqslant 1000$, $y \leqslant \frac{3}{2}x$

b



c £6900