A5.1 Exponential growth and decay

Before you start

You need to be able to:

- work out scale factors for percentage increase and decrease.
- draw a graph given y in terms of x.

Why do this?

In science, population growth is often an exponential function of time. The size of an investment in a bank account will grow exponentially if the interest rate remains constant. Radioactive decay is an example of exponential decay.

Objectives

- You can understand the meaning of exponential growth and decay.
- You can use multipliers to explore exponential growth and decay.
- You can use exponential growth in real life problems.

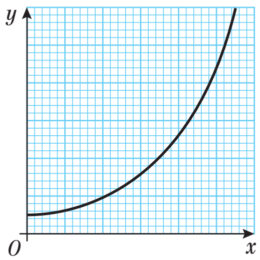
Get Ready

Work out:

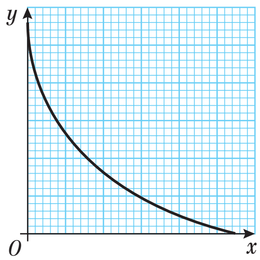
1 5^4 **2** 2^7 **3** 0.8^2 **4** 56^0

Key Points

- Exponential growth occurs when a function keeps increasing by the same scale factor. For example, the size of a population of bacteria may double every hour, the amount invested in a bank account will increase each year by the same scale factor as long as the interest rate remains constant.
- Exponential decay occurs when a function keeps decreasing by the same scale factor. For example, the mass of a radioactive element may halve every hour.
- All exponential growth and decay functions can be represented by the equation $y = ka^x$
- For exponential growth, $a > 1$
- For exponential decay, $0 < a < 1$
- The value of a, called the multiplier, is the scale factor by which the function grows or decays.
- y represents the size of the population or amount at time x
- k represents the initial value of y



$y = ka^x$ ($a > 0$)



$y = ka^x$ ($0 < a < 1$)

AO3

Example 1

A scientist is studying a population of flies.

The size of the population, P, after t days is given by the equation $P = 60 \times 2^t$.

a Work out the size of the population of flies at the beginning of the study.

b How many flies will there be after 10 days?

c Draw a graph to show the size of the population for the first 5 days of the study.

d What happens to the size of the population every day?

a $P = 60 \times 2^0$
$= 60$

At the beginning of the study $t = 0$ so substitute this into the equation.

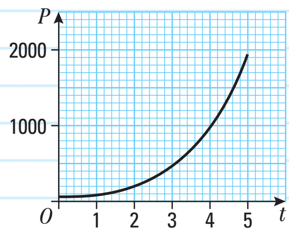
b $P = 60 \times 2^{10}$
$= 61440$

Substitute $t = 10$ into the equation.

c

t	0	1	2	3	4	5
P	60	120	240	480	960	1920

Work out the size of the population for the first 5 days. Use a table to organise your results.



Plot your results on a graph.

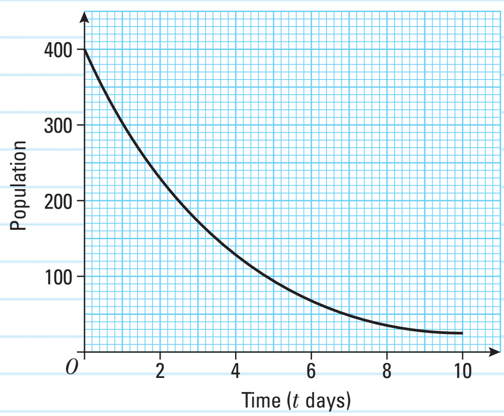
d $P = 60 \times 2^t$
The multiplier is 2.
The population doubles every day.

Compare the equation with $y = ka^x$

AO3

Example 2

A scientist recorded the size of a population of small mammals for a number of days. The mammals are suffering from a disease and so the population is decreasing exponentially. The graph shows the size of the population t days after the start of the experiment.



a How many small mammals were present at the beginning of the experiment?

b Work out the decrease in the number of small mammals during the 2nd day of the experiment.

c Work out the percentage change in the number of small mammals each day.

d Work out the multiplier.

a 400 insects

Read the value of P from the graph when $t = 0$.

b When $t = 1, P = 300$.

When $t = 2, P = 225$.

The 2nd day of the experiment is between $t = 1$ and $t = 2$. Take readings from the graph at these two values.

Decrease in number $= 300 - 225$

$= 75$ small mammals

Subtract these values to find the decrease in the number of small mammals.

c Percentage change $= \frac{75}{300} \times 100$

$= 25\%$ decrease

As this is an exponential relationship, the percentage change will be the same each day.

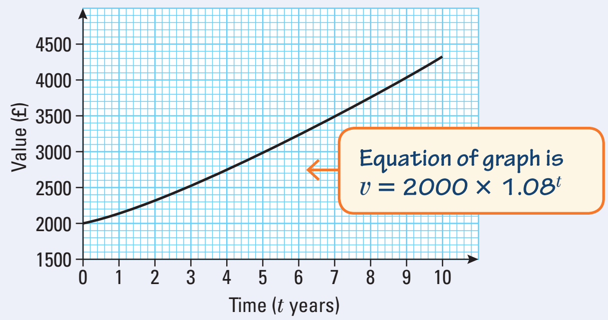
d Multiplier $= 100\% - 25\%$

$= 75\%$

$= 0.75$

Exercise 5A

1 The graph shows the value, v, of an investment t years after the original amount was invested. The value of the investment increases exponentially.



a What was the original amount invested?

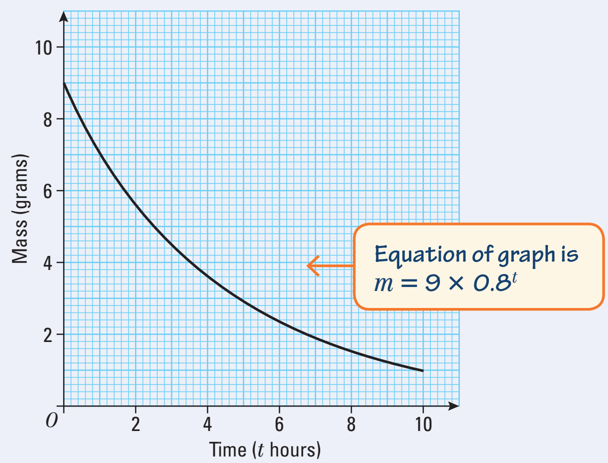
b How much did the investment grow by in the 4th year?

c **i** Work out the multiplier.

ii Work out the interest rate paid.

A03

2 The mass, m grams, of a radioactive substance decreases exponentially as shown in this graph.



a Work out the original mass of the substance.

b Work out the mass of the substance after 6 hours.

c **i** Work out the multiplier.

ii Work out the percentage rate of decrease.

A03

3 The value of a car, £C, t years after the car was bought is given by the equation:

$C = 30\,000 \times 0.7^t$

a Work out the original price paid for the car.

b Draw a graph to show the value of the car for the first five years after the car was bought.

c By what percentage does the price of the car decrease every year?

A03

A

A* A03

4 The values in the table show the size of a population that is known to be increasing exponentially.

Year	2005	2006	2007	2008	2009
Size of population	43 600	48 832	54 692	61 255	68 605

a Work out the multiplier.

b Work out the likely size of the population in 2015.

Key Points

- An alternative form of the equation $y = ka^x$ is: $A = P\left(\frac{100 + r}{100}\right)^n$
  where: P is the original population (or amount)
  n is the number of years (or hours etc)
  r is the percentage by which the population is increasing (or decreasing)
  A is the population (or amount) after n years.

A01

Example 3

An initial investment of £P grows exponentially at a rate of r% per year.
The size of the investment, A, after n years is given by:

$$A = P\left(\frac{100 + r}{100}\right)^n$$

a An investment is worth £11 576.25 after 3 years. Given that the interest rate was 5% per annum, work out the initial value of this investment.

b Harry invests £2000, after 9 years the value of his investment is £2726. Work out the annual interest rate. Give your answer correct to two significant figures.

a $11576.25 = P\left(\frac{100 + 5}{100}\right)^3$ ← Substitute the information into the equation.

$11576.25 = P \times 1.05^3$ ← Work out the sum in the brackets.

$P = \frac{11576.25}{1.05^3}$ ← Rearrange the equation.

$= £10\,000$

b $2726 = 2000\left(\frac{100 + r}{100}\right)^9$ ← Substitute the information into the equation.

$\sqrt[9]{\frac{2726}{2000}} = \frac{100 + r}{100}$ ← Rearrange the equation.

$100 \times \sqrt[9]{\frac{2726}{2000}} - 100 = r$

$r = 3.5\%$

A03

Example 4

The population of an island is increasing exponentially. In 2 years the population increased from 6900 to 8400. Assuming that the population continues to increase at the same rate, what is the population of the island likely to be 5 years after the population was 6900?

$8400 = 6900 \times a^2$ ← As the population is increasing exponentially it will satisfy the equation $y = ka^x$

$\frac{8400}{6900} = a^2$

$a = \sqrt{\frac{8400}{6900}}$ ← Solve the equation to work out the value of the multiplier a.

$= 1.1033\ldots$

Multiplier is 1.1033...

$P = 6900 \times (1.1033\ldots)^5$ ← Use the original equation with the value for the multiplier and $x = 5$ to work out the likely population.

$= 11282.99\ldots$

$= 11\,300$ correct to 3 significant figures

ResultsPlus
Examiner's Tip

The equation $A = P\left(\frac{100 + r}{100}\right)^n$ could also be used to solve this problem. However, as the percentage rate of increase was not required, it is more efficient to use $y = ka^x$

Exercise 5B

1 An initial population, P, grows exponentially at a rate of r% per year. A03

The size of the population, A, after n years is given by:

$A = P\left(\frac{100 + r}{100}\right)^n$

a Given that a population is initially 4000 and is growing exponentially at a rate of 7%, find the size of the population after 10 years.

b Another population grows exponentially from 16 500 to 19 000 in 3 years. Work out the percentage rate of growth.

2 The value of a machine in a factory decreases exponentially from its initial value, £P, at a rate of r% per year. The value of the machine, A, after n years is given by: A03

$A = P\left(\frac{100 - r}{100}\right)^n$

a Given that a machine cost £180 000 initially and its value is decreasing by 14% per annum, find the value of the machine after 10 years.

b Another machine is initially worth £78 000; its value has dropped to £49 000 after 4 years. Find its percentage rate of decrease.

3 An initial investment of £P grows exponentially at a rate of r% per year. A03

The size of the investment, A, after n years is given by:

$A = P\left(\frac{100 + r}{100}\right)^n$

a Ali wants to invest £3000 for 5 years. Bank A offers an interest rate of 3.6%. Bank B offers an interest rate of 3.75%. How much more interest will she earn in 5 years if she invests her money in Bank B?

b The value of an investment at another bank doubles in 15 years. Work out the interest rate.

A

A* A03

4 The mass, m grams, of a radioactive substance decreases exponentially. It takes 3 days for the mass of the substance to halve. If there is initially 38 grams of the substance, work out how much will remain after 5 days.

A03

5 The value of an investment is increasing exponentially. In 3 years the value of the investment increases from £15 000 to £18 119. Assuming that the value of the investment continues to increase at the same rate, what is the value likely to be after it has been invested for a total of 8 years?

A03

6 The size of a population is increasing exponentially. Given that it takes 10 years for the population to double, work out the percentage rate at which the population is increasing.

Review

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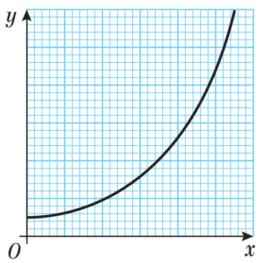
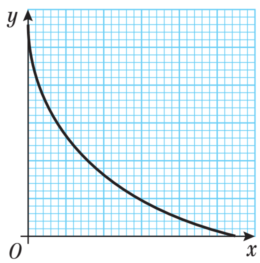


$y = ka^x$ $(a > 0)$



$y = ka^x$ $(0 < a < 1)$

- An alternative form of the equation $y = ka^x$ is: $A = P\left(\frac{100 + r}{100}\right)^n$
  where: P is the original population (or amount)
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  A is the population (or amount) after n years.

Answers

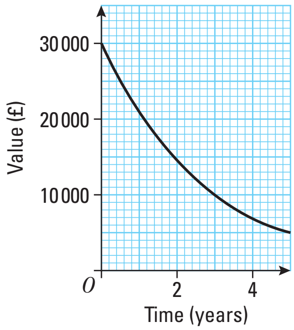
Chapter 5

A5.1 Get Ready answers

1 625
2 128
3 0.64
4 1

Exercise 5A

1 **a** £2000 **b** £201.55 **c** **i** 1.08 **ii** 8%
2 **a** 9 grams **b** 2.4 grams **c** **i** 0.8 **ii** 20%
3 **a** £30 000
b



c 30%
4 **a** 1.12 or 12% **b** 135 415

Exercise 5B

1 **a** 7869 **b** 4.8%
2 **a** £39 834 **b** 11%
3 **a** £25.99 **b** 4.73%
4 11.97 grams
5 £24 824
6 7.2%