# M3.1 The Midpoint and Intercept theorems

## Before you start

You should be able to:

- prove two triangles congruent
- prove two triangles similar
- use properties of similar triangles.

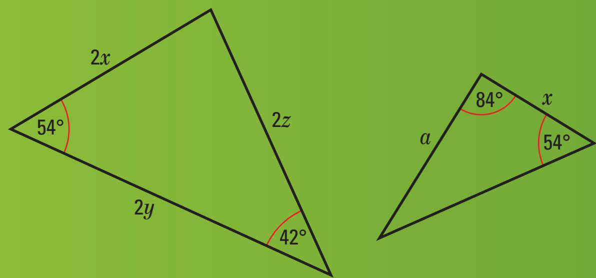
## Why do this?

Architects have to have an excellentknowledge of geometry.

## Objective

- You can understand the midpoint theorem and the intercept theorem.
- You can use the midpoint theorem and the intercept theorem.

## Get Ready




**1** **a** The two triangles are similar – explain why.

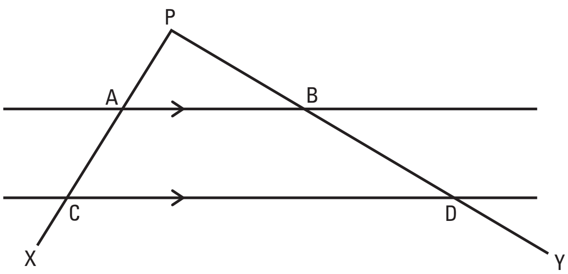
**b** Write $a$ in terms of $x$ or $y$ or $z$.

**2** Solve these equations:

**a** $\frac{t}{4} = \frac{9}{2}$ **b** $\frac{5}{s} = \frac{4}{9}$

## Key Points

- The Intercept theorem

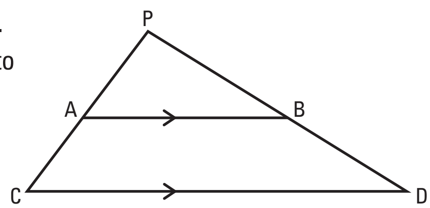



The lines PX and PY cut a pair of parallel lines at A, C and B, D respectively.

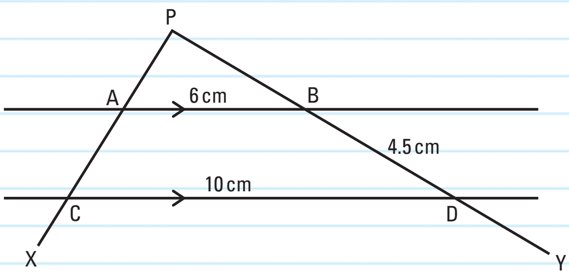
Then: $\frac{PA}{PC} = \frac{PB}{PD} = \frac{AB}{CD}$

- The Midpoint theorem

A special case of the converse of the intercept theorem is the midpoint theorem.
In the triangle PAB, A and B are the midpoints of the sides AB and CD.
Then the line AB is parallel to the side CD of the triangle and is equal to half its length.



**Example 1**




Find the length of PB.

*This problem should be solved algebraically.*

Using the intercept theorem: $\frac{PA}{PC} = \frac{PB}{PD} = \frac{AB}{CD}$

Let the length of PB $= x$ cm.

$$\frac{x}{x + 4.5} = \frac{6}{10}$$

*Use the last two fractions with PB $= x$ and so PD $= x + 4.5$*

$$10x = 6(x + 4.5)$$

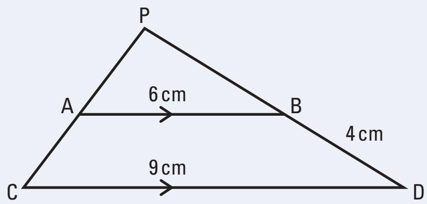
*Multiply through by $10(x + 4.5)$ and cancel.*

$$x = 6 \times 4.5 \div 4 = 6.75$$

**Exercise 3A** (All diagrams not to scale)

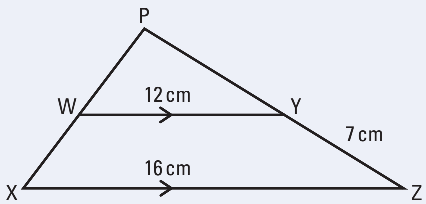
A

**1**



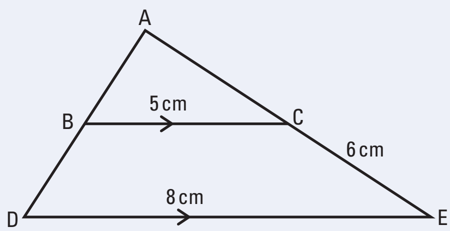

Find the length of PB.

**2**



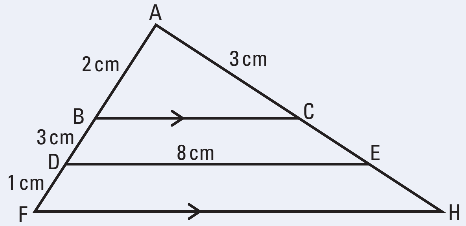

Find the length of PY.

**3**




Find the length of AE.

**4**



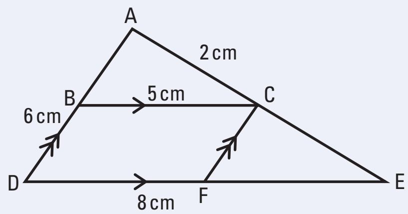

**a** Work out the length of BC.

**b** Work out the length of FH.

**c** Work out the length of CE.

**5** ABCD is a rectangle. K, L, M and N are the midpoints of AB, BC, CD and DA respectively. Use the midpoint theorem to show that the quadrilateral KLMN is a parallelogram.

**6**



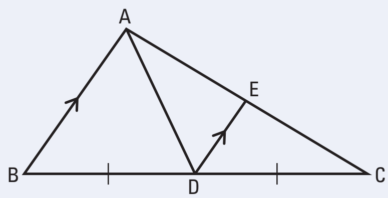

DE = 8 cm AD = 6 cm AC = 2 cm BC = 5 cm

**a** Work out the length of CE.

**b** Work out the length of CF.

**7** A kite has the property that its diagonals intersect at right angles.
ABCD is a kite. E, F, G, H are the midpoints of the sides AB, BC, CD and DA respectively.
Prove that EFGH is a rectangle.

**8**

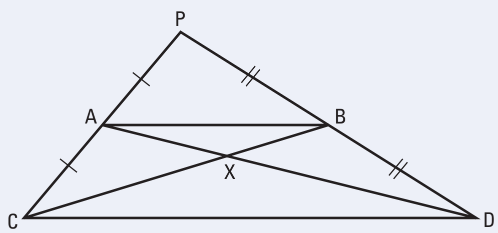



D is the midpoint of BC. DE is parallel to BA.
Show that:

**a** area triangle AED = area triangle EDC

**b** area triangle EDC : area triangle ABD = 1 : 2

**9**




A is the midpoint of PC and B is the midpoint of PD.
AD and BC intersect at X.
Prove that AX : AD = 1 : 3

**10** ABCD is a parallelogram. M is the midpoint of the diagonal BD and L is the midpoint of BM. X is the point on BC such that LX is parallel to DC. Y is the point on CD such that LY is parallel to BC.

Find area LXCY : area ABCD

# M3.2 Intersecting Chords

## Before you start

You should be able to:

- prove that two triangles are similar
- recall the angles in the same segment (or angles subtended at the circumference by equal arcs) theorem
- recall the properties of tangents to circles.

## Why do this?

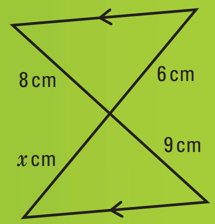
Architects study the properties of circles so that they can design structures.

## Objectives

- You can solve problems which involve intersecting chords of circles.
- You can solve problems which involve a chord and a tangent to a circle.

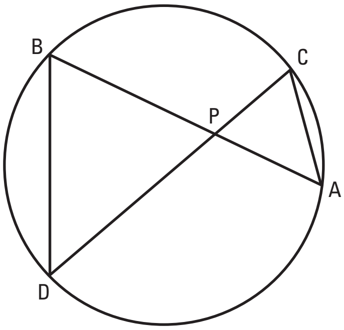
## Get Ready

1. Show that the two triangles are similar.
2. Find the value of $x$.

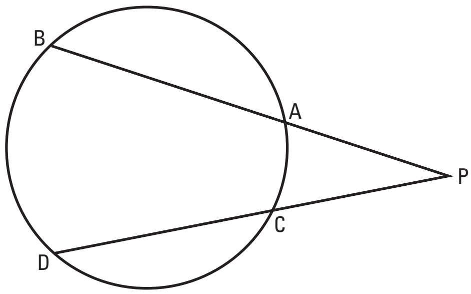



## Key Points

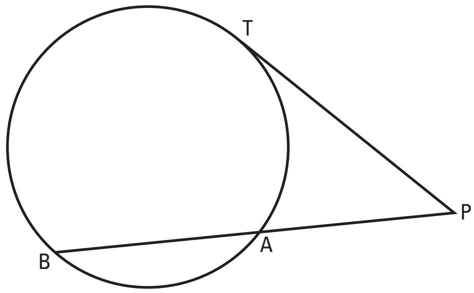
- For any chords AB and CD intersecting inside a circle: PA $\times$ PB $=$ PC $\times$ PD.




PA $\times$PB $=$ PC $\times$ PD

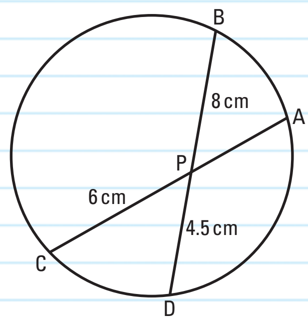



- For any chords AB and CD intersecting outside a circle: PA $\times$PB $=$ PC $\times$ PD.
  Let A, B and T be points on a circle. P is the point of intersection of the tangent at T and the chord BA.
  PA $\times$ PB $=$ PT$^2$



## Example 2

A, B, C and D are 4 points on a circle. DPA and BPC are straight lines.
Work out the length of AP.

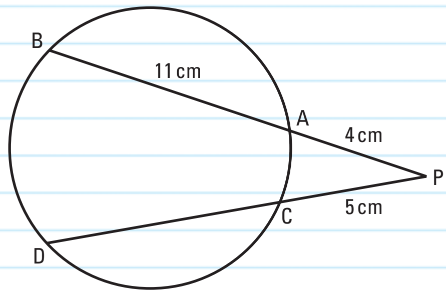



$AP \times 6 = 8 \times 4.5 \qquad AP = 36 \div 6 = 5.5$

Use $AP \times PC = BP \times PD$

## Example 3

A, B, C and D are points on a circle. PAB and PCD are straight lines.
PA = 4 cm. AB = 11 cm. PC = 5 cm.
Work out the length of DC.



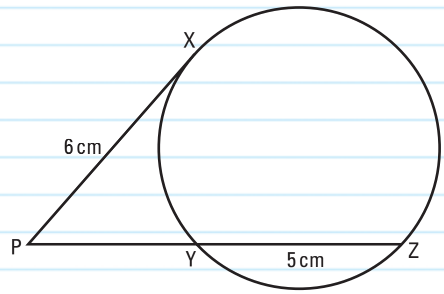

$4 \times 15 = 5 \times PD \qquad PD = 60 \div 5 = 12$

$CD = 12 - 5 = 7$ cm

Use $PA \times PB = PC \times PD$

## Example 4

X, Y and Z are 3 points on a circle.
PYZ is a straight line. PX is a tangent to the circle at X. PX = 6 cm,
YZ = 5 cm.
Find the length of PZ.




Let $PY = x$ cm ← Use $PY \times PZ = PX^2$; $PZ = PY + 5 = x + 5$

$x(x + 5) = 6^2 \quad x^2 + 5x - 36 = 0$ ← Expand the brackets and collect the terms on the left hand side.

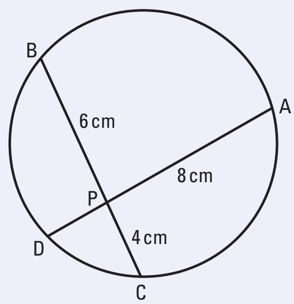
$(x + 9)(x - 4) = 0 \qquad x = -9$ or $x = 4$ ← Solve by factorising or using the quadratic formula.

$PY = 4, PZ = 9$ ← Select the positive value of $x$.

## Exercise 3B

(All diagrams not to scale)

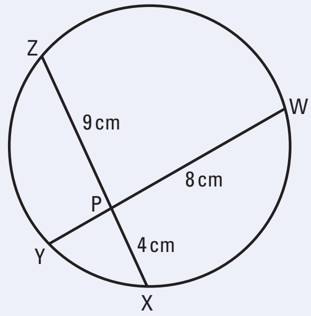
1 A, B, C and D are 4 points on a circle. DPA and BPC are straight lines.
Work out the length of PD.




B

A

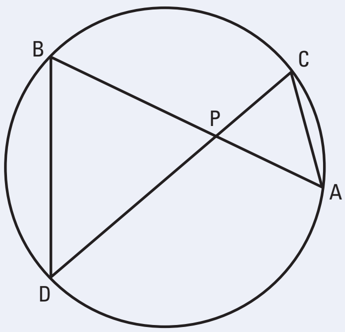
**2** W, X, Y and Z are 4 points on a circle. P is the point of intersection of the chords WY and XZ.
Work out the length of WY.



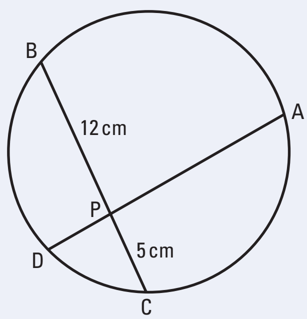

**3** W, X, Y and Z are points on a circle. The chords WY and XZ meet inside the circle at P.
WP = 8 cm, WY = 12 cm, XP = 5 cm. Work out the length of PZ.

**4** Copy and complete the table for the given diagram.

| | AP | BP | CP | DP | AB | CD |
|---|---|---|---|---|---|---|
| **a** | 9 | 8 | 6 | | | |
| **b** | 10 | 6 | 4 | | | |
| **c** | 8 | | 6 | | 20 | |
| **d** | 4 | | 5 | | 29 | |
| **e** | 5 | | 4 | 15 | | |
| **f** | | 4 | | 7 | | 19 |
| **g** | | 12 | | 9 | | 25 |
| **h** | | 9 | 6 | | | 24 |



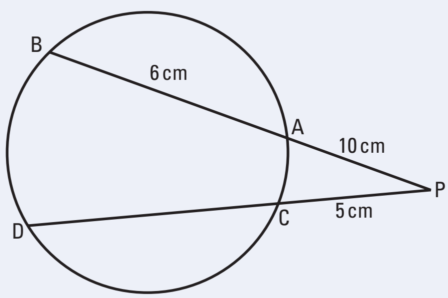

**5** A, B, C and D are 4 points on a circle. DA = 23 cm.
DP < PA. Work out the length of AP.




**6** A, B, C and D are 4 points on a circle. PAB and PCD are straight lines.

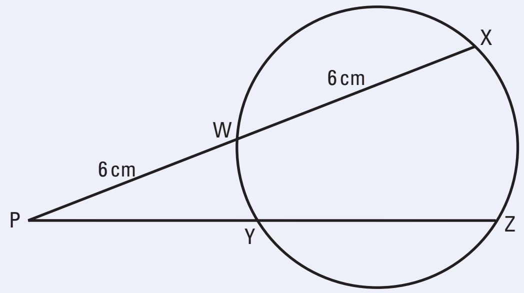
**a** Work out the length of PD.

**b** Write down the length of CD.



A

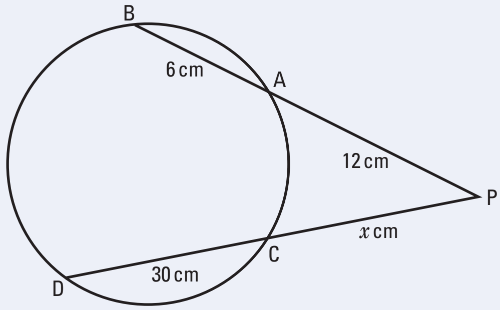
**7** W, X, Y and Z are 4 points on a circle.
PZ = 10 cm. Work out the length of YZ.




**8** A, B C and D are points on a circle.

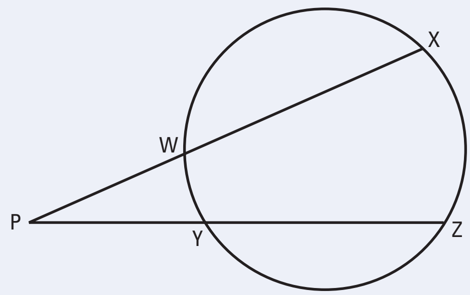
**a** Show that $x$ satisfies the equation $x^2 + 30x = 216$

**b** Find the value of $x$.

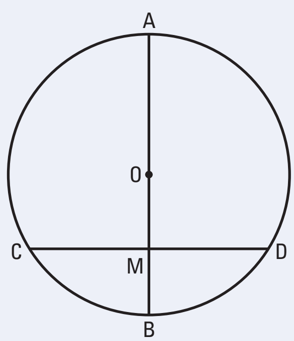



**9** W, X, Y and Z are 4 points on a circle.
The lengths of some parts of the diagram are given in the table.
Copy and complete the table.

| | PW | PX | PY | PZ | WX | YZ |
|---|---|---|---|---|---|---|
| **a** | 8 | 15 | 10 | | | |
| **b** | 7 | | 8 | | 9 | |
| **c** | 6 | | | 18 | 6 | |
| **d** | 15 | | 7 | | | 38 |
| **e** | | 18 | | 36 | | 30 |
| **f** | | 16 | 8 | | | 12 |
| **g** | 9 | | | | 6 | 22 |
| **h** | 12 | 24 | | | | 2 |

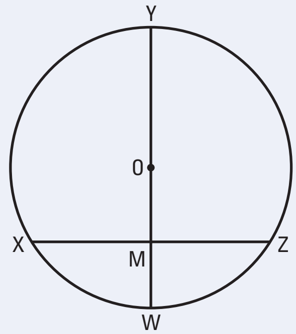



**10** O is the centre of the circle. AOB is a diameter. CM = MD.
AB = 10 cm. OM = 2 cm. Find the length of CM.

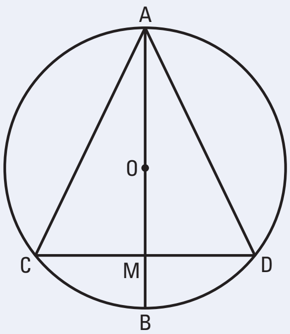


A

**11** O is the centre of the circle. YOW is a diameter. M is the midpoint of XZ.
The radius of the circle is $c$ cm. OM = $a$ cm. MZ = $b$ cm.
Show that: $c^2 = a^2 + b^2$

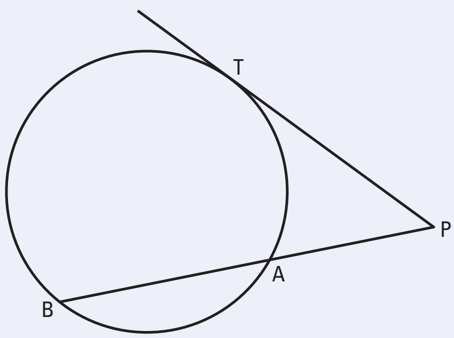



**12** O is the centre of the circle. AOB is a diameter. CM = MD.
MB = 4 cm. OM = 6 cm. Find the length of AD.

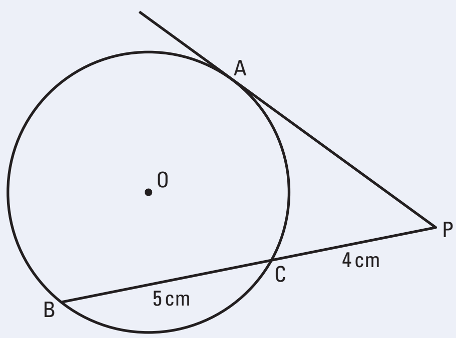



**13** In each of the cases copy and complete the table.

| | PT | PA | PB | AB |
|---|---|---|---|---|
| **a** | | 9 | 16 | |
| **b** | | 4 | | 5 |
| **c** | 10 | 5 | | |
| **d** | 8 | | 16 | |
| **e** | 14 | | 28 | |
| **f** | | | 25 | 16 |
| **g** | 18 | | 27 | |




**14** A, B and C are 3 points on a circle, centre O, radius 4.5 cm.
PCB is a straight line. PA is a tangent to the circle at A.
Work out the length of OP.



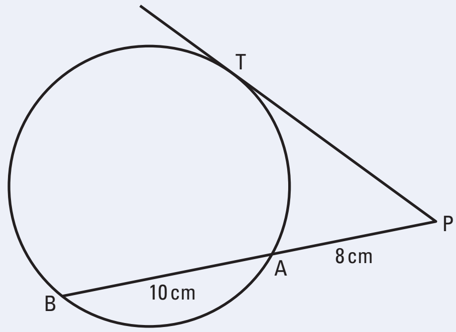
A

**15** A, B and T are 3 points on a circle.
PAB is a straight line. PT is a tangent to the circle at T.

**a** Work out the length of PT.

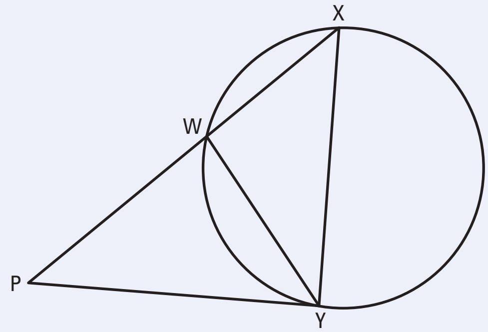
**b** Given that TB is a diameter of the circle, work out the length of TB.

**c** Work out the length of TA.

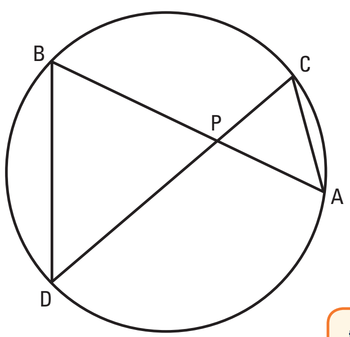


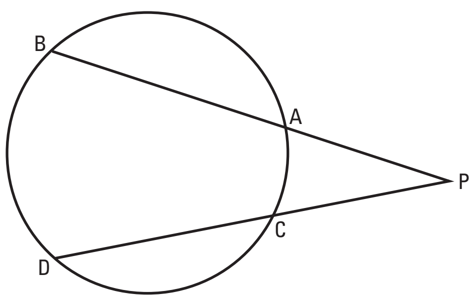

A*

**16** W, X and Y are 3 points on a circle.
PY is the tangent to the circle at Y. PWX is a straight line.
YX is a diameter.
Prove that PW × WX = $YW^2$

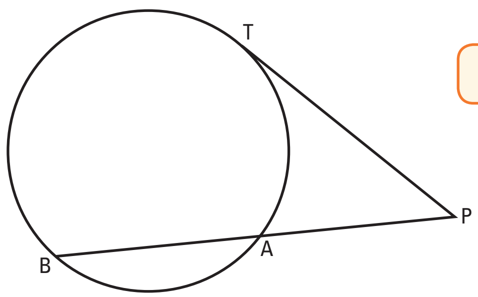



## Review




PA × PB = PC × PD




PA × PB = $PT^2$

# Answers

## Chapter 3

### M3.1 Get Ready answers

1 **a** Missing angles in the triangles are 84° and 42° respectively. The triangles are similar because they are equiangular.
**b** $a = z$
2 **a** $t = 18$ **b** $s = 11.25$

### Exercise 3A answers

1 8 cm
2 21 cm
3 16 cm
4 **a** 3.2 cm **b** 9.6 cm **c** 4.5 cm
5 Join A to C. In triangle ABC, KL is half of AC and parallel to AC. In triangle ADC, MN is half of AC and parallel to AC. Hence KLMN is a parallelogram.
6 **a** 1.2 cm **b** 2.25 cm
7 Join A to C. EF is parallel to AC, HG is parallel to AC. Therefore EF is parallel to HG. Similarly HE is parallel to DB and GF. But AC is perpendicular to DB, so HE is perpendicular to EF and EFGH is a rectangle.
8 **a** AE = EC (E is the midpoint of AC) and ED is common. Since the triangles have the same base and the same height they have the same area.
**b** Triangles ABD and ADC have equal bases and the same height (AD is common). Hence, they have the same area. Hence, Area triangle EDC : area triangle ABD = 1 : 2
9 AB is parallel to CD; angle BAD = angle ADC, angle ABC = angle BCD. Hence triangles ABX and CXD are similar. $\frac{AB}{CD} = \frac{BX}{XC} = \frac{AX}{XD}$ Now AB $= \frac{1}{2}$DC so $\frac{AX}{XD} = \frac{1}{2}$
XD = 2AX, so 3AX = AD
10 3 : 16

### M3.2 Get Ready answers

1 The angles in the two triangles are equal in pairs (alternate angles and vertically opposite angles)
2 $x = 6.75$

### Exercise 3B answers

1 PD = 4 × 6 ÷ 8 = 3 cm
2 9 × 4 = WP × YP
YP = 36 ÷ 8 = 4.5
YW = 12.5 cm
3 5 × PZ = 4 × 8 PZ = 32 ÷ 5 = 6.4 cm

4

| | AP | BP | CP | DP | AB | CD |
|---|---|---|---|---|---|---|
| **a** | 9 | 8 | 6 | 12 | 17 | 18 |
| **b** | 10 | 6 | 4 | 15 | 16 | 19 |
| **c** | 8 | 12 | 6 | 16 | 20 | 22 |
| **d** | 4 | 25 | 5 | 20 | 29 | 25 |
| **e** | 5 | 12 | 4 | 15 | 17 | 19 |
| **f** | 21 | 4 | 12 | 7 | 25 | 19 |
| **g** | 12 | 12 | 16 | 9 | 24 | 25 |
| **h** | 12 | 9 | 6 | 18 | 21 | 24 |

5 Let AP = $x$ $x(23 - x) = 5 \times 12$
$23x - x^2 = 60$
$x^2 - 23x + 60 = 0$
$x = 3$ or $x = 20$
AP = 20 cm
6 **a** PD $= \frac{10 \times 16}{5} = 32$ **b** CD = 27 cm
7 YZ $= 10 - \frac{6 \times 12}{10} = 2.8$ cm
8 $x(x + 30) = 12 \times 18$
$x^2 + 30x = 216$
$x^2 + 30x - 216 = 0$
$(x + 36)(x - 6) = 0$, $x = -36$ or 6.
Here $x = 6$

9

| | PW | PX | PY | PZ | WX | YZ |
|---|---|---|---|---|---|---|
| **a** | 8 | 15 | 10 | 12 | 7 | 2 |
| **b** | 7 | 16 | 8 | 14 | 9 | 6 |
| **c** | 6 | 12 | 4 | 18 | 6 | 14 |
| **d** | 15 | 21 | 7 | 45 | 6 | 38 |
| **e** | 12 | 18 | 6 | 36 | 6 | 30 |
| **f** | 10 | 16 | 8 | 20 | 6 | 12 |
| **g** | 9 | 15 | 5 | 27 | 6 | 22 |
| **h** | 12 | 24 | 16 | 18 | 12 | 2 |

10 $(5 - 2) \times (5 + 2) = 21$ $CM^2 = 21$ $CM = \sqrt{21}$
11 $(c - a) \times (c + a) =$ XM × MZ = b × b
So $c^2 - a^2 = b^2$ So $c^2 = a^2 + b^2$
12 AM × MB = CM × MD
$(10 + 6) \times 4 = MD^2$
MD = 8
$AD^2 = 8^2 + 16^2$
AD $= \sqrt{320}$ cm

**13**

| | PT | PA | PB | AB |
|---|---|---|---|---|
| **a** | 12 | 9 | 16 | 7 |
| **b** | 6 | 4 | 9 | 5 |
| **c** | 10 | 5 | 20 | 15 |
| **d** | 8 | 4 | 16 | 12 |
| **e** | 14 | 7 | 28 | 21 |
| **f** | 15 | 9 | 25 | 16 |
| **g** | 18 | 12 | 27 | 15 |

**14** $PA^2 = 4 \times (4 + 5) = 36$
$PA = 6$
$OP^2 = 4.5^2 + 6^2 = 56.25$
$OP = 7.5$ cm

**15** **a** $8 \times 18 = PT^2$ $PT = 12$ cm
**b** $TB^2 = 18^2 - 12^2$ $TB = \sqrt{180}$ cm
**c** $(\sqrt{180})^2 - 10^2 = 80$ $TA = \sqrt{80}$ cm

**16** $PW \times PX = PY^2$
$PW \times (PW + WX) = PY^2$
$PW^2 + PW \times WX = PY^2$
$PW \times WX = PY^2 - PW^2 = YW^2$