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# Pearson Edexcel Level 3 Advanced Subsidiary and Advanced GCE Mathematics and Further Mathematics

## **Mathematical formulae and statistical tables**

### **First certification from 2018**

Advanced Subsidiary GCE in Mathematics (8MA0)

Advanced GCE in Mathematics (9MA0)

Advanced Subsidiary GCE in Further Mathematics (8FM0)

### **First certification from 2019**

Advanced GCE in Further Mathematics (9FM0)

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ISBN 978 1 4469 4857 6

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## Introduction

The formulae in this booklet have been arranged by qualification. Students sitting AS or A Level Further Mathematics papers may be required to use the formulae that were introduced in AS or A Level Mathematics papers.

It may also be the case that students sitting Mechanics and Statistics papers will need to use formulae introduced in the appropriate Pure Mathematics papers for the qualification they are sitting.



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# 1 AS Mathematics

# 1

## Pure Mathematics

### Mensuration

Surface area of sphere =  $4\pi r^2$

Area of curved surface of cone =  $\pi r \times \text{slant height}$

### Binomial series

$$(a + b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{r}a^{n-r}b^r + \dots + b^n \quad (n \in \mathbb{N})$$

$$\text{where } \binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

### Logarithms and exponentials

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$e^{x \ln a} = a^x$$

### Differentiation

First Principles

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

## Statistics

### Probability

$$P(A') = 1 - P(A)$$

### Standard deviation

Standard deviation =  $\sqrt{\text{Variance}}$

Interquartile range = IQR =  $Q_3 - Q_1$

For a set of  $n$  values  $x_1, x_2, \dots, x_i, \dots, x_n$

$$S_{xx} = \sum (x_i - \bar{x})^2 = \sum x_i^2 - \frac{(\sum x_i)^2}{n}$$

$$\text{Standard deviation} = \sqrt{\frac{S_{xx}}{n}} \text{ or } \sqrt{\frac{\sum x^2}{n} - \bar{x}^2}$$

### Statistical tables

The following statistical tables are required for A Level Mathematics:

*Binomial Cumulative Distribution Function* (see page 29)

*Random Numbers* (see page 38)

## Mechanics

### Kinematics

For motion in a straight line with constant acceleration:

$$v = u + at$$

$$s = ut + \frac{1}{2}at^2$$

$$s = vt - \frac{1}{2}at^2$$

$$v^2 = u^2 + 2as$$

$$s = \frac{1}{2}(u + v)t$$

## 2 A Level Mathematics

### Pure Mathematics

#### Mensuration

Surface area of sphere =  $4\pi r^2$

Area of curved surface of cone =  $\pi r \times \text{slant height}$

#### Arithmetic series

$$S_n = \frac{1}{2}n(a + l) = \frac{1}{2}n[2a + (n - 1)d]$$

#### Binomial series

$$(a + b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{r}a^{n-r}b^r + \dots + b^n \quad (n \in \mathbb{N})$$

$$\text{where } \binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1 \times 2}x^2 + \dots + \frac{n(n-1)\dots(n-r+1)}{1 \times 2 \times \dots \times r}x^r + \dots \quad (|x| < 1, n \in \mathbb{R})$$

#### Logarithms and exponentials

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$e^{x \ln a} = a^x$$

#### Geometric series

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$S_\infty = \frac{a}{1 - r} \text{ for } |r| < 1$$

## Trigonometric identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \quad (A \pm B \neq (k + \frac{1}{2})\pi)$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

Small angle approximations

$$\sin \theta \approx \theta$$

$$\cos \theta \approx 1 - \frac{\theta^2}{2}$$

$$\tan \theta \approx \theta$$

where  $\theta$  is measured in radians

## Differentiation

First Principles

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

| $f(x)$ | $f'(x)$ |
|--------|---------|
|--------|---------|

|           |               |
|-----------|---------------|
| $\tan kx$ | $k \sec^2 kx$ |
|-----------|---------------|

|           |                     |
|-----------|---------------------|
| $\sec kx$ | $k \sec kx \tan kx$ |
|-----------|---------------------|

|           |                                |
|-----------|--------------------------------|
| $\cot kx$ | $-k \operatorname{cosec}^2 kx$ |
|-----------|--------------------------------|

|                           |                                      |
|---------------------------|--------------------------------------|
| $\operatorname{cosec} kx$ | $-k \operatorname{cosec} kx \cot kx$ |
|---------------------------|--------------------------------------|

|                     |                                          |
|---------------------|------------------------------------------|
| $\frac{f(x)}{g(x)}$ | $\frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$ |
|---------------------|------------------------------------------|

## Integration (+ constant)

$$f(x) \quad \int f(x) \, dx$$

$$\sec^2 kx \quad \frac{1}{k} \tan kx$$

$$\tan kx \quad \frac{1}{k} \ln |\sec kx|$$

$$\cot kx \quad \frac{1}{k} \ln |\sin kx|$$

$$\operatorname{cosec} kx \quad -\frac{1}{k} \ln |\operatorname{cosec} kx + \cot kx|, \quad \frac{1}{k} \ln |\tan(\frac{1}{2} kx)|$$

$$\sec kx \quad \frac{1}{k} \ln |\sec kx + \tan kx|, \quad \frac{1}{k} \ln |\tan(\frac{1}{2} kx + \frac{1}{4} \pi)|$$

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

## Numerical Methods

The trapezium rule:  $\int_a^b y \, dx \approx \frac{1}{2} h \{(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})\}$ , where  $h = \frac{b-a}{n}$

The Newton-Raphson iteration for solving  $f(x) = 0$ :  $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

## Statistics

### Probability

$$P(A') = 1 - P(A)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cap B) = P(A)P(B | A)$$

$$P(A | B) = \frac{P(B | A)P(A)}{P(B | A)P(A) + P(B | A')P(A')}$$

For independent events  $A$  and  $B$ ,

$$P(B | A) = P(B)$$

$$P(A | B) = P(A)$$

$$P(A \cap B) = P(A) P(B)$$

## Standard deviation

Standard deviation =  $\sqrt{\text{Variance}}$

Interquartile range = IQR =  $Q_3 - Q_1$

For a set of  $n$  values  $x_1, x_2, \dots, x_i, \dots, x_n$

$$S_{xx} = \sum (x_i - \bar{x})^2 = \sum x_i^2 - \frac{(\sum x_i)^2}{n}$$

$$\text{Standard deviation} = \sqrt{\frac{S_{xx}}{n}} \text{ or } \sqrt{\frac{\sum x^2}{n} - \bar{x}^2}$$

## Discrete distributions

| Distribution of $X$ | $P(X = x)$                     | Mean | Variance  |
|---------------------|--------------------------------|------|-----------|
| Binomial $B(n, p)$  | $\binom{n}{x} p^x (1-p)^{n-x}$ | $np$ | $np(1-p)$ |

## Sampling distributions

For a random sample of  $n$  observations from  $N(\mu, \sigma^2)$

$$\frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \sim N(0, 1)$$

## Statistical tables

The following statistical tables are required for A Level Mathematics:

*Binomial Cumulative Distribution Function* (see page 29)

*Percentage Points of The Normal Distribution* (see page 34)

*Critical Values for Correlation Coefficients: Product Moment Coefficient* (see page 37)

*Random Numbers* (see page 38)

## Mechanics

### Kinematics

For motion in a straight line with constant acceleration:

$$v = u + at$$

$$s = ut + \frac{1}{2}at^2$$

$$s = vt - \frac{1}{2}at^2$$

$$v^2 = u^2 + 2as$$

$$s = \frac{1}{2}(u + v)t$$

## 3 AS Further Mathematics

Students sitting an AS Level Further Mathematics paper may also require those formulae listed for A Level Mathematics in Section 2.

### Pure Mathematics

#### Summations

$$\sum_{r=1}^n r^2 = \frac{1}{6} n(n+1)(2n+1)$$

$$\sum_{r=1}^n r^3 = \frac{1}{4} n^2(n+1)^2$$

#### Matrix transformations

Anticlockwise rotation through  $\theta$  about  $O$ :  $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$

Reflection in the line  $y = (\tan \theta)x$ :  $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

#### Area of a sector

$$A = \frac{1}{2} \int r^2 d\theta \quad (\text{polar coordinates})$$

## Maclaurin's and Taylor's Series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^r}{r!} f^{(r)}(0) + \dots$$

$$e^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^r}{r!} + \dots \quad \text{for all } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{r+1} \frac{x^r}{r} + \dots \quad (-1 < x \leq 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \dots \quad \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^r \frac{x^{2r}}{(2r)!} + \dots \quad \text{for all } x$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^r \frac{x^{2r+1}}{2r+1} + \dots \quad (-1 \leq x \leq 1)$$

## Vectors

$$\text{Vector product: } \mathbf{a} \times \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \sin \theta \hat{\mathbf{n}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

If  $A$  is the point with position vector  $\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$  and the direction vector  $\mathbf{b}$  is given by  $\mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k}$ , then the straight line through  $A$  with direction vector  $\mathbf{b}$  has cartesian equation

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3} (= \lambda)$$

The plane through  $A$  with normal vector  $\mathbf{n} = n_1 \mathbf{i} + n_2 \mathbf{j} + n_3 \mathbf{k}$  has cartesian equation

$$n_1 x + n_2 y + n_3 z + d = 0 \quad \text{where } d = -\mathbf{a} \cdot \mathbf{n}$$

The plane through non-collinear points  $A$ ,  $B$  and  $C$  has vector equation

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a}) = (1 - \lambda - \mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$

The plane through the point with position vector  $\mathbf{a}$  and parallel to  $\mathbf{b}$  and  $\mathbf{c}$  has equation

$$\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$$

The perpendicular distance of  $(\alpha, \beta, \gamma)$  from  $n_1 x + n_2 y + n_3 z + d = 0$  is  $\frac{|n_1 \alpha + n_2 \beta + n_3 \gamma + d|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$ .

## Hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\operatorname{arcosh} x = \ln\{x + \sqrt{x^2 - 1}\} \quad (x \geq 1)$$

$$\operatorname{arsinh} x = \ln\{x + \sqrt{x^2 + 1}\}$$

$$\operatorname{artanh} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \quad (|x| < 1)$$

## Differentiation

| $f(x)$                    | $f'(x)$                   |
|---------------------------|---------------------------|
| $\arcsin x$               | $\frac{1}{\sqrt{1-x^2}}$  |
| $\arccos x$               | $-\frac{1}{\sqrt{1-x^2}}$ |
| $\arctan x$               | $\frac{1}{1+x^2}$         |
| $\sinh x$                 | $\cosh x$                 |
| $\cosh x$                 | $\sinh x$                 |
| $\tanh x$                 | $\operatorname{sech}^2 x$ |
| $\operatorname{arsinh} x$ | $\frac{1}{\sqrt{1+x^2}}$  |
| $\operatorname{arcosh} x$ | $\frac{1}{\sqrt{x^2-1}}$  |
| $\operatorname{artanh} x$ | $\frac{1}{1-x^2}$         |

**Integration (+ constant;  $a > 0$  where relevant)**

|                              |                                                                                                                                  |
|------------------------------|----------------------------------------------------------------------------------------------------------------------------------|
| $f(x)$                       | $\int f(x) \, dx$                                                                                                                |
| $\sinh x$                    | $\cosh x$                                                                                                                        |
| $\cosh x$                    | $\sinh x$                                                                                                                        |
| $\tanh x$                    | $\ln \cosh x$                                                                                                                    |
| $\frac{1}{\sqrt{a^2 - x^2}}$ | $\arcsin \left( \frac{x}{a} \right) \quad ( x  < a)$                                                                             |
| $\frac{1}{a^2 + x^2}$        | $\frac{1}{a} \arctan \left( \frac{x}{a} \right)$                                                                                 |
| $\frac{1}{\sqrt{x^2 - a^2}}$ | $\operatorname{arcosh} \left( \frac{x}{a} \right), \quad \ln \{x + \sqrt{x^2 - a^2}\} \quad (x > a)$                             |
| $\frac{1}{\sqrt{a^2 + x^2}}$ | $\operatorname{arsinh} \left( \frac{x}{a} \right), \quad \ln \{x + \sqrt{x^2 + a^2}\}$                                           |
| $\frac{1}{a^2 - x^2}$        | $\frac{1}{2a} \ln \left  \frac{a+x}{a-x} \right  = \frac{1}{a} \operatorname{artanh} \left( \frac{x}{a} \right) \quad ( x  < a)$ |
| $\frac{1}{x^2 - a^2}$        | $\frac{1}{2a} \ln \left  \frac{x-a}{x+a} \right $                                                                                |

## Statistics

### Discrete distributions

For a discrete random variable  $X$  taking values  $x_i$  with probabilities  $P(X = x_i)$

Expectation (mean):  $E(X) = \mu = \sum x_i P(X = x_i)$

Variance:  $\text{Var}(X) = \sigma^2 = \sum (x_i - \mu)^2 P(X = x_i) = \sum x_i^2 P(X = x_i) - \mu^2$

### Discrete distributions

Standard discrete distributions:

| Distribution of $X$   | $P(X = x)$                          | Mean      | Variance  |
|-----------------------|-------------------------------------|-----------|-----------|
| Binomial $B(n, p)$    | $\binom{n}{x} p^x (1-p)^{n-x}$      | $np$      | $np(1-p)$ |
| Poisson $Po(\lambda)$ | $e^{-\lambda} \frac{\lambda^x}{x!}$ | $\lambda$ | $\lambda$ |

### Continuous distributions

For a continuous random variable  $X$  having probability density function  $f$

Expectation (mean):  $E(X) = \mu = \int x f(x) dx$

Variance:  $\text{Var}(X) = \sigma^2 = \int (x - \mu)^2 f(x) dx = \int x^2 f(x) dx - \mu^2$

For a function  $g(X)$ :  $E(g(X)) = \int g(x) f(x) dx$

Cumulative distribution function:  $F(x_0) = P(X \leq x_0) = \int_{-\infty}^{x_0} f(t) dt$

Standard continuous distribution:

| Distribution of $X$               | P.D.F.                                                                            | Mean               | Variance              |
|-----------------------------------|-----------------------------------------------------------------------------------|--------------------|-----------------------|
| Normal $N(\mu, \sigma^2)$         | $\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ | $\mu$              | $\sigma^2$            |
| Uniform (Rectangular) on $[a, b]$ | $\frac{1}{b-a}$                                                                   | $\frac{1}{2}(a+b)$ | $\frac{1}{12}(b-a)^2$ |

## Correlation and regression

For a set of  $n$  pairs of values  $(x_i, y_i)$

$$S_{xx} = \Sigma(x_i - \bar{x})^2 = \Sigma x_i^2 - \frac{(\Sigma x_i)^2}{n}$$

$$S_{yy} = \Sigma(y_i - \bar{y})^2 = \Sigma y_i^2 - \frac{(\Sigma y_i)^2}{n}$$

$$S_{xy} = \Sigma(x_i - \bar{x})(y_i - \bar{y}) = \Sigma x_i y_i - \frac{(\Sigma x_i)(\Sigma y_i)}{n}$$

The product moment correlation coefficient is

$$r = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}} = \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\left\{\Sigma(x_i - \bar{x})^2\right\}\left\{\Sigma(y_i - \bar{y})^2\right\}}} = \frac{\Sigma x_i y_i - \frac{(\Sigma x_i)(\Sigma y_i)}{n}}{\sqrt{\left(\Sigma x_i^2 - \frac{(\Sigma x_i)^2}{n}\right)\left(\Sigma y_i^2 - \frac{(\Sigma y_i)^2}{n}\right)}}$$

The regression coefficient of  $y$  on  $x$  is  $b = \frac{S_{xy}}{S_{xx}} = \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\Sigma(x_i - \bar{x})^2}$

Least squares regression line of  $y$  on  $x$  is  $y = a + bx$  where  $a = \bar{y} - b\bar{x}$

$$\text{Residual Sum of Squares (RSS)} = S_{yy} - \frac{(S_{xy})^2}{S_{xx}} = S_{yy} (1 - r^2)$$

Spearman's rank correlation coefficient is  $r_s = 1 - \frac{6\Sigma d^2}{n(n^2 - 1)}$

## Non-parametric tests

Goodness-of-fit test and contingency tables:  $\sum \frac{(O_i - E_i)^2}{E_i} \sim \chi_v^2$

## Statistical tables

The following statistical tables are required for AS Level Further Mathematics:

*Binomial Cumulative Distribution Function* (see page 29)

*Poisson Cumulative Distribution Function* (see page 35)

*Percentage Points of the  $\chi^2$  Distribution* (see page 36)

*Critical Values for Correlation Coefficients: Product Moment Coefficient and Spearman's Coefficient* (see page 37)

*Random Numbers* (see page 38)

## Mechanics

### Centres of mass

For uniform bodies:

Triangular lamina:  $\frac{2}{3}$  along median from vertex

Circular arc, radius  $r$ , angle at centre  $2\alpha$  :  $\frac{r \sin \alpha}{\alpha}$  from centre

Sector of circle, radius  $r$ , angle at centre  $2\alpha$  :  $\frac{2r \sin \alpha}{3\alpha}$  from centre



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## 4 A Level Further Mathematics

Students sitting an A Level Further Mathematics paper may also require those formulae listed for A Level Mathematics in Section 2.

### Pure Mathematics

#### Summations

$$\sum_{r=1}^n r^2 = \frac{1}{6} n(n+1)(2n+1)$$

$$\sum_{r=1}^n r^3 = \frac{1}{4} n^2(n+1)^2$$

#### Matrix transformations

Anticlockwise rotation through  $\theta$  about  $O$ :  $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$

Reflection in the line  $y = (\tan \theta)x$ :  $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

#### Area of a sector

$$A = \frac{1}{2} \int r^2 d\theta \quad (\text{polar coordinates})$$

#### Complex numbers

$$\{r(\cos \theta + i \sin \theta)\}^n = r^n (\cos n\theta + i \sin n\theta)$$

The roots of  $z^n = 1$  are given by  $z = e^{\frac{2\pi ki}{n}}$ , for  $k = 0, 1, 2, \dots, n-1$

## Maclaurin's and Taylor's Series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^r}{r!} f^{(r)}(0) + \dots$$

$$e^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^r}{r!} + \dots \quad \text{for all } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{r+1} \frac{x^r}{r} + \dots \quad (-1 < x \leq 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \dots \quad \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^r \frac{x^{2r}}{(2r)!} + \dots \quad \text{for all } x$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^r \frac{x^{2r+1}}{2r+1} + \dots \quad (-1 \leq x \leq 1)$$

## Vectors

$$\text{Vector product: } \mathbf{a} \times \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \sin \theta \hat{\mathbf{n}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

If  $A$  is the point with position vector  $\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$  and the direction vector  $\mathbf{b}$  is given by  $\mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k}$ , then the straight line through  $A$  with direction vector  $\mathbf{b}$  has cartesian equation

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3} (= \lambda)$$

The plane through  $A$  with normal vector  $\mathbf{n} = n_1 \mathbf{i} + n_2 \mathbf{j} + n_3 \mathbf{k}$  has cartesian equation

$$n_1 x + n_2 y + n_3 z + d = 0 \quad \text{where } d = -\mathbf{a} \cdot \mathbf{n}$$

The plane through non-collinear points  $A$ ,  $B$  and  $C$  has vector equation

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a}) = (1 - \lambda - \mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$

The plane through the point with position vector  $\mathbf{a}$  and parallel to  $\mathbf{b}$  and  $\mathbf{c}$  has equation

$$\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$$

The perpendicular distance of  $(\alpha, \beta, \gamma)$  from  $n_1 x + n_2 y + n_3 z + d = 0$  is  $\frac{|n_1 \alpha + n_2 \beta + n_3 \gamma + d|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$ .

## Hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\operatorname{arcosh} x = \ln\{x + \sqrt{x^2 - 1}\} \quad (x \geq 1)$$

$$\operatorname{arsinh} x = \ln\{x + \sqrt{x^2 + 1}\}$$

$$\operatorname{artanh} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \quad (|x| < 1)$$

## Conics

|                 | Ellipse                                 | Parabola      | Hyperbola                                                                  | Rectangular Hyperbola          |
|-----------------|-----------------------------------------|---------------|----------------------------------------------------------------------------|--------------------------------|
| Standard Form   | $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ | $y^2 = 4ax$   | $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$                                    | $xy = c^2$                     |
| Parametric Form | $(a \cos \theta, b \sin \theta)$        | $(at^2, 2at)$ | $(a \sec \theta, b \tan \theta)$<br>$(\pm a \cosh \theta, b \sinh \theta)$ | $\left(ct, \frac{c}{t}\right)$ |
| Eccentricity    | $e < 1$<br>$b^2 = a^2(1 - e^2)$         | $e = 1$       | $e > 1$<br>$b^2 = a^2(e^2 - 1)$                                            | $e = \sqrt{2}$                 |
| Foci            | $(\pm ae, 0)$                           | $(a, 0)$      | $(\pm ae, 0)$                                                              | $(\pm\sqrt{2}c, \pm\sqrt{2}c)$ |
| Directrices     | $x = \pm \frac{a}{e}$                   | $x = -a$      | $x = \pm \frac{a}{e}$                                                      | $x + y = \pm\sqrt{2}c$         |
| Asymptotes      | none                                    | none          | $\frac{x}{a} = \pm \frac{y}{b}$                                            | $x = 0, y = 0$                 |



## Differentiation

| $f(x)$                    | $f'(x)$                   |
|---------------------------|---------------------------|
| $\arcsin x$               | $\frac{1}{\sqrt{1-x^2}}$  |
| $\arccos x$               | $-\frac{1}{\sqrt{1-x^2}}$ |
| $\arctan x$               | $\frac{1}{1+x^2}$         |
| $\sinh x$                 | $\cosh x$                 |
| $\cosh x$                 | $\sinh x$                 |
| $\tanh x$                 | $\operatorname{sech}^2 x$ |
| $\operatorname{arsinh} x$ | $\frac{1}{\sqrt{1+x^2}}$  |
| $\operatorname{arcosh} x$ | $\frac{1}{\sqrt{x^2-1}}$  |
| $\operatorname{artanh} x$ | $\frac{1}{1-x^2}$         |



### Integration (+ constant; $a > 0$ where relevant)

|                              |                                                                                                                                  |
|------------------------------|----------------------------------------------------------------------------------------------------------------------------------|
| $f(x)$                       | $\int f(x) \, dx$                                                                                                                |
| $\sinh x$                    | $\cosh x$                                                                                                                        |
| $\cosh x$                    | $\sinh x$                                                                                                                        |
| $\tanh x$                    | $\ln \cosh x$                                                                                                                    |
| $\frac{1}{\sqrt{a^2 - x^2}}$ | $\arcsin \left( \frac{x}{a} \right) \quad ( x  < a)$                                                                             |
| $\frac{1}{a^2 + x^2}$        | $\frac{1}{a} \arctan \left( \frac{x}{a} \right)$                                                                                 |
| $\frac{1}{\sqrt{x^2 - a^2}}$ | $\operatorname{arcosh} \left( \frac{x}{a} \right), \ln \{x + \sqrt{x^2 - a^2}\} \quad (x > a)$                                   |
| $\frac{1}{\sqrt{a^2 + x^2}}$ | $\operatorname{arsinh} \left( \frac{x}{a} \right), \ln \{x + \sqrt{x^2 + a^2}\}$                                                 |
| $\frac{1}{a^2 - x^2}$        | $\frac{1}{2a} \ln \left  \frac{a+x}{a-x} \right  = \frac{1}{a} \operatorname{artanh} \left( \frac{x}{a} \right) \quad ( x  < a)$ |
| $\frac{1}{x^2 - a^2}$        | $\frac{1}{2a} \ln \left  \frac{x-a}{x+a} \right $                                                                                |

### Arc length

$$s = \int \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \quad (\text{cartesian coordinates})$$
$$s = \int \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt \quad (\text{parametric form})$$
$$s = \int \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} \, d\theta \quad (\text{polar form})$$

### Surface area of revolution

$$s_x = 2\pi \int y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad (\text{cartesian coordinates})$$

$$s_x = 2\pi \int y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \quad (\text{parametric form})$$

$$s_x = 2\pi \int r \sin \theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \quad (\text{polar form})$$

## Statistics

### Discrete distributions

For a discrete random variable  $X$  taking values  $x_i$  with probabilities  $P(X = x_i)$

Expectation (mean):  $E(X) = \mu = \sum x_i P(X = x_i)$

Variance:  $\text{Var}(X) = \sigma^2 = \sum (x_i - \mu)^2 P(X = x_i) = \sum x_i^2 P(X = x_i) - \mu^2$

For a function  $g(X)$ :  $E(g(X)) = \sum g(x_i) P(X = x_i)$

The probability generating function of  $X$  is  $G_X(t) = E(t^X)$  and

$E(X) = G'_X(1)$  and  $\text{Var}(X) = G''_X(1) + G'_X(1) - [G'_X(1)]^2$

For  $Z = X + Y$ , where  $X$  and  $Y$  are independent:  $G_Z(t) = G_X(t) \times G_Y(t)$

### Discrete distributions

Standard discrete distributions:

| Distribution of $X$                           | $P(X = x)$                          | Mean          | Variance             | P.G.F.                                 |
|-----------------------------------------------|-------------------------------------|---------------|----------------------|----------------------------------------|
| Binomial $B(n, p)$                            | $\binom{n}{x} p^x (1-p)^{n-x}$      | $np$          | $np(1-p)$            | $(1-p+pt)^n$                           |
| Poisson $\text{Po}(\lambda)$                  | $e^{-\lambda} \frac{\lambda^x}{x!}$ | $\lambda$     | $\lambda$            | $e^{\lambda(t-1)}$                     |
| Geometric $\text{Geo}(p)$<br>on $1, 2, \dots$ | $p(1-p)^{x-1}$                      | $\frac{1}{p}$ | $\frac{1-p}{p^2}$    | $\frac{pt}{1-(1-p)t}$                  |
| Negative binomial<br>on $r, r+1, \dots$       | $\binom{x-1}{r-1} p^r (1-p)^{x-r}$  | $\frac{r}{p}$ | $\frac{r(1-p)}{p^2}$ | $\left( \frac{pt}{1-(1-p)t} \right)^r$ |

### Continuous distributions

For a continuous random variable  $X$  having probability density function  $f$

Expectation (mean):  $E(X) = \mu = \int x f(x) dx$

Variance:  $\text{Var}(X) = \sigma^2 = \int (x - \mu)^2 f(x) dx = \int x^2 f(x) dx - \mu^2$

For a function  $g(X)$ :  $E(g(X)) = \int g(x) f(x) dx$

Cumulative distribution function:  $F(x_0) = P(X \leq x_0) = \int_{-\infty}^{x_0} f(t) dt$

Standard continuous distribution:

| Distribution of $X$               | P.D.F.                                                                            | Mean               | Variance              |
|-----------------------------------|-----------------------------------------------------------------------------------|--------------------|-----------------------|
| Normal $N(\mu, \sigma^2)$         | $\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ | $\mu$              | $\sigma^2$            |
| Uniform (Rectangular) on $[a, b]$ | $\frac{1}{b-a}$                                                                   | $\frac{1}{2}(a+b)$ | $\frac{1}{12}(b-a)^2$ |

### Correlation and regression

For a set of  $n$  pairs of values  $(x_i, y_i)$

$$S_{xx} = \Sigma(x_i - \bar{x})^2 = \Sigma x_i^2 - \frac{(\Sigma x_i)^2}{n}$$

$$S_{yy} = \Sigma(y_i - \bar{y})^2 = \Sigma y_i^2 - \frac{(\Sigma y_i)^2}{n}$$

$$S_{xy} = \Sigma(x_i - \bar{x})(y_i - \bar{y}) = \Sigma x_i y_i - \frac{(\Sigma x_i)(\Sigma y_i)}{n}$$

The product moment correlation coefficient is

$$r = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}} = \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\left\{\Sigma(x_i - \bar{x})^2\right\}\left\{\Sigma(y_i - \bar{y})^2\right\}}} = \frac{\Sigma x_i y_i - \frac{(\Sigma x_i)(\Sigma y_i)}{n}}{\sqrt{\left(\Sigma x_i^2 - \frac{(\Sigma x_i)^2}{n}\right)\left(\Sigma y_i^2 - \frac{(\Sigma y_i)^2}{n}\right)}}$$

$$\text{The regression coefficient of } y \text{ on } x \text{ is } b = \frac{S_{xy}}{S_{xx}} = \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\Sigma(x_i - \bar{x})^2}$$

Least squares regression line of  $y$  on  $x$  is  $y = a + bx$  where  $a = \bar{y} - b\bar{x}$

$$\text{Residual Sum of Squares (RSS)} = S_{yy} - \frac{(S_{xy})^2}{S_{xx}} = S_{yy} (1 - r^2)$$

$$\text{Spearman's rank correlation coefficient is } r_s = 1 - \frac{6\Sigma d^2}{n(n^2 - 1)}$$

### Expectation algebra

For independent random variables  $X$  and  $Y$

$$E(XY) = E(X)E(Y), \quad \text{Var}(aX \pm bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y)$$

## Sampling distributions

### (i) Tests for mean when $\sigma$ is known

For a random sample  $X_1, X_2, \dots, X_n$  of  $n$  independent observations from a distribution having mean  $\mu$  and variance  $\sigma^2$ :

$\bar{X}$  is an unbiased estimator of  $\mu$ , with  $\text{Var}(\bar{X}) = \frac{\sigma^2}{n}$

$S^2$  is an unbiased estimator of  $\sigma^2$ , where  $S^2 = \frac{\sum (X_i - \bar{X})^2}{n - 1}$

For a random sample of  $n$  observations from  $N(\mu, \sigma^2)$ ,  $\frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \sim N(0, 1)$

For a random sample of  $n_x$  observations from  $N(\mu_x, \sigma_x^2)$  and, independently, a random sample of  $n_y$  observations from  $N(\mu_y, \sigma_y^2)$ ,  $\frac{(\bar{X} - \bar{Y}) - (\mu_x - \mu_y)}{\sqrt{\frac{\sigma_x^2}{n_x} + \frac{\sigma_y^2}{n_y}}} \sim N(0, 1)$

### (ii) Tests for variance and mean when $\sigma$ is not known

For a random sample of  $n$  observations from  $N(\mu, \sigma^2)$

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$$

$$\frac{\bar{X} - \mu}{S / \sqrt{n}} \sim t_{n-1} \quad (\text{also valid in matched-pairs situations})$$

For a random sample of  $n_x$  observations from  $N(\mu_x, \sigma_x^2)$  and, independently, a random sample of  $n_y$  observations from  $N(\mu_y, \sigma_y^2)$

$$\frac{S_x^2 / \sigma_x^2}{S_y^2 / \sigma_y^2} \sim F_{n_x-1, n_y-1}$$

If  $\sigma_x^2 = \sigma_y^2 = \sigma^2$  (unknown) then

$$\frac{(\bar{X} - \bar{Y}) - (\mu_x - \mu_y)}{\sqrt{S_p^2 \left( \frac{1}{n_x} + \frac{1}{n_y} \right)}} \sim t_{n_x+n_y-2} \quad \text{where} \quad S_p^2 = \frac{(n_x-1)S_x^2 + (n_y-1)S_y^2}{n_x + n_y - 2}$$

## Non-parametric tests

Goodness-of-fit test and contingency tables:  $\sum \frac{(O_i - E_i)^2}{E_i} \sim \chi^2_v$

## Statistical tables

The following statistical tables are required for A Level Further Mathematics:

*Binomial Cumulative Distribution Function* (see page 29)

*Poisson Cumulative Distribution Function* (see page 35)

*Percentage Points of the  $\chi^2$  Distribution* (see page 36)

*Critical Values for Correlation Coefficients: Product Moment Coefficient and Spearman's Coefficient* (see page 37)

*Random Numbers* (see page 38)

*Percentage Points of Student's  $t$  Distribution* (see page 39)

*Percentage Points of the  $F$  Distribution* (see page 40)

## Mechanics

### Centres of mass

For uniform bodies:

Triangular lamina:  $\frac{2}{3}$  along median from vertex

Circular arc, radius  $r$ , angle at centre  $2\alpha$  :  $\frac{r \sin \alpha}{\alpha}$  from centre

Sector of circle, radius  $r$ , angle at centre  $2\alpha$  :  $\frac{2r \sin \alpha}{3\alpha}$  from centre

Solid hemisphere, radius  $r$ :  $\frac{3}{8}r$  from centre

Hemispherical shell, radius  $r$ :  $\frac{1}{2}r$  from centre

Solid cone or pyramid of height  $h$ :  $\frac{1}{4}h$  above the base on the line from centre of base to vertex

Conical shell of height  $h$ :  $\frac{1}{3}h$  above the base on the line from centre of base to vertex

### Motion in a circle

Transverse velocity:  $v = r\dot{\theta}$

Transverse acceleration:  $\dot{v} = r\ddot{\theta}$

Radial acceleration:  $-r\dot{\theta}^2 = -\frac{v^2}{r}$



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## 5 Statistical Tables

### Binomial Cumulative Distribution Function

The tabulated value is  $P(X \leq x)$ , where  $X$  has a binomial distribution with index  $n$  and parameter  $p$ .

| $p =$           | 0.05   | 0.10   | 0.15   | 0.20   | 0.25   | 0.30   | 0.35   | 0.40   | 0.45   | 0.50   |
|-----------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $n = 5, x = 0$  | 0.7738 | 0.5905 | 0.4437 | 0.3277 | 0.2373 | 0.1681 | 0.1160 | 0.0778 | 0.0503 | 0.0312 |
| 1               | 0.9774 | 0.9185 | 0.8352 | 0.7373 | 0.6328 | 0.5282 | 0.4284 | 0.3370 | 0.2562 | 0.1875 |
| 2               | 0.9988 | 0.9914 | 0.9734 | 0.9421 | 0.8965 | 0.8369 | 0.7648 | 0.6826 | 0.5931 | 0.5000 |
| 3               | 1.0000 | 0.9995 | 0.9978 | 0.9933 | 0.9844 | 0.9692 | 0.9460 | 0.9130 | 0.8688 | 0.8125 |
| 4               | 1.0000 | 1.0000 | 0.9999 | 0.9997 | 0.9990 | 0.9976 | 0.9947 | 0.9898 | 0.9815 | 0.9688 |
| $n = 6, x = 0$  | 0.7351 | 0.5314 | 0.3771 | 0.2621 | 0.1780 | 0.1176 | 0.0754 | 0.0467 | 0.0277 | 0.0156 |
| 1               | 0.9672 | 0.8857 | 0.7765 | 0.6554 | 0.5339 | 0.4202 | 0.3191 | 0.2333 | 0.1636 | 0.1094 |
| 2               | 0.9978 | 0.9842 | 0.9527 | 0.9011 | 0.8306 | 0.7443 | 0.6471 | 0.5443 | 0.4415 | 0.3438 |
| 3               | 0.9999 | 0.9987 | 0.9941 | 0.9830 | 0.9624 | 0.9295 | 0.8826 | 0.8208 | 0.7447 | 0.6563 |
| 4               | 1.0000 | 0.9999 | 0.9996 | 0.9984 | 0.9954 | 0.9891 | 0.9777 | 0.9590 | 0.9308 | 0.8906 |
| 5               | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9998 | 0.9993 | 0.9982 | 0.9959 | 0.9917 | 0.9844 |
| $n = 7, x = 0$  | 0.6983 | 0.4783 | 0.3206 | 0.2097 | 0.1335 | 0.0824 | 0.0490 | 0.0280 | 0.0152 | 0.0078 |
| 1               | 0.9556 | 0.8503 | 0.7166 | 0.5767 | 0.4449 | 0.3294 | 0.2338 | 0.1586 | 0.1024 | 0.0625 |
| 2               | 0.9962 | 0.9743 | 0.9262 | 0.8520 | 0.7564 | 0.6471 | 0.5323 | 0.4199 | 0.3164 | 0.2266 |
| 3               | 0.9998 | 0.9973 | 0.9879 | 0.9667 | 0.9294 | 0.8740 | 0.8002 | 0.7102 | 0.6083 | 0.5000 |
| 4               | 1.0000 | 0.9998 | 0.9988 | 0.9953 | 0.9871 | 0.9712 | 0.9444 | 0.9037 | 0.8471 | 0.7734 |
| 5               | 1.0000 | 1.0000 | 0.9999 | 0.9996 | 0.9987 | 0.9962 | 0.9910 | 0.9812 | 0.9643 | 0.9375 |
| 6               | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9998 | 0.9994 | 0.9984 | 0.9963 | 0.9922 |
| $n = 8, x = 0$  | 0.6634 | 0.4305 | 0.2725 | 0.1678 | 0.1001 | 0.0576 | 0.0319 | 0.0168 | 0.0084 | 0.0039 |
| 1               | 0.9428 | 0.8131 | 0.6572 | 0.5033 | 0.3671 | 0.2553 | 0.1691 | 0.1064 | 0.0632 | 0.0352 |
| 2               | 0.9942 | 0.9619 | 0.8948 | 0.7969 | 0.6785 | 0.5518 | 0.4278 | 0.3154 | 0.2201 | 0.1445 |
| 3               | 0.9996 | 0.9950 | 0.9786 | 0.9437 | 0.8862 | 0.8059 | 0.7064 | 0.5941 | 0.4770 | 0.3633 |
| 4               | 1.0000 | 0.9996 | 0.9971 | 0.9896 | 0.9727 | 0.9420 | 0.8939 | 0.8263 | 0.7396 | 0.6367 |
| 5               | 1.0000 | 1.0000 | 0.9998 | 0.9988 | 0.9958 | 0.9887 | 0.9747 | 0.9502 | 0.9115 | 0.8555 |
| 6               | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9996 | 0.9987 | 0.9964 | 0.9915 | 0.9819 | 0.9648 |
| 7               | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9998 | 0.9993 | 0.9983 | 0.9961 |
| $n = 9, x = 0$  | 0.6302 | 0.3874 | 0.2316 | 0.1342 | 0.0751 | 0.0404 | 0.0207 | 0.0101 | 0.0046 | 0.0020 |
| 1               | 0.9288 | 0.7748 | 0.5995 | 0.4362 | 0.3003 | 0.1960 | 0.1211 | 0.0705 | 0.0385 | 0.0195 |
| 2               | 0.9916 | 0.9470 | 0.8591 | 0.7382 | 0.6007 | 0.4628 | 0.3373 | 0.2318 | 0.1495 | 0.0898 |
| 3               | 0.9994 | 0.9917 | 0.9661 | 0.9144 | 0.8343 | 0.7297 | 0.6089 | 0.4826 | 0.3614 | 0.2539 |
| 4               | 1.0000 | 0.9991 | 0.9944 | 0.9804 | 0.9511 | 0.9012 | 0.8283 | 0.7334 | 0.6214 | 0.5000 |
| 5               | 1.0000 | 0.9999 | 0.9994 | 0.9969 | 0.9900 | 0.9747 | 0.9464 | 0.9006 | 0.8342 | 0.7461 |
| 6               | 1.0000 | 1.0000 | 1.0000 | 0.9997 | 0.9987 | 0.9957 | 0.9888 | 0.9750 | 0.9502 | 0.9102 |
| 7               | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9996 | 0.9986 | 0.9962 | 0.9909 | 0.9805 |
| 8               | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9997 | 0.9992 | 0.9980 |
| $n = 10, x = 0$ | 0.5987 | 0.3487 | 0.1969 | 0.1074 | 0.0563 | 0.0282 | 0.0135 | 0.0060 | 0.0025 | 0.0010 |
| 1               | 0.9139 | 0.7361 | 0.5443 | 0.3758 | 0.2440 | 0.1493 | 0.0860 | 0.0464 | 0.0233 | 0.0107 |
| 2               | 0.9885 | 0.9298 | 0.8202 | 0.6778 | 0.5256 | 0.3828 | 0.2616 | 0.1673 | 0.0996 | 0.0547 |
| 3               | 0.9990 | 0.9872 | 0.9500 | 0.8791 | 0.7759 | 0.6496 | 0.5138 | 0.3823 | 0.2660 | 0.1719 |
| 4               | 0.9999 | 0.9984 | 0.9901 | 0.9672 | 0.9219 | 0.8497 | 0.7515 | 0.6331 | 0.5044 | 0.3770 |
| 5               | 1.0000 | 0.9999 | 0.9986 | 0.9936 | 0.9803 | 0.9527 | 0.9051 | 0.8338 | 0.7384 | 0.6230 |
| 6               | 1.0000 | 1.0000 | 0.9999 | 0.9991 | 0.9965 | 0.9894 | 0.9740 | 0.9452 | 0.8980 | 0.8281 |
| 7               | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9996 | 0.9984 | 0.9952 | 0.9877 | 0.9726 | 0.9453 |
| 8               | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9995 | 0.9983 | 0.9955 | 0.9893 |
| 9               | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9997 | 0.9990 |

| $p =$           | 0.05   | 0.10   | 0.15   | 0.20   | 0.25   | 0.30   | 0.35   | 0.40   | 0.45   | 0.50   |
|-----------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $n = 12, x = 0$ | 0.5404 | 0.2824 | 0.1422 | 0.0687 | 0.0317 | 0.0138 | 0.0057 | 0.0022 | 0.0008 | 0.0002 |
| 1               | 0.8816 | 0.6590 | 0.4435 | 0.2749 | 0.1584 | 0.0850 | 0.0424 | 0.0196 | 0.0083 | 0.0032 |
| 2               | 0.9804 | 0.8891 | 0.7358 | 0.5583 | 0.3907 | 0.2528 | 0.1513 | 0.0834 | 0.0421 | 0.0193 |
| 3               | 0.9978 | 0.9744 | 0.9078 | 0.7946 | 0.6488 | 0.4925 | 0.3467 | 0.2253 | 0.1345 | 0.0730 |
| 4               | 0.9998 | 0.9957 | 0.9761 | 0.9274 | 0.8424 | 0.7237 | 0.5833 | 0.4382 | 0.3044 | 0.1938 |
| 5               | 1.0000 | 0.9995 | 0.9954 | 0.9806 | 0.9456 | 0.8822 | 0.7873 | 0.6652 | 0.5269 | 0.3872 |
| 6               | 1.0000 | 0.9999 | 0.9993 | 0.9961 | 0.9857 | 0.9614 | 0.9154 | 0.8418 | 0.7393 | 0.6128 |
| 7               | 1.0000 | 1.0000 | 0.9999 | 0.9994 | 0.9972 | 0.9905 | 0.9745 | 0.9427 | 0.8883 | 0.8062 |
| 8               | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9996 | 0.9983 | 0.9944 | 0.9847 | 0.9644 | 0.9270 |
| 9               | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9998 | 0.9992 | 0.9972 | 0.9921 | 0.9807 |
| 10              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9997 | 0.9989 | 0.9968 |
| 11              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9998 |
| $n = 15, x = 0$ | 0.4633 | 0.2059 | 0.0874 | 0.0352 | 0.0134 | 0.0047 | 0.0016 | 0.0005 | 0.0001 | 0.0000 |
| 1               | 0.8290 | 0.5490 | 0.3186 | 0.1671 | 0.0802 | 0.0353 | 0.0142 | 0.0052 | 0.0017 | 0.0005 |
| 2               | 0.9638 | 0.8159 | 0.6042 | 0.3980 | 0.2361 | 0.1268 | 0.0617 | 0.0271 | 0.0107 | 0.0037 |
| 3               | 0.9945 | 0.9444 | 0.8227 | 0.6482 | 0.4613 | 0.2969 | 0.1727 | 0.0905 | 0.0424 | 0.0176 |
| 4               | 0.9994 | 0.9873 | 0.9383 | 0.8358 | 0.6865 | 0.5155 | 0.3519 | 0.2173 | 0.1204 | 0.0592 |
| 5               | 0.9999 | 0.9978 | 0.9832 | 0.9389 | 0.8516 | 0.7216 | 0.5643 | 0.4032 | 0.2608 | 0.1509 |
| 6               | 1.0000 | 0.9997 | 0.9964 | 0.9819 | 0.9434 | 0.8689 | 0.7548 | 0.6098 | 0.4522 | 0.3036 |
| 7               | 1.0000 | 1.0000 | 0.9994 | 0.9958 | 0.9827 | 0.9500 | 0.8868 | 0.7869 | 0.6535 | 0.5000 |
| 8               | 1.0000 | 1.0000 | 0.9999 | 0.9992 | 0.9958 | 0.9848 | 0.9578 | 0.9050 | 0.8182 | 0.6964 |
| 9               | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9992 | 0.9963 | 0.9876 | 0.9662 | 0.9231 | 0.8491 |
| 10              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9993 | 0.9972 | 0.9907 | 0.9745 | 0.9408 |
| 11              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9995 | 0.9981 | 0.9937 | 0.9824 |
| 12              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9997 | 0.9989 | 0.9963 |
| 13              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9995 |
| 14              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| $n = 20, x = 0$ | 0.3585 | 0.1216 | 0.0388 | 0.0115 | 0.0032 | 0.0008 | 0.0002 | 0.0000 | 0.0000 | 0.0000 |
| 1               | 0.7358 | 0.3917 | 0.1756 | 0.0692 | 0.0243 | 0.0076 | 0.0021 | 0.0005 | 0.0001 | 0.0000 |
| 2               | 0.9245 | 0.6769 | 0.4049 | 0.2061 | 0.0913 | 0.0355 | 0.0121 | 0.0036 | 0.0009 | 0.0002 |
| 3               | 0.9841 | 0.8670 | 0.6477 | 0.4114 | 0.2252 | 0.1071 | 0.0444 | 0.0160 | 0.0049 | 0.0013 |
| 4               | 0.9974 | 0.9568 | 0.8298 | 0.6296 | 0.4148 | 0.2375 | 0.1182 | 0.0510 | 0.0189 | 0.0059 |
| 5               | 0.9997 | 0.9887 | 0.9327 | 0.8042 | 0.6172 | 0.4164 | 0.2454 | 0.1256 | 0.0553 | 0.0207 |
| 6               | 1.0000 | 0.9976 | 0.9781 | 0.9133 | 0.7858 | 0.6080 | 0.4166 | 0.2500 | 0.1299 | 0.0577 |
| 7               | 1.0000 | 0.9996 | 0.9941 | 0.9679 | 0.8982 | 0.7723 | 0.6010 | 0.4159 | 0.2520 | 0.1316 |
| 8               | 1.0000 | 0.9999 | 0.9987 | 0.9900 | 0.9591 | 0.8867 | 0.7624 | 0.5956 | 0.4143 | 0.2517 |
| 9               | 1.0000 | 1.0000 | 0.9998 | 0.9974 | 0.9861 | 0.9520 | 0.8782 | 0.7553 | 0.5914 | 0.4119 |
| 10              | 1.0000 | 1.0000 | 1.0000 | 0.9994 | 0.9961 | 0.9829 | 0.9468 | 0.8725 | 0.7507 | 0.5881 |
| 11              | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9991 | 0.9949 | 0.9804 | 0.9435 | 0.8692 | 0.7483 |
| 12              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9998 | 0.9987 | 0.9940 | 0.9790 | 0.9420 | 0.8684 |
| 13              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9997 | 0.9985 | 0.9935 | 0.9786 | 0.9423 |
| 14              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9997 | 0.9984 | 0.9936 | 0.9793 |
| 15              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9997 | 0.9985 | 0.9941 |
| 16              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9997 | 0.9987 |
| 17              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9998 |
| 18              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

| $p =$           | 0.05   | 0.10   | 0.15   | 0.20   | 0.25   | 0.30   | 0.35   | 0.40   | 0.45   | 0.50   |
|-----------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $n = 25, x = 0$ | 0.2774 | 0.0718 | 0.0172 | 0.0038 | 0.0008 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1               | 0.6424 | 0.2712 | 0.0931 | 0.0274 | 0.0070 | 0.0016 | 0.0003 | 0.0001 | 0.0000 | 0.0000 |
| 2               | 0.8729 | 0.5371 | 0.2537 | 0.0982 | 0.0321 | 0.0090 | 0.0021 | 0.0004 | 0.0001 | 0.0000 |
| 3               | 0.9659 | 0.7636 | 0.4711 | 0.2340 | 0.0962 | 0.0332 | 0.0097 | 0.0024 | 0.0005 | 0.0001 |
| 4               | 0.9928 | 0.9020 | 0.6821 | 0.4207 | 0.2137 | 0.0905 | 0.0320 | 0.0095 | 0.0023 | 0.0005 |
| 5               | 0.9988 | 0.9666 | 0.8385 | 0.6167 | 0.3783 | 0.1935 | 0.0826 | 0.0294 | 0.0086 | 0.0020 |
| 6               | 0.9998 | 0.9905 | 0.9305 | 0.7800 | 0.5611 | 0.3407 | 0.1734 | 0.0736 | 0.0258 | 0.0073 |
| 7               | 1.0000 | 0.9977 | 0.9745 | 0.8909 | 0.7265 | 0.5118 | 0.3061 | 0.1536 | 0.0639 | 0.0216 |
| 8               | 1.0000 | 0.9995 | 0.9920 | 0.9532 | 0.8506 | 0.6769 | 0.4668 | 0.2735 | 0.1340 | 0.0539 |
| 9               | 1.0000 | 0.9999 | 0.9979 | 0.9827 | 0.9287 | 0.8106 | 0.6303 | 0.4246 | 0.2424 | 0.1148 |
| 10              | 1.0000 | 1.0000 | 0.9995 | 0.9944 | 0.9703 | 0.9022 | 0.7712 | 0.5858 | 0.3843 | 0.2122 |
| 11              | 1.0000 | 1.0000 | 0.9999 | 0.9985 | 0.9893 | 0.9558 | 0.8746 | 0.7323 | 0.5426 | 0.3450 |
| 12              | 1.0000 | 1.0000 | 1.0000 | 0.9996 | 0.9966 | 0.9825 | 0.9396 | 0.8462 | 0.6937 | 0.5000 |
| 13              | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9991 | 0.9940 | 0.9745 | 0.9222 | 0.8173 | 0.6550 |
| 14              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9998 | 0.9982 | 0.9907 | 0.9656 | 0.9040 | 0.7878 |
| 15              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9995 | 0.9971 | 0.9868 | 0.9560 | 0.8852 |
| 16              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9992 | 0.9957 | 0.9826 | 0.9461 |
| 17              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9998 | 0.9988 | 0.9942 | 0.9784 |
| 18              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9997 | 0.9984 | 0.9927 |
| 19              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9996 | 0.9980 |
| 20              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9995 |
| 21              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 |
| 22              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| $n = 30, x = 0$ | 0.2146 | 0.0424 | 0.0076 | 0.0012 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1               | 0.5535 | 0.1837 | 0.0480 | 0.0105 | 0.0020 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 2               | 0.8122 | 0.4114 | 0.1514 | 0.0442 | 0.0106 | 0.0021 | 0.0003 | 0.0000 | 0.0000 | 0.0000 |
| 3               | 0.9392 | 0.6474 | 0.3217 | 0.1227 | 0.0374 | 0.0093 | 0.0019 | 0.0003 | 0.0000 | 0.0000 |
| 4               | 0.9844 | 0.8245 | 0.5245 | 0.2552 | 0.0979 | 0.0302 | 0.0075 | 0.0015 | 0.0002 | 0.0000 |
| 5               | 0.9967 | 0.9268 | 0.7106 | 0.4275 | 0.2026 | 0.0766 | 0.0233 | 0.0057 | 0.0011 | 0.0002 |
| 6               | 0.9994 | 0.9742 | 0.8474 | 0.6070 | 0.3481 | 0.1595 | 0.0586 | 0.0172 | 0.0040 | 0.0007 |
| 7               | 0.9999 | 0.9922 | 0.9302 | 0.7608 | 0.5143 | 0.2814 | 0.1238 | 0.0435 | 0.0121 | 0.0026 |
| 8               | 1.0000 | 0.9980 | 0.9722 | 0.8713 | 0.6736 | 0.4315 | 0.2247 | 0.0940 | 0.0312 | 0.0081 |
| 9               | 1.0000 | 0.9995 | 0.9903 | 0.9389 | 0.8034 | 0.5888 | 0.3575 | 0.1763 | 0.0694 | 0.0214 |
| 10              | 1.0000 | 0.9999 | 0.9971 | 0.9744 | 0.8943 | 0.7304 | 0.5078 | 0.2915 | 0.1350 | 0.0494 |
| 11              | 1.0000 | 1.0000 | 0.9992 | 0.9905 | 0.9493 | 0.8407 | 0.6548 | 0.4311 | 0.2327 | 0.1002 |
| 12              | 1.0000 | 1.0000 | 0.9998 | 0.9969 | 0.9784 | 0.9155 | 0.7802 | 0.5785 | 0.3592 | 0.1808 |
| 13              | 1.0000 | 1.0000 | 1.0000 | 0.9991 | 0.9918 | 0.9599 | 0.8737 | 0.7145 | 0.5025 | 0.2923 |
| 14              | 1.0000 | 1.0000 | 1.0000 | 0.9998 | 0.9973 | 0.9831 | 0.9348 | 0.8246 | 0.6448 | 0.4278 |
| 15              | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9992 | 0.9936 | 0.9699 | 0.9029 | 0.7691 | 0.5722 |
| 16              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9998 | 0.9979 | 0.9876 | 0.9519 | 0.8644 | 0.7077 |
| 17              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9994 | 0.9955 | 0.9788 | 0.9286 | 0.8192 |
| 18              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9998 | 0.9986 | 0.9917 | 0.9666 | 0.8998 |
| 19              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9996 | 0.9971 | 0.9862 | 0.9506 |
| 20              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9991 | 0.9950 | 0.9786 |
| 21              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9998 | 0.9984 | 0.9919 |
| 22              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9996 | 0.9974 |
| 23              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9993 |
| 24              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9998 |
| 25              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

| $p =$           | 0.05   | 0.10   | 0.15   | 0.20   | 0.25   | 0.30   | 0.35   | 0.40   | 0.45   | 0.50   |
|-----------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $n = 40, x = 0$ | 0.1285 | 0.0148 | 0.0015 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1               | 0.3991 | 0.0805 | 0.0121 | 0.0015 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 2               | 0.6767 | 0.2228 | 0.0486 | 0.0079 | 0.0010 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 3               | 0.8619 | 0.4231 | 0.1302 | 0.0285 | 0.0047 | 0.0006 | 0.0001 | 0.0000 | 0.0000 | 0.0000 |
| 4               | 0.9520 | 0.6290 | 0.2633 | 0.0759 | 0.0160 | 0.0026 | 0.0003 | 0.0000 | 0.0000 | 0.0000 |
| 5               | 0.9861 | 0.7937 | 0.4325 | 0.1613 | 0.0433 | 0.0086 | 0.0013 | 0.0001 | 0.0000 | 0.0000 |
| 6               | 0.9966 | 0.9005 | 0.6067 | 0.2859 | 0.0962 | 0.0238 | 0.0044 | 0.0006 | 0.0001 | 0.0000 |
| 7               | 0.9993 | 0.9581 | 0.7559 | 0.4371 | 0.1820 | 0.0553 | 0.0124 | 0.0021 | 0.0002 | 0.0000 |
| 8               | 0.9999 | 0.9845 | 0.8646 | 0.5931 | 0.2998 | 0.1110 | 0.0303 | 0.0061 | 0.0009 | 0.0001 |
| 9               | 1.0000 | 0.9949 | 0.9328 | 0.7318 | 0.4395 | 0.1959 | 0.0644 | 0.0156 | 0.0027 | 0.0003 |
| 10              | 1.0000 | 0.9985 | 0.9701 | 0.8392 | 0.5839 | 0.3087 | 0.1215 | 0.0352 | 0.0074 | 0.0011 |
| 11              | 1.0000 | 0.9996 | 0.9880 | 0.9125 | 0.7151 | 0.4406 | 0.2053 | 0.0709 | 0.0179 | 0.0032 |
| 12              | 1.0000 | 0.9999 | 0.9957 | 0.9568 | 0.8209 | 0.5772 | 0.3143 | 0.1285 | 0.0386 | 0.0083 |
| 13              | 1.0000 | 1.0000 | 0.9986 | 0.9806 | 0.8968 | 0.7032 | 0.4408 | 0.2112 | 0.0751 | 0.0192 |
| 14              | 1.0000 | 1.0000 | 0.9996 | 0.9921 | 0.9456 | 0.8074 | 0.5721 | 0.3174 | 0.1326 | 0.0403 |
| 15              | 1.0000 | 1.0000 | 0.9999 | 0.9971 | 0.9738 | 0.8849 | 0.6946 | 0.4402 | 0.2142 | 0.0769 |
| 16              | 1.0000 | 1.0000 | 1.0000 | 0.9990 | 0.9884 | 0.9367 | 0.7978 | 0.5681 | 0.3185 | 0.1341 |
| 17              | 1.0000 | 1.0000 | 1.0000 | 0.9997 | 0.9953 | 0.9680 | 0.8761 | 0.6885 | 0.4391 | 0.2148 |
| 18              | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9983 | 0.9852 | 0.9301 | 0.7911 | 0.5651 | 0.3179 |
| 19              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9994 | 0.9937 | 0.9637 | 0.8702 | 0.6844 | 0.4373 |
| 20              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9998 | 0.9976 | 0.9827 | 0.9256 | 0.7870 | 0.5627 |
| 21              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9991 | 0.9925 | 0.9608 | 0.8669 | 0.6821 |
| 22              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9997 | 0.9970 | 0.9811 | 0.9233 | 0.7852 |
| 23              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9989 | 0.9917 | 0.9595 | 0.8659 |
| 24              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9996 | 0.9966 | 0.9804 | 0.9231 |
| 25              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9988 | 0.9914 | 0.9597 |
| 26              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9996 | 0.9966 | 0.9808 |
| 27              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9988 | 0.9917 |
| 28              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9996 | 0.9968 |
| 29              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9989 |
| 30              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9997 |
| 31              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 |
| 32              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

| $p =$           | 0.05   | 0.10   | 0.15   | 0.20   | 0.25   | 0.30   | 0.35   | 0.40   | 0.45   | 0.50   |
|-----------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $n = 50, x = 0$ | 0.0769 | 0.0052 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1               | 0.2794 | 0.0338 | 0.0029 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 2               | 0.5405 | 0.1117 | 0.0142 | 0.0013 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 3               | 0.7604 | 0.2503 | 0.0460 | 0.0057 | 0.0005 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 4               | 0.8964 | 0.4312 | 0.1121 | 0.0185 | 0.0021 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 5               | 0.9622 | 0.6161 | 0.2194 | 0.0480 | 0.0070 | 0.0007 | 0.0001 | 0.0000 | 0.0000 | 0.0000 |
| 6               | 0.9882 | 0.7702 | 0.3613 | 0.1034 | 0.0194 | 0.0025 | 0.0002 | 0.0000 | 0.0000 | 0.0000 |
| 7               | 0.9968 | 0.8779 | 0.5188 | 0.1904 | 0.0453 | 0.0073 | 0.0008 | 0.0001 | 0.0000 | 0.0000 |
| 8               | 0.9992 | 0.9421 | 0.6681 | 0.3073 | 0.0916 | 0.0183 | 0.0025 | 0.0002 | 0.0000 | 0.0000 |
| 9               | 0.9998 | 0.9755 | 0.7911 | 0.4437 | 0.1637 | 0.0402 | 0.0067 | 0.0008 | 0.0001 | 0.0000 |
| 10              | 1.0000 | 0.9906 | 0.8801 | 0.5836 | 0.2622 | 0.0789 | 0.0160 | 0.0022 | 0.0002 | 0.0000 |
| 11              | 1.0000 | 0.9968 | 0.9372 | 0.7107 | 0.3816 | 0.1390 | 0.0342 | 0.0057 | 0.0006 | 0.0000 |
| 12              | 1.0000 | 0.9990 | 0.9699 | 0.8139 | 0.5110 | 0.2229 | 0.0661 | 0.0133 | 0.0018 | 0.0002 |
| 13              | 1.0000 | 0.9997 | 0.9868 | 0.8894 | 0.6370 | 0.3279 | 0.1163 | 0.0280 | 0.0045 | 0.0005 |
| 14              | 1.0000 | 0.9999 | 0.9947 | 0.9393 | 0.7481 | 0.4468 | 0.1878 | 0.0540 | 0.0104 | 0.0013 |
| 15              | 1.0000 | 1.0000 | 0.9981 | 0.9692 | 0.8369 | 0.5692 | 0.2801 | 0.0955 | 0.0220 | 0.0033 |
| 16              | 1.0000 | 1.0000 | 0.9993 | 0.9856 | 0.9017 | 0.6839 | 0.3889 | 0.1561 | 0.0427 | 0.0077 |
| 17              | 1.0000 | 1.0000 | 0.9998 | 0.9937 | 0.9449 | 0.7822 | 0.5060 | 0.2369 | 0.0765 | 0.0164 |
| 18              | 1.0000 | 1.0000 | 0.9999 | 0.9975 | 0.9713 | 0.8594 | 0.6216 | 0.3356 | 0.1273 | 0.0325 |
| 19              | 1.0000 | 1.0000 | 1.0000 | 0.9991 | 0.9861 | 0.9152 | 0.7264 | 0.4465 | 0.1974 | 0.0595 |
| 20              | 1.0000 | 1.0000 | 1.0000 | 0.9997 | 0.9937 | 0.9522 | 0.8139 | 0.5610 | 0.2862 | 0.1013 |
| 21              | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9974 | 0.9749 | 0.8813 | 0.6701 | 0.3900 | 0.1611 |
| 22              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9990 | 0.9877 | 0.9290 | 0.7660 | 0.5019 | 0.2399 |
| 23              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9996 | 0.9944 | 0.9604 | 0.8438 | 0.6134 | 0.3359 |
| 24              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9976 | 0.9793 | 0.9022 | 0.7160 | 0.4439 |
| 25              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9991 | 0.9900 | 0.9427 | 0.8034 | 0.5561 |
| 26              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9997 | 0.9955 | 0.9686 | 0.8721 | 0.6641 |
| 27              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9981 | 0.9840 | 0.9220 | 0.7601 |
| 28              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9993 | 0.9924 | 0.9556 | 0.8389 |
| 29              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9997 | 0.9966 | 0.9765 | 0.8987 |
| 30              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9986 | 0.9884 | 0.9405 |
| 31              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9995 | 0.9947 | 0.9675 |
| 32              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9998 | 0.9978 | 0.9836 |
| 33              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9991 | 0.9923 |
| 34              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9997 | 0.9967 |
| 35              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9987 |
| 36              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9995 |
| 37              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9998 |
| 38              | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

# Percentage Points of The Normal Distribution

The values  $z$  in the table are those which a random variable  $Z \sim N(0, 1)$  exceeds with probability  $p$ ; that is,  $P(Z > z) = 1 - \Phi(z) = p$ .

| $p$    | $z$    | $p$    | $z$    |
|--------|--------|--------|--------|
| 0.5000 | 0.0000 | 0.0500 | 1.6449 |
| 0.4000 | 0.2533 | 0.0250 | 1.9600 |
| 0.3000 | 0.5244 | 0.0100 | 2.3263 |
| 0.2000 | 0.8416 | 0.0050 | 2.5758 |
| 0.1500 | 1.0364 | 0.0010 | 3.0902 |
| 0.1000 | 1.2816 | 0.0005 | 3.2905 |

## Poisson Cumulative Distribution Function

The tabulated value is  $P(X \leq x)$ , where  $X$  has a Poisson distribution with parameter  $\lambda$ .

| $\lambda =$ | 0.5    | 1.0    | 1.5    | 2.0    | 2.5    | 3.0    | 3.5    | 4.0    | 4.5    | 5.0    |
|-------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $x = 0$     | 0.6065 | 0.3679 | 0.2231 | 0.1353 | 0.0821 | 0.0498 | 0.0302 | 0.0183 | 0.0111 | 0.0067 |
| 1           | 0.9098 | 0.7358 | 0.5578 | 0.4060 | 0.2873 | 0.1991 | 0.1359 | 0.0916 | 0.0611 | 0.0404 |
| 2           | 0.9856 | 0.9197 | 0.8088 | 0.6767 | 0.5438 | 0.4232 | 0.3208 | 0.2381 | 0.1736 | 0.1247 |
| 3           | 0.9982 | 0.9810 | 0.9344 | 0.8571 | 0.7576 | 0.6472 | 0.5366 | 0.4335 | 0.3423 | 0.2650 |
| 4           | 0.9998 | 0.9963 | 0.9814 | 0.9473 | 0.8912 | 0.8153 | 0.7254 | 0.6288 | 0.5321 | 0.4405 |
| 5           | 1.0000 | 0.9994 | 0.9955 | 0.9834 | 0.9580 | 0.9161 | 0.8576 | 0.7851 | 0.7029 | 0.6160 |
| 6           | 1.0000 | 0.9999 | 0.9991 | 0.9955 | 0.9858 | 0.9665 | 0.9347 | 0.8893 | 0.8311 | 0.7622 |
| 7           | 1.0000 | 1.0000 | 0.9998 | 0.9989 | 0.9958 | 0.9881 | 0.9733 | 0.9489 | 0.9134 | 0.8666 |
| 8           | 1.0000 | 1.0000 | 1.0000 | 0.9998 | 0.9989 | 0.9962 | 0.9901 | 0.9786 | 0.9597 | 0.9319 |
| 9           | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9997 | 0.9989 | 0.9967 | 0.9919 | 0.9829 | 0.9682 |
| 10          | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9997 | 0.9990 | 0.9972 | 0.9933 | 0.9863 |
| 11          | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9997 | 0.9991 | 0.9976 | 0.9945 |
| 12          | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9997 | 0.9992 | 0.9980 |
| 13          | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9997 | 0.9993 |
| 14          | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9998 |
| 15          | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 |
| 16          | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 17          | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 18          | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 19          | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| $\lambda =$ | 5.5    | 6.0    | 6.5    | 7.0    | 7.5    | 8.0    | 8.5    | 9.0    | 9.5    | 10.0   |
| $x = 0$     | 0.0041 | 0.0025 | 0.0015 | 0.0009 | 0.0006 | 0.0003 | 0.0002 | 0.0001 | 0.0001 | 0.0000 |
| 1           | 0.0266 | 0.0174 | 0.0113 | 0.0073 | 0.0047 | 0.0030 | 0.0019 | 0.0012 | 0.0008 | 0.0005 |
| 2           | 0.0884 | 0.0620 | 0.0430 | 0.0296 | 0.0203 | 0.0138 | 0.0093 | 0.0062 | 0.0042 | 0.0028 |
| 3           | 0.2017 | 0.1512 | 0.1118 | 0.0818 | 0.0591 | 0.0424 | 0.0301 | 0.0212 | 0.0149 | 0.0103 |
| 4           | 0.3575 | 0.2851 | 0.2237 | 0.1730 | 0.1321 | 0.0996 | 0.0744 | 0.0550 | 0.0403 | 0.0293 |
| 5           | 0.5289 | 0.4457 | 0.3690 | 0.3007 | 0.2414 | 0.1912 | 0.1496 | 0.1157 | 0.0885 | 0.0671 |
| 6           | 0.6860 | 0.6063 | 0.5265 | 0.4497 | 0.3782 | 0.3134 | 0.2562 | 0.2068 | 0.1649 | 0.1301 |
| 7           | 0.8095 | 0.7440 | 0.6728 | 0.5987 | 0.5246 | 0.4530 | 0.3856 | 0.3239 | 0.2687 | 0.2202 |
| 8           | 0.8944 | 0.8472 | 0.7916 | 0.7291 | 0.6620 | 0.5925 | 0.5231 | 0.4557 | 0.3918 | 0.3328 |
| 9           | 0.9462 | 0.9161 | 0.8774 | 0.8305 | 0.7764 | 0.7166 | 0.6530 | 0.5874 | 0.5218 | 0.4579 |
| 10          | 0.9747 | 0.9574 | 0.9332 | 0.9015 | 0.8622 | 0.8159 | 0.7634 | 0.7060 | 0.6453 | 0.5830 |
| 11          | 0.9890 | 0.9799 | 0.9661 | 0.9467 | 0.9208 | 0.8881 | 0.8487 | 0.8030 | 0.7520 | 0.6968 |
| 12          | 0.9955 | 0.9912 | 0.9840 | 0.9730 | 0.9573 | 0.9362 | 0.9091 | 0.8758 | 0.8364 | 0.7916 |
| 13          | 0.9983 | 0.9964 | 0.9929 | 0.9872 | 0.9784 | 0.9658 | 0.9486 | 0.9261 | 0.8981 | 0.8645 |
| 14          | 0.9994 | 0.9986 | 0.9970 | 0.9943 | 0.9897 | 0.9827 | 0.9726 | 0.9585 | 0.9400 | 0.9165 |
| 15          | 0.9998 | 0.9995 | 0.9988 | 0.9976 | 0.9954 | 0.9918 | 0.9862 | 0.9780 | 0.9665 | 0.9513 |
| 16          | 0.9999 | 0.9998 | 0.9996 | 0.9990 | 0.9980 | 0.9963 | 0.9934 | 0.9889 | 0.9823 | 0.9730 |
| 17          | 1.0000 | 0.9999 | 0.9998 | 0.9996 | 0.9992 | 0.9984 | 0.9970 | 0.9947 | 0.9911 | 0.9857 |
| 18          | 1.0000 | 1.0000 | 0.9999 | 0.9999 | 0.9997 | 0.9993 | 0.9987 | 0.9976 | 0.9957 | 0.9928 |
| 19          | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9997 | 0.9995 | 0.9989 | 0.9980 | 0.9965 |
| 20          | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9998 | 0.9996 | 0.9991 | 0.9984 |
| 21          | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9998 | 0.9996 | 0.9993 |
| 22          | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9999 | 0.9997 |

## Percentage Points of the $\chi^2$ Distribution

The values in the table are those which a random variable with the  $\chi^2$  distribution on  $\nu$  degrees of freedom exceeds with the probability shown.

| $\nu$ | 0.995  | 0.990  | 0.975  | 0.950  | 0.900  | 0.100  | 0.050  | 0.025  | 0.010  | 0.005  |
|-------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 1     | 0.000  | 0.000  | 0.001  | 0.004  | 0.016  | 2.705  | 3.841  | 5.024  | 6.635  | 7.879  |
| 2     | 0.010  | 0.020  | 0.051  | 0.103  | 0.211  | 4.605  | 5.991  | 7.378  | 9.210  | 10.597 |
| 3     | 0.072  | 0.115  | 0.216  | 0.352  | 0.584  | 6.251  | 7.815  | 9.348  | 11.345 | 12.838 |
| 4     | 0.207  | 0.297  | 0.484  | 0.711  | 1.064  | 7.779  | 9.488  | 11.143 | 13.277 | 14.860 |
| 5     | 0.412  | 0.554  | 0.831  | 1.145  | 1.610  | 9.236  | 11.070 | 12.832 | 15.086 | 16.750 |
| 6     | 0.676  | 0.872  | 1.237  | 1.635  | 2.204  | 10.645 | 12.592 | 14.449 | 16.812 | 18.548 |
| 7     | 0.989  | 1.239  | 1.690  | 2.167  | 2.833  | 12.017 | 14.067 | 16.013 | 18.475 | 20.278 |
| 8     | 1.344  | 1.646  | 2.180  | 2.733  | 3.490  | 13.362 | 15.507 | 17.535 | 20.090 | 21.955 |
| 9     | 1.735  | 2.088  | 2.700  | 3.325  | 4.168  | 14.684 | 16.919 | 19.023 | 21.666 | 23.589 |
| 10    | 2.156  | 2.558  | 3.247  | 3.940  | 4.865  | 15.987 | 18.307 | 20.483 | 23.209 | 25.188 |
| 11    | 2.603  | 3.053  | 3.816  | 4.575  | 5.580  | 17.275 | 19.675 | 21.920 | 24.725 | 26.757 |
| 12    | 3.074  | 3.571  | 4.404  | 5.226  | 6.304  | 18.549 | 21.026 | 23.337 | 26.217 | 28.300 |
| 13    | 3.565  | 4.107  | 5.009  | 5.892  | 7.042  | 19.812 | 22.362 | 24.736 | 27.688 | 29.819 |
| 14    | 4.075  | 4.660  | 5.629  | 6.571  | 7.790  | 21.064 | 23.685 | 26.119 | 29.141 | 31.319 |
| 15    | 4.601  | 5.229  | 6.262  | 7.261  | 8.547  | 22.307 | 24.996 | 27.488 | 30.578 | 32.801 |
| 16    | 5.142  | 5.812  | 6.908  | 7.962  | 9.312  | 23.542 | 26.296 | 28.845 | 32.000 | 34.267 |
| 17    | 5.697  | 6.408  | 7.564  | 8.672  | 10.085 | 24.769 | 27.587 | 30.191 | 33.409 | 35.718 |
| 18    | 6.265  | 7.015  | 8.231  | 9.390  | 10.865 | 25.989 | 28.869 | 31.526 | 34.805 | 37.156 |
| 19    | 6.844  | 7.633  | 8.907  | 10.117 | 11.651 | 27.204 | 30.144 | 32.852 | 36.191 | 38.582 |
| 20    | 7.434  | 8.260  | 9.591  | 10.851 | 12.443 | 28.412 | 31.410 | 34.170 | 37.566 | 39.997 |
| 21    | 8.034  | 8.897  | 10.283 | 11.591 | 13.240 | 29.615 | 32.671 | 35.479 | 38.932 | 41.401 |
| 22    | 8.643  | 9.542  | 10.982 | 12.338 | 14.042 | 30.813 | 33.924 | 36.781 | 40.289 | 42.796 |
| 23    | 9.260  | 10.196 | 11.689 | 13.091 | 14.848 | 32.007 | 35.172 | 38.076 | 41.638 | 44.181 |
| 24    | 9.886  | 10.856 | 12.401 | 13.848 | 15.659 | 33.196 | 36.415 | 39.364 | 42.980 | 45.558 |
| 25    | 10.520 | 11.524 | 13.120 | 14.611 | 16.473 | 34.382 | 37.652 | 40.646 | 44.314 | 46.928 |
| 26    | 11.160 | 12.198 | 13.844 | 15.379 | 17.292 | 35.563 | 38.885 | 41.923 | 45.642 | 48.290 |
| 27    | 11.808 | 12.879 | 14.573 | 16.151 | 18.114 | 36.741 | 40.113 | 43.194 | 46.963 | 49.645 |
| 28    | 12.461 | 13.565 | 15.308 | 16.928 | 18.939 | 37.916 | 41.337 | 44.461 | 48.278 | 50.993 |
| 29    | 13.121 | 14.256 | 16.047 | 17.708 | 19.768 | 39.088 | 42.557 | 45.722 | 49.588 | 52.336 |
| 30    | 13.787 | 14.953 | 16.791 | 18.493 | 20.599 | 40.256 | 43.773 | 46.979 | 50.892 | 53.672 |

## Critical Values for Correlation Coefficients

These tables concern tests of the hypothesis that a population correlation coefficient  $\rho$  is 0. The values in the tables are the minimum values which need to be reached by a sample correlation coefficient in order to be significant at the level shown, on a one-tailed test.

| Product Moment Coefficient |        |        |        |        | Sample<br>size, $n$ | Spearman's Coefficient |        |        |
|----------------------------|--------|--------|--------|--------|---------------------|------------------------|--------|--------|
| Level                      |        |        |        |        |                     | Level                  |        |        |
| 0.10                       | 0.05   | 0.025  | 0.01   | 0.005  |                     | 0.05                   | 0.025  | 0.01   |
| 0.8000                     | 0.9000 | 0.9500 | 0.9800 | 0.9900 | 4                   | 1.0000                 | —      | —      |
| 0.6870                     | 0.8054 | 0.8783 | 0.9343 | 0.9587 | 5                   | 0.9000                 | 1.0000 | 1.0000 |
| 0.6084                     | 0.7293 | 0.8114 | 0.8822 | 0.9172 | 6                   | 0.8286                 | 0.8857 | 0.9429 |
| 0.5509                     | 0.6694 | 0.7545 | 0.8329 | 0.8745 | 7                   | 0.7143                 | 0.7857 | 0.8929 |
| 0.5067                     | 0.6215 | 0.7067 | 0.7887 | 0.8343 | 8                   | 0.6429                 | 0.7381 | 0.8333 |
| 0.4716                     | 0.5822 | 0.6664 | 0.7498 | 0.7977 | 9                   | 0.6000                 | 0.7000 | 0.7833 |
| 0.4428                     | 0.5494 | 0.6319 | 0.7155 | 0.7646 | 10                  | 0.5636                 | 0.6485 | 0.7455 |
| 0.4187                     | 0.5214 | 0.6021 | 0.6851 | 0.7348 | 11                  | 0.5364                 | 0.6182 | 0.7091 |
| 0.3981                     | 0.4973 | 0.5760 | 0.6581 | 0.7079 | 12                  | 0.5035                 | 0.5874 | 0.6783 |
| 0.3802                     | 0.4762 | 0.5529 | 0.6339 | 0.6835 | 13                  | 0.4835                 | 0.5604 | 0.6484 |
| 0.3646                     | 0.4575 | 0.5324 | 0.6120 | 0.6614 | 14                  | 0.4637                 | 0.5385 | 0.6264 |
| 0.3507                     | 0.4409 | 0.5140 | 0.5923 | 0.6411 | 15                  | 0.4464                 | 0.5214 | 0.6036 |
| 0.3383                     | 0.4259 | 0.4973 | 0.5742 | 0.6226 | 16                  | 0.4294                 | 0.5029 | 0.5824 |
| 0.3271                     | 0.4124 | 0.4821 | 0.5577 | 0.6055 | 17                  | 0.4142                 | 0.4877 | 0.5662 |
| 0.3170                     | 0.4000 | 0.4683 | 0.5425 | 0.5897 | 18                  | 0.4014                 | 0.4716 | 0.5501 |
| 0.3077                     | 0.3887 | 0.4555 | 0.5285 | 0.5751 | 19                  | 0.3912                 | 0.4596 | 0.5351 |
| 0.2992                     | 0.3783 | 0.4438 | 0.5155 | 0.5614 | 20                  | 0.3805                 | 0.4466 | 0.5218 |
| 0.2914                     | 0.3687 | 0.4329 | 0.5034 | 0.5487 | 21                  | 0.3701                 | 0.4364 | 0.5091 |
| 0.2841                     | 0.3598 | 0.4227 | 0.4921 | 0.5368 | 22                  | 0.3608                 | 0.4252 | 0.4975 |
| 0.2774                     | 0.3515 | 0.4133 | 0.4815 | 0.5256 | 23                  | 0.3528                 | 0.4160 | 0.4862 |
| 0.2711                     | 0.3438 | 0.4044 | 0.4716 | 0.5151 | 24                  | 0.3443                 | 0.4070 | 0.4757 |
| 0.2653                     | 0.3365 | 0.3961 | 0.4622 | 0.5052 | 25                  | 0.3369                 | 0.3977 | 0.4662 |
| 0.2598                     | 0.3297 | 0.3882 | 0.4534 | 0.4958 | 26                  | 0.3306                 | 0.3901 | 0.4571 |
| 0.2546                     | 0.3233 | 0.3809 | 0.4451 | 0.4869 | 27                  | 0.3242                 | 0.3828 | 0.4487 |
| 0.2497                     | 0.3172 | 0.3739 | 0.4372 | 0.4785 | 28                  | 0.3180                 | 0.3755 | 0.4401 |
| 0.2451                     | 0.3115 | 0.3673 | 0.4297 | 0.4705 | 29                  | 0.3118                 | 0.3685 | 0.4325 |
| 0.2407                     | 0.3061 | 0.3610 | 0.4226 | 0.4629 | 30                  | 0.3063                 | 0.3624 | 0.4251 |
| 0.2070                     | 0.2638 | 0.3120 | 0.3665 | 0.4026 | 40                  | 0.2640                 | 0.3128 | 0.3681 |
| 0.1843                     | 0.2353 | 0.2787 | 0.3281 | 0.3610 | 50                  | 0.2353                 | 0.2791 | 0.3293 |
| 0.1678                     | 0.2144 | 0.2542 | 0.2997 | 0.3301 | 60                  | 0.2144                 | 0.2545 | 0.3005 |
| 0.1550                     | 0.1982 | 0.2352 | 0.2776 | 0.3060 | 70                  | 0.1982                 | 0.2354 | 0.2782 |
| 0.1448                     | 0.1852 | 0.2199 | 0.2597 | 0.2864 | 80                  | 0.1852                 | 0.2201 | 0.2602 |
| 0.1364                     | 0.1745 | 0.2072 | 0.2449 | 0.2702 | 90                  | 0.1745                 | 0.2074 | 0.2453 |
| 0.1292                     | 0.1654 | 0.1966 | 0.2324 | 0.2565 | 100                 | 0.1654                 | 0.1967 | 0.2327 |

## Random Numbers

|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 86 | 13 | 84 | 10 | 07 | 30 | 39 | 05 | 97 | 96 | 88 | 07 | 37 | 26 | 04 | 89 | 13 | 48 | 19 | 20 |
| 60 | 78 | 48 | 12 | 99 | 47 | 09 | 46 | 91 | 33 | 17 | 21 | 03 | 94 | 79 | 00 | 08 | 50 | 40 | 16 |
| 78 | 48 | 06 | 37 | 82 | 26 | 01 | 06 | 64 | 65 | 94 | 41 | 17 | 26 | 74 | 66 | 61 | 93 | 24 | 97 |
| 80 | 56 | 90 | 79 | 66 | 94 | 18 | 40 | 97 | 79 | 93 | 20 | 41 | 51 | 25 | 04 | 20 | 71 | 76 | 04 |
| 99 | 09 | 39 | 25 | 66 | 31 | 70 | 56 | 30 | 15 | 52 | 17 | 87 | 55 | 31 | 11 | 10 | 68 | 98 | 23 |
| 56 | 32 | 32 | 72 | 91 | 65 | 97 | 36 | 56 | 61 | 12 | 79 | 95 | 17 | 57 | 16 | 53 | 58 | 96 | 36 |
| 66 | 02 | 49 | 93 | 97 | 44 | 99 | 15 | 56 | 86 | 80 | 57 | 11 | 78 | 40 | 23 | 58 | 40 | 86 | 14 |
| 31 | 77 | 53 | 94 | 05 | 93 | 56 | 14 | 71 | 23 | 60 | 46 | 05 | 33 | 23 | 72 | 93 | 10 | 81 | 23 |
| 98 | 79 | 72 | 43 | 14 | 76 | 54 | 77 | 66 | 29 | 84 | 09 | 88 | 56 | 75 | 86 | 41 | 67 | 04 | 42 |
| 50 | 97 | 92 | 15 | 10 | 01 | 57 | 01 | 87 | 33 | 73 | 17 | 70 | 18 | 40 | 21 | 24 | 20 | 66 | 62 |
| 90 | 51 | 94 | 50 | 12 | 48 | 88 | 95 | 09 | 34 | 09 | 30 | 22 | 27 | 25 | 56 | 40 | 76 | 01 | 59 |
| 31 | 99 | 52 | 24 | 13 | 43 | 27 | 88 | 11 | 39 | 41 | 65 | 00 | 84 | 13 | 06 | 31 | 79 | 74 | 97 |
| 22 | 96 | 23 | 34 | 46 | 12 | 67 | 11 | 48 | 06 | 99 | 24 | 14 | 83 | 78 | 37 | 65 | 73 | 39 | 47 |
| 06 | 84 | 55 | 41 | 27 | 06 | 74 | 59 | 14 | 29 | 20 | 14 | 45 | 75 | 31 | 16 | 05 | 41 | 22 | 96 |
| 08 | 64 | 89 | 30 | 25 | 25 | 71 | 35 | 33 | 31 | 04 | 56 | 12 | 67 | 03 | 74 | 07 | 16 | 49 | 32 |
| 86 | 87 | 62 | 43 | 15 | 11 | 76 | 49 | 79 | 13 | 78 | 80 | 93 | 89 | 09 | 57 | 07 | 14 | 40 | 74 |
| 94 | 44 | 97 | 13 | 77 | 04 | 35 | 02 | 12 | 76 | 60 | 91 | 93 | 40 | 81 | 06 | 85 | 85 | 72 | 84 |
| 63 | 25 | 55 | 14 | 66 | 47 | 99 | 90 | 02 | 90 | 83 | 43 | 16 | 01 | 19 | 69 | 11 | 78 | 87 | 16 |
| 11 | 22 | 83 | 98 | 15 | 21 | 18 | 57 | 53 | 42 | 91 | 91 | 26 | 52 | 89 | 13 | 86 | 00 | 47 | 61 |
| 01 | 70 | 10 | 83 | 94 | 71 | 13 | 67 | 11 | 12 | 36 | 54 | 53 | 32 | 90 | 43 | 79 | 01 | 95 | 15 |

## Percentage Points of Student's $t$ Distribution

The values in the table are those which a random variable with Student's  $t$  distribution on  $\nu$  degrees of freedom exceeds with the probability shown.

| $\nu$ | 0.10  | 0.05  | 0.025  | 0.01   | 0.005  |
|-------|-------|-------|--------|--------|--------|
| 1     | 3.078 | 6.314 | 12.706 | 31.821 | 63.657 |
| 2     | 1.886 | 2.920 | 4.303  | 6.965  | 9.925  |
| 3     | 1.638 | 2.353 | 3.182  | 4.541  | 5.841  |
| 4     | 1.533 | 2.132 | 2.776  | 3.747  | 4.604  |
| 5     | 1.476 | 2.015 | 2.571  | 3.365  | 4.032  |
| 6     | 1.440 | 1.943 | 2.447  | 3.143  | 3.707  |
| 7     | 1.415 | 1.895 | 2.365  | 2.998  | 3.499  |
| 8     | 1.397 | 1.860 | 2.306  | 2.896  | 3.355  |
| 9     | 1.383 | 1.833 | 2.262  | 2.821  | 3.250  |
| 10    | 1.372 | 1.812 | 2.228  | 2.764  | 3.169  |
| 11    | 1.363 | 1.796 | 2.201  | 2.718  | 3.106  |
| 12    | 1.356 | 1.782 | 2.179  | 2.681  | 3.055  |
| 13    | 1.350 | 1.771 | 2.160  | 2.650  | 3.012  |
| 14    | 1.345 | 1.761 | 2.145  | 2.624  | 2.977  |
| 15    | 1.341 | 1.753 | 2.131  | 2.602  | 2.947  |
| 16    | 1.337 | 1.746 | 2.120  | 2.583  | 2.921  |
| 17    | 1.333 | 1.740 | 2.110  | 2.567  | 2.898  |
| 18    | 1.330 | 1.734 | 2.101  | 2.552  | 2.878  |
| 19    | 1.328 | 1.729 | 2.093  | 2.539  | 2.861  |
| 20    | 1.325 | 1.725 | 2.086  | 2.528  | 2.845  |
| 21    | 1.323 | 1.721 | 2.080  | 2.518  | 2.831  |
| 22    | 1.321 | 1.717 | 2.074  | 2.508  | 2.819  |
| 23    | 1.319 | 1.714 | 2.069  | 2.500  | 2.807  |
| 24    | 1.318 | 1.711 | 2.064  | 2.492  | 2.797  |
| 25    | 1.316 | 1.708 | 2.060  | 2.485  | 2.787  |
| 26    | 1.315 | 1.706 | 2.056  | 2.479  | 2.779  |
| 27    | 1.314 | 1.703 | 2.052  | 2.473  | 2.771  |
| 28    | 1.313 | 1.701 | 2.048  | 2.467  | 2.763  |
| 29    | 1.311 | 1.699 | 2.045  | 2.462  | 2.756  |
| 30    | 1.310 | 1.697 | 2.042  | 2.457  | 2.750  |
| 32    | 1.309 | 1.694 | 2.037  | 2.449  | 2.738  |
| 34    | 1.307 | 1.691 | 2.032  | 2.441  | 2.728  |
| 36    | 1.306 | 1.688 | 2.028  | 2.435  | 2.719  |
| 38    | 1.304 | 1.686 | 2.024  | 2.429  | 2.712  |
| 40    | 1.303 | 1.684 | 2.021  | 2.423  | 2.704  |
| 45    | 1.301 | 1.679 | 2.014  | 2.412  | 2.690  |
| 50    | 1.299 | 1.676 | 2.009  | 2.403  | 2.678  |
| 55    | 1.297 | 1.673 | 2.004  | 2.396  | 2.668  |
| 60    | 1.296 | 1.671 | 2.000  | 2.390  | 2.660  |
| 70    | 1.294 | 1.667 | 1.994  | 2.381  | 2.648  |
| 80    | 1.292 | 1.664 | 1.990  | 2.374  | 2.639  |
| 90    | 1.291 | 1.662 | 1.987  | 2.369  | 2.632  |
| 100   | 1.290 | 1.660 | 1.984  | 2.364  | 2.626  |
| 110   | 1.289 | 1.659 | 1.982  | 2.361  | 2.621  |
| 120   | 1.289 | 1.658 | 1.980  | 2.358  | 2.617  |

## Percentage Points of the $F$ Distribution

The values in the table are those which a random variable with the  $F$  distribution on  $\nu_1$  and  $\nu_2$  degrees of freedom exceeds with probability 0.05 or 0.01.

| Probability | $\nu_1 \backslash \nu_2$ | 1     | 2     | 3     | 4     | 5     | 6     | 8     | 10    | 12    | 24    | $\infty$ |
|-------------|--------------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|
| 0.05        | 1                        | 161.4 | 199.5 | 215.7 | 224.6 | 230.2 | 234.0 | 238.9 | 241.9 | 243.9 | 249.1 | 254.3    |
|             | 2                        | 18.51 | 19.00 | 19.16 | 19.25 | 19.30 | 19.33 | 19.37 | 19.40 | 19.41 | 19.46 | 19.50    |
|             | 3                        | 10.13 | 9.55  | 9.28  | 9.12  | 9.01  | 8.94  | 8.85  | 8.79  | 8.74  | 8.64  | 8.53     |
|             | 4                        | 7.71  | 6.94  | 6.59  | 6.39  | 6.26  | 6.16  | 6.04  | 5.96  | 5.91  | 5.77  | 5.63     |
|             | 5                        | 6.61  | 5.79  | 5.41  | 5.19  | 5.05  | 4.95  | 4.82  | 4.74  | 4.68  | 4.53  | 4.37     |
|             | 6                        | 5.99  | 5.14  | 4.76  | 4.53  | 4.39  | 4.28  | 4.15  | 4.06  | 4.00  | 3.84  | 3.67     |
|             | 7                        | 5.59  | 4.74  | 4.35  | 4.12  | 3.97  | 3.87  | 3.73  | 3.64  | 3.57  | 3.41  | 3.23     |
|             | 8                        | 5.32  | 4.46  | 4.07  | 3.84  | 3.69  | 3.58  | 3.44  | 3.35  | 3.28  | 3.12  | 2.93     |
|             | 9                        | 5.12  | 4.26  | 3.86  | 3.63  | 3.48  | 3.37  | 3.23  | 3.14  | 3.07  | 2.90  | 2.71     |
|             | 10                       | 4.96  | 4.10  | 3.71  | 3.48  | 3.33  | 3.22  | 3.07  | 2.98  | 2.91  | 2.74  | 2.54     |
|             | 11                       | 4.84  | 3.98  | 3.59  | 3.36  | 3.20  | 3.09  | 2.95  | 2.85  | 2.79  | 2.61  | 2.40     |
|             | 12                       | 4.75  | 3.89  | 3.49  | 3.26  | 3.11  | 3.00  | 2.85  | 2.75  | 2.69  | 2.51  | 2.30     |
|             | 14                       | 4.60  | 3.74  | 3.34  | 3.11  | 2.96  | 2.85  | 2.70  | 2.60  | 2.53  | 2.35  | 2.13     |
|             | 16                       | 4.49  | 3.63  | 3.24  | 3.01  | 2.85  | 2.74  | 2.59  | 2.49  | 2.42  | 2.24  | 2.01     |
|             | 18                       | 4.41  | 3.55  | 3.16  | 2.93  | 2.77  | 2.66  | 2.51  | 2.41  | 2.34  | 2.15  | 1.92     |
|             | 20                       | 4.35  | 3.49  | 3.10  | 2.87  | 2.71  | 2.60  | 2.45  | 2.35  | 2.28  | 2.08  | 1.84     |
|             | 25                       | 4.24  | 3.39  | 2.99  | 2.76  | 2.60  | 2.49  | 2.34  | 2.24  | 2.16  | 1.96  | 1.71     |
|             | 30                       | 4.17  | 3.32  | 2.92  | 2.69  | 2.53  | 2.42  | 2.27  | 2.16  | 2.09  | 1.89  | 1.62     |
|             | 40                       | 4.08  | 3.23  | 2.84  | 2.61  | 2.45  | 2.34  | 2.18  | 2.08  | 2.00  | 1.79  | 1.51     |
|             | 60                       | 4.00  | 3.15  | 2.76  | 2.53  | 2.37  | 2.25  | 2.10  | 1.99  | 1.92  | 1.70  | 1.39     |
|             | 120                      | 3.92  | 3.07  | 2.68  | 2.45  | 2.29  | 2.18  | 2.02  | 1.91  | 1.83  | 1.61  | 1.25     |
|             | $\infty$                 | 3.84  | 3.00  | 2.60  | 2.37  | 2.21  | 2.10  | 1.94  | 1.83  | 1.75  | 1.52  | 1.00     |
| 0.01        | 1                        | 4052. | 5000. | 5403. | 5625. | 5764. | 5859. | 5982. | 6056. | 6106. | 6235. | 6366.    |
|             | 2                        | 98.50 | 99.00 | 99.17 | 99.25 | 99.30 | 99.33 | 99.37 | 99.40 | 99.42 | 99.46 | 99.50    |
|             | 3                        | 34.12 | 30.82 | 29.46 | 28.71 | 28.24 | 27.91 | 27.49 | 27.23 | 27.05 | 26.60 | 26.13    |
|             | 4                        | 21.20 | 18.00 | 16.69 | 15.98 | 15.52 | 15.21 | 14.80 | 14.55 | 14.37 | 13.93 | 13.45    |
|             | 5                        | 16.26 | 13.27 | 12.06 | 11.39 | 10.97 | 10.67 | 10.29 | 10.05 | 9.89  | 9.47  | 9.02     |
|             | 6                        | 13.70 | 10.90 | 9.78  | 9.15  | 8.75  | 8.47  | 8.10  | 7.87  | 7.72  | 7.31  | 6.88     |
|             | 7                        | 12.20 | 9.55  | 8.45  | 7.85  | 7.46  | 7.19  | 6.84  | 6.62  | 6.47  | 6.07  | 5.65     |
|             | 8                        | 11.30 | 8.65  | 7.59  | 7.01  | 6.63  | 6.37  | 6.03  | 5.81  | 5.67  | 5.28  | 4.86     |
|             | 9                        | 10.60 | 8.02  | 6.99  | 6.42  | 6.06  | 5.80  | 5.47  | 5.26  | 5.11  | 4.73  | 4.31     |
|             | 10                       | 10.00 | 7.56  | 6.55  | 5.99  | 5.64  | 5.39  | 5.06  | 4.85  | 4.71  | 4.33  | 3.91     |
|             | 11                       | 9.65  | 7.21  | 6.22  | 5.67  | 5.32  | 5.07  | 4.74  | 4.54  | 4.40  | 4.02  | 3.60     |
|             | 12                       | 9.33  | 6.93  | 5.95  | 5.41  | 5.06  | 4.82  | 4.50  | 4.30  | 4.16  | 3.78  | 3.36     |
|             | 14                       | 8.86  | 6.51  | 5.56  | 5.04  | 4.70  | 4.46  | 4.14  | 3.94  | 3.80  | 3.43  | 3.00     |
|             | 16                       | 8.53  | 6.23  | 5.29  | 4.77  | 4.44  | 4.20  | 3.89  | 3.69  | 3.55  | 3.18  | 2.75     |
|             | 18                       | 8.29  | 6.01  | 5.09  | 4.58  | 4.25  | 4.01  | 3.71  | 3.51  | 3.37  | 3.00  | 2.57     |
|             | 20                       | 8.10  | 5.85  | 4.94  | 4.43  | 4.10  | 3.87  | 3.56  | 3.37  | 3.23  | 2.86  | 2.42     |
|             | 25                       | 7.77  | 5.57  | 4.68  | 4.18  | 3.86  | 3.63  | 3.32  | 3.13  | 2.99  | 2.62  | 2.17     |
|             | 30                       | 7.56  | 5.39  | 4.51  | 4.02  | 3.70  | 3.47  | 3.17  | 2.98  | 2.84  | 2.47  | 2.01     |
|             | 40                       | 7.31  | 5.18  | 4.31  | 3.83  | 3.51  | 3.29  | 2.99  | 2.80  | 2.66  | 2.29  | 1.80     |
|             | 60                       | 7.08  | 4.98  | 4.13  | 3.65  | 3.34  | 3.12  | 2.82  | 2.63  | 2.50  | 2.12  | 1.60     |
|             | 120                      | 6.85  | 4.79  | 3.95  | 3.48  | 3.17  | 2.96  | 2.66  | 2.47  | 2.34  | 1.95  | 1.38     |
|             | $\infty$                 | 6.63  | 4.61  | 3.78  | 3.32  | 3.02  | 2.80  | 2.51  | 2.32  | 2.18  | 1.79  | 1.00     |

If an *upper* percentage point of the  $F$  distribution on  $\nu_1$  and  $\nu_2$  degrees of freedom is  $f$ , then the corresponding *lower* percentage point of the  $F$  distribution on  $\nu_2$  and  $\nu_1$  degrees of freedom is  $1/f$ .



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