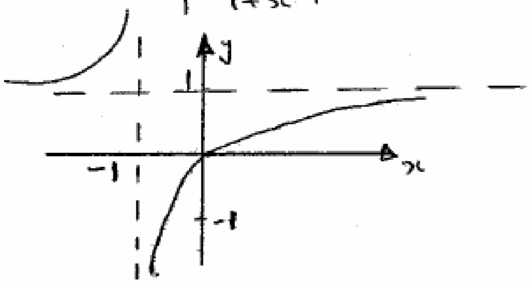
June 2006
9801 Advanced Extension Award
Mark Scheme

Question Number	Scheme	Marks
1 (a)	$(1-y)^{-2} = 1 + 2y + 3y^2 + 4y^3 + \dots$	B1 (1)
(b)	$S = 1 + 2\left(\frac{x}{1+x}\right) + 3\left(\frac{x}{1+x}\right)^2 + \dots$ Identify $y = \frac{x}{x+1}$	M1
	$\Rightarrow S = 1 + 2y + 3y^2 + \dots$	A1
	$= \left(1 - \frac{x}{1+x}\right)^{-2}$	
	$= \underline{(1+x)^{+2}}$, so $\underline{a = 1, n = 2}$	M1, A1 (4)
(c)	Need $\left\lvert \frac{x}{1+x} \right\rvert < 1$ correct condition	B1
	 Critical value is $\frac{x}{x+1} = -1$	M1
	$\Rightarrow x = -\frac{1}{2}$	
	$\therefore \underline{x > -\frac{1}{2}}$	A1 (3)

Question Number	Scheme	Marks
2	$\left(\sin 2\theta - \sqrt{3}\cos 2\theta\right)\left[\dfrac{2\cos 2\theta}{\sin\theta + \cos\theta} - \sqrt{6}\right] = 0$	M1 (factor)
	$\sin 2\theta - \sqrt{3}\cos 2\theta = 0 \Rightarrow \tan 2\theta = \sqrt{3}$ or $\sin(2\theta - 60°) = 0$	A1
	$\Rightarrow 2\theta = 60°, 240°, 420°, 600°$	M1
	$\underline{\theta = 30°, 120°, 210°, 300°}$	A1
	$\dfrac{2\cos 2\theta}{\sin\theta + \cos\theta} - \sqrt{6} = 0 \Rightarrow \dfrac{2(\cos^2\theta - \sin^2\theta)}{\sin\theta + \cos\theta} = \sqrt{6}$ (use of $\cos 2\theta = \ldots$)	M1
	$= \dfrac{2(\cos\theta - \sin\theta)\cancel{(\cos\theta + \sin\theta)}}{\cancel{(\cos\theta + \sin\theta)}} = \sqrt{6}$ (factors + cancel) $(\because \cos\theta + \sin\theta \neq 0)$	M1
	$\therefore \cos(\theta + 45°) = \dfrac{\sqrt{6}}{2}\cdot\dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{3}}{2}$	M1, A1 $R\cos(\theta + \alpha)$
	$\theta + 45° = (30), 330, 390$ (one correct value for $(\theta + 45°)$)	M1
	$\underline{\theta = 285°, 345°}$ (cao)	A1
		(10)

<table>
<tr><td>Qu. 3. (a)</td><td>$\log_y x = z \quad \therefore x = y^z$
$\therefore y = x^{1/z} \Rightarrow \log_x y = \frac{1}{z} = \frac{1}{\log_y x}$
<u>or</u> $\log_x x = \log_x y^z = z\log_x y = 1$
$\therefore \log_x y = \frac{1}{\log_y x}$ (Answer)</td><td>M1 (conv. of logs)
A1 c.s.o
(2)</td></tr>
<tr><td>(b)</td><td>$\log_x y = \log_y x = \frac{1}{\log_x y} \Rightarrow (\log_x y)^2 = 1$
$\therefore \log_x y = \pm 1$
$\log_x y \neq 1 \because x \neq y. \quad \therefore \log_x y = -1$
$\therefore \underline{y = \frac{1}{x}}$</td><td>M1
A1 c.s.o
(2)</td></tr>
<tr><td>(c)</td><td>First equation $\Rightarrow y = 1/x$
Second equation $\Rightarrow \log_x\left(x-\frac{1}{x}\right) = \log_{\frac{1}{x}}\left(x+\frac{1}{x}\right) = z$
$\therefore x^z = x - \frac{1}{x}$; $\left(\frac{1}{x}\right)^z = x + \frac{1}{x}$
$\therefore x^z\left(\frac{1}{x}\right)^z = 1 = \left(x-\frac{1}{x}\right)\left(x+\frac{1}{x}\right)$ (eliminate z)
$\Rightarrow x^2 = (x^2-1)(x^2+1)$
$\Rightarrow x^4 - x^2 - 1 = 0$ (quadratic)
$x^2 = (1 \pm \sqrt{5})/2$
$x^2 > 0$, hence $x^2 = \frac{1+\sqrt{5}}{2}$ (quad. formula + $x^2>0$)
$\underline{x = \sqrt{\frac{1+\sqrt{5}}{2}}}$ (value of +ve root)
$y = \frac{1}{x} = \underline{\sqrt{\frac{2}{1+\sqrt{5}}}}$ $\left(\text{or } \sqrt{\frac{\sqrt{5}-1}{2}}\right)$</td><td>M1
M1
M1, A1
M1
A1
A1✓
(7)</td></tr>
</table>

Qn 4

(a) $(x+4)^2 + (y-7)^2 = 13$ and $y = mx$

$\therefore (x^2+8x+16) + (m^2x^2 - 14mx + 49) = 13$ — M1

$(1+m^2)x^2 + (8-14m)x + 52 = 0$ — (3 t-quad) A1

Touches, so "$b^2 = 4ac$" $\quad (8-14m)^2 = 4 \times 52 \times (1+m^2)$ — M1 ($b^2=4ac$)

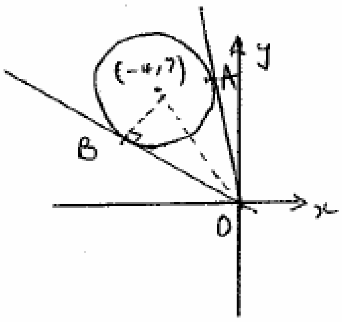
$(4-7m)^2 = 52 + 52m^2$

$\therefore \underline{\underline{3m^2 + 56m + 36 = 0}}$ * — A1 cso

(4)

(b) $(3m+2)(m+18) = 0$ — M1

$\therefore m = -2/3$ or -18 — (both m) A1



Let A or B be (X, Y)

then $(X^2+Y^2) + 13 = 4^2 + 7^2 = 65$

$X^2 + Y^2 = 52$ — M1, A1

$Y = -\frac{2}{3}X \Rightarrow \frac{13}{9}X^2 = 52 \Rightarrow X = \pm 6$

From the configuration $X_B = -6 \therefore Y_B = +4$: $\underline{\underline{B \text{ is } (-6, 4)}}$ — M1, A1

$Y = -18X \Rightarrow 325X^2 = 52 \Rightarrow X^2 = \frac{4}{25}$

Again $X < 0$ for A $\therefore X_A = -\frac{2}{5}$; $Y_A = \frac{36}{5}$

$\underline{\underline{A = \left(-\frac{2}{5}, \frac{36}{5}\right)}}$ — M1, A1

(8)

(c) Situation is a translation of problem in (b) by $\begin{pmatrix} 4 \\ -7 \end{pmatrix}$ — M1

So P, Q are $\begin{pmatrix} -6 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 \\ -7 \end{pmatrix}$ and $\left(-\frac{2}{5}, \frac{36}{5}\right) + (4, -7)$

$= \underline{\underline{(-2, -3)}}$ and $\underline{\underline{\left(\frac{18}{5}, \frac{1}{5}\right)}}$ — A1 (either)

(2)

Qu 5

(a) If L_1, L_2 intersect, then $\underline{r}_1 = \underline{r}_2$

$\underline{i} \Rightarrow -2+3\lambda = 11.5+7\mu \Rightarrow -13.5+3\lambda = 7\mu$ (1)

$\underline{j} \Rightarrow -11.5-4\lambda = 3+8\mu \Rightarrow -14.5-4\lambda = 8\mu$ (2)

$\underline{k} \Rightarrow -\lambda = 8.5-11\mu \Rightarrow 8.5+\lambda = 11\mu$ (3)

(Any one eqn) M1

Solve any pair of these (Attempt to solve 2) M1

(1) & (2) $\Rightarrow \lambda = 1/8,\ \mu = -15/8$

(1) & (3) $\Rightarrow \lambda = 8,\ \mu = 3/2$ (Soln for any 2 eqns) A1

(2) & (3) $\Rightarrow \lambda = \frac{-35}{8},\ \mu = 3/8$

Check in third equation $\Rightarrow$ inconsistency (check) M1

Hence L_1, L_2 do not intersect (comment) B1 **(5)**

(b) $\begin{pmatrix}3\\-4\\-1\end{pmatrix}\cdot\begin{pmatrix}2\\1\\2\end{pmatrix} = 6-4-2 = 0 \therefore \begin{pmatrix}2\\1\\2\end{pmatrix} \perp^r L_1$ (A correct product) M1

$\begin{pmatrix}7\\8\\-11\end{pmatrix}\cdot\begin{pmatrix}2\\1\\2\end{pmatrix} = 14+8-22 = 0 \therefore \begin{pmatrix}2\\1\\2\end{pmatrix} \perp^r L_2$ (both) A1 **(2)**

(c) $\overrightarrow{AB} = \begin{pmatrix}7\mu+11.5-(-2+3\lambda)\\8\mu+3-(-11.5-4\lambda)\\-11\mu+8.5-(-\lambda)\end{pmatrix} = \begin{pmatrix}7\mu-3\lambda+13.5\\8\mu+4\lambda+14.5\\-11\mu+\lambda+8.5\end{pmatrix}$ (Form $\overrightarrow{AB}$) M1

$\overrightarrow{AB} \parallel \begin{pmatrix}2\\1\\2\end{pmatrix} \quad \therefore$ "$\underline{i}$" = "$\underline{k}$" $\Rightarrow 7\mu-3\lambda+\frac{27}{2} = -11\mu+\lambda+\frac{17}{2}$ M1

$\Rightarrow 18\mu-4\lambda+5=0$ (4) A1

"$2\underline{j}$" = "$\underline{i}$" $\Rightarrow 16\mu+8\lambda+29 = 7\mu-3\lambda+13.5$ M1

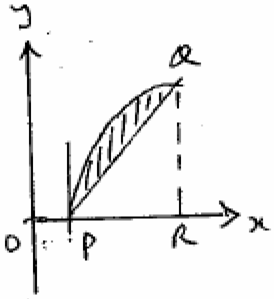
$\Rightarrow 9\mu+11\lambda+15.5=0$ (5) A1

[Note "$2\underline{j}$" = "$\underline{k}$" $\Rightarrow 27\mu+7\lambda+20.5=0$]

Solve (4) & (5) (or any 2 eqns) $\Rightarrow \underline{\lambda = -1\ ;\ \mu = -1/2}$ (solving) M1

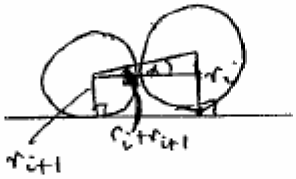
$\therefore \overrightarrow{OA} = \begin{pmatrix}-5\\-7.5\\1\end{pmatrix} \qquad \overrightarrow{OB} = \begin{pmatrix}8\\-1\\14\end{pmatrix}$ A1, A1

(8)

Qn 6	(a) $x=1$; $y=\sin(\ln 1)=\sin 0=0$ $\therefore$ $P=(1,0)$ and P lies on C	B1 c.s.o (1
(b)	$y'=\frac{1}{x}\cos(\ln x)$	M1, A1
	$y'=0$ at Q $\therefore \cos(\ln x)=0$ $\therefore$ $\ln x=\pi/2$ $x=e^{\pi/2}$	M1
	$\therefore Q=\left(e^{\pi/2}, \sin(\ln e^{\pi/2})\right)$	A1
	$=\left(e^{\pi/2},1\right)$	A1 (5
(c)	 $\text{Area}=\int_1^{e^{\pi/2}}\sin(\ln x)\,dx-\text{Area }\Delta PQR$ (correct approach)	M1
	$\text{Area }\Delta PQR=\frac{1}{2}\times 1\times\left(e^{\pi/2}-1\right)$	B1
	For integral; let $\ln x=u$ $\therefore x=e^u$ $\frac{1}{x}dx=du$ $\therefore$ $dx=e^u\,du$ (subst.)	M1
	$I=\int_0^{\pi/2}\sin u.(e^u\,du)$ $(\int \text{in } u)$	A1
	$=\left[e^u\sin u\right]_0^{\pi/2}-\int e^u\cos u\,du$ (limits) (parts)	M1 M1
	$=e^{\pi/2}-\left[e^u\cos u\right]_0^{\pi/2}-\int e^u\sin u\,du$ (second parts)	M1
	$\therefore 2I=e^{\pi/2}+1$ $I=\frac{1}{2}\left(1+e^{\pi/2}\right)$ (I)	A1
	$\therefore \text{Area}=\frac{1}{2}\left(1+e^{\pi/2}\right)-\frac{1}{2}\left(-1+e^{\pi/2}\right)=\underline{\underline{1}}$	A1
		(9)

Qu 7.

(a)



Appropriate figure — M1

$\Rightarrow \sin\alpha = \dfrac{r_i - r_{i+1}}{r_i + r_{i+1}}$ (exp. for $\sin\alpha$) — M1, A1

$\therefore (r_i + r_{i+1}) \sin\alpha = r_i - r_{i+1}$

$\therefore r_{i+1}(1+\sin\alpha) = r_i(1-\sin\alpha)$ $\left(\frac{r_{i+1}}{r_i}\right)$ — M1

$\therefore$ ratio of radii $= \underline{\underline{\dfrac{1-\sin\alpha}{1+\sin\alpha}}}$ (∗) $(=r)$ — A1 cso

(5)

(b) Total area $= \pi R^2 + \pi r_2^2 + \pi r_3^2 + \cdots$

$= \pi R^2 (1 + r^2 + r^4 + \cdots)$ (correct "r") — B1

$= \dfrac{\pi R^2}{1-r^2} = \pi R^2 \dfrac{1}{1 - \left(\frac{1-\sin\alpha}{1+\sin\alpha}\right)^2}$ — M1

$= \dfrac{\pi R^2 (1+\sin\alpha)^2}{(1+\sin\alpha)^2 - (1-\sin\alpha)^2} = \underline{\underline{\dfrac{\pi R^2 (1+\sin\alpha)^2}{4\sin\alpha}}}$ — A1

(3)

(c) Required area $= 2 \times$ Area ΔPOA + Area major sector AOB — M1

$-$ Area found in (b).

Area $\Delta POA = \frac{1}{2} R . (R\cot\alpha)$ — B1

$\angle POA = \pi/2 - \alpha$ $\therefore$ angle of major sector $AOB = \pi + 2\alpha$ — M1

$\therefore$ Area sector $AOB = \frac{1}{2} R^2 (\pi + 2\alpha)$ — A1

$\therefore$ Required area $= R^2\left(\cot\alpha + \frac{\pi}{2} + \alpha - \frac{\pi}{4}\left(\frac{1 + 2\sin\alpha + \sin^2\alpha}{\sin\alpha}\right)\right)$

$= \underline{\underline{R^2\left(\alpha + \cot\alpha - \frac{\pi}{4}\operatorname{cosec}\alpha - \frac{\pi}{4}\sin\alpha\right)}}$ (∗) — A1 cso

(5)

Qn 7 (cont.)	(d) $\frac{dS}{d\alpha} = R^2\left(1 - \operatorname{cosec}^2\alpha + \frac{\pi}{4}\operatorname{cosec}\alpha \cot\alpha - \frac{\pi}{4}\cos\alpha\right)$	M1, A1
	$= R^2\left(-\cot^2\alpha + \frac{\pi}{4}\frac{\cos\alpha}{\sin^2\alpha} - \frac{\pi}{4}\cos\alpha\right)$	
	$= R^2\left(-\cot^2\alpha + \frac{\pi}{4}\cos\alpha\left(\operatorname{cosec}^2\alpha - 1\right)\right)$	M1
	$= \underline{R^2\cot^2\alpha\left(\frac{\pi}{4}\cos\alpha - 1\right)}$ (use of $\cot^2\alpha = \operatorname{cosec}^2\alpha - 1$) o.e.	
		(cso) A1 (4)
	(e) In the given range $R^2\cot^2\alpha > 0$	
	In the interval $(0, \pi/4)$; $\frac{\pi}{4}\cos\alpha - 1$ is a decreasing function ($\because \cos\alpha$ is decreasing).	
	At $\alpha = 0$, $\pi/4 \cos\alpha - 1 = \pi/4 - 1 < 0$	
	$\therefore$ $\pi/4\cos\alpha - 1 < 0$ in $(\pi/6, \pi/4)$	
	$\therefore$ $\frac{dS}{d\alpha} < 0$ throughout the interval. (convincing argument)	M1
	$\therefore$ Least value of S occurs at $\alpha = \pi/4$	A1
	Min $S = R^2\left(\frac{\pi}{4} + 1 - \frac{\pi}{4}\cdot\sqrt{2} - \frac{\pi}{4}\cdot\frac{1}{\sqrt{2}}\right)$	
	$= \underline{R^2\left(1 - \frac{\pi}{4}\left(-1 + \sqrt{2} + \frac{1}{\sqrt{2}}\right)\right)}$ o.e.	A1
		(3)