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Introduction

The Pearson Edexcel Level 3 Advanced GCE in Further Mathematics is designed for use in schools and colleges. It is part of a suite of AS/A Level qualifications offered by Pearson. These sample assessment materials have been developed to support this qualification and will be used as the benchmark to develop the assessment students will take.

The booklet ‘Mathematical Formulae and Statistical Tables’ will be provided for use with these assessments and can be downloaded from our website, qualifications.pearson.com.
General marking guidance

- All candidates must receive the same treatment. Examiners must mark the last candidate in exactly the same way as they mark the first.
- Mark schemes should be applied positively. Candidates must be rewarded for what they have shown they can do rather than be penalised for omissions.
- Examiners should mark according to the mark scheme – not according to their perception of where the grade boundaries may lie.
- All the marks on the mark scheme are designed to be awarded. Examiners should always award full marks if deserved, i.e. if the answer matches the mark scheme. Examiners should also be prepared to award zero marks if the candidate’s response is not worthy of credit according to the mark scheme.
- Where some judgement is required, mark schemes will provide the principles by which marks will be awarded and exemplification/indicative content will not be exhaustive. However different examples of responses will be provided at standardisation.
- When examiners are in doubt regarding the application of the mark scheme to a candidate’s response, a senior examiner must be consulted before a mark is given.
- Crossed-out work should be marked unless the candidate has replaced it with an alternative response.

Specific guidance for mathematics

1. These mark schemes use the following types of marks:
   - **M** marks: Method marks are awarded for ‘knowing a method and attempting to apply it’, unless otherwise indicated.
   - **A** marks: Accuracy marks can only be awarded if the relevant method (M) marks have been earned.
   - **B** marks are unconditional accuracy marks (independent of M marks)
   - Marks should not be subdivided.

2. Abbreviations
   These are some of the traditional marking abbreviations that may appear in the mark schemes.
   - **bod** benefit of doubt
   - **ft** follow through
   - **✓** this symbol is used for correct ft
   - **cao** correct answer only
   - **cso** correct solution only. There must be no errors in this part of the question to obtain this mark
   - **isw** ignore subsequent working
   - **awrt** answers which round to
   - **SC:** special case
   - **o.e.** or equivalent (and appropriate)
   - **d...** dependent or dep
   - **indep** independent
   - **dp** decimal places
   - **sf** significant figures
   - *** The answer is printed on the paper or ag- answer given

Pearson Edexcel Level 3 Advanced GCE in Further Mathematics
3. All M marks are follow through.

All A marks are ‘correct answer only’ (cao.), unless shown, for example, as A1 ft to indicate that previous wrong working is to be followed through. After a misread however, the subsequent A marks affected are treated as A ft, but answers that don’t logically make sense e.g. if an answer given for a probability is >1 or <0, should never be awarded A marks.

4. For misreading which does not alter the character of a question or materially simplify it, deduct two from any A or B marks gained, in that part of the question affected.

5. Where a candidate has made multiple responses and indicates which response they wish to submit, examiners should mark this response. If there are several attempts at a question which have not been crossed out, examiners should mark the final answer which is the answer that is the most complete.

6. Ignore wrong working or incorrect statements following a correct answer.

7. Mark schemes will firstly show the solution judged to be the most common response expected from candidates. Where appropriate, alternative answers are provided in the notes. If examiners are not sure if an answer is acceptable, they will check the mark scheme to see if an alternative answer is given for the method used. If no such alternative answer is provided but deemed to be valid, examiners must escalate the response to a senior examiner to review.
Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use black ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- Fill in the boxes at the top of this page with your name, centre number and candidate number.
- Answer all questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided — there may be more space than you need.
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information

- A booklet ‘Mathematical Formulae and Statistical Tables’ is provided.
- There are 9 questions in this question paper. The total mark for this paper is 75.
- The marks for each question are shown in brackets — use this as a guide as to how much time to spend on each question.

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
Answer ALL questions. Write your answers in the spaces provided.

1. Prove that

$$\sum_{r=1}^{n} \frac{1}{(r+1)(r+3)} = \frac{n(an + b)}{12(n + 2)(n + 3)}$$

where $a$ and $b$ are constants to be found.

(5)
2. Prove by induction that for all positive integers \( n \),

\[
f(n) = 2^{3n+1} + 3(5^{2n+1})
\]

is divisible by 17
is divisible by 17

2. Prove by induction that for all positive integers \( n \),

\[
f(n, n+1) = \frac{4}{3}n^3 + \frac{1}{3}n^2 + \frac{1}{3}n + \frac{2}{3}
\]

(Total for Question 2 is 6 marks)
3. \( f(z) = z^4 + az^3 + 6z^2 + bz + 65 \)
where \( a \) and \( b \) are real constants.

Given that \( z = 3 + 2i \) is a root of the equation \( f(z) = 0 \), show the roots of \( f(z) = 0 \) on a single Argand diagram.

(9)
where single Argand diagram. 

$z = 3 + 2i$ and $a$ and $b$ are roots of the equation $f(z) = z^4 + az^2 + 65 = 0$, show the roots of $f(z) = z^2 + b)$ on a

(Total for Question 3 is 9 marks)
4.

The curve $C$ shown in Figure 1 has polar equation

$$r = 4 + \cos 2\theta \quad 0 \leq \theta \leq \frac{\pi}{2}$$

At the point $A$ on $C$, the value of $r$ is $\frac{9}{2}$.

The point $N$ lies on the initial line and $AN$ is perpendicular to the initial line.

The finite region $R$, shown shaded in Figure 1, is bounded by the curve $C$, the initial line and the line $AN$.

Find the exact area of the shaded region $R$, giving your answer in the form $p\pi + q\sqrt{3}$ where $p$ and $q$ are rational numbers to be found.

(9)
The curve shown in Figure 1 has polar equation

\[ C + R \angle \theta \]

giving your answer in the form

\[ AN \]

and the line, shown shaded in Figure 1, is bounded by the curve

\[ R \angle \theta \]

lies on the initial line and

\[ \theta = \frac{\pi}{2} \]

\[ \theta = 4 + \cos 2 \theta \]

Figure 1

Initial line

Question 4 continued

(Total for Question 4 is 9 marks)
5. A pond initially contains 1000 litres of unpolluted water.

The pond is leaking at a constant rate of 20 litres per day.

It is suspected that contaminated water flows into the pond at a constant rate of 25 litres per day and that the contaminated water contains 2 grams of pollutant in every litre of water.

It is assumed that the pollutant instantly dissolves throughout the pond upon entry.

Given that there are $x$ grams of the pollutant in the pond after $t$ days,

(a) show that the situation can be modelled by the differential equation,

\[
\frac{dx}{dt} = 50 - \frac{4x}{200 + t}
\]

(b) Hence find the number of grams of pollutant in the pond after 8 days.

(c) Explain how the model could be refined.
Question 5 continued
6. 

\[ f(x) = \frac{x + 2}{x^2 + 9} \]

(a) Show that

\[ \int f(x)\,dx = A \ln(x^2 + 9) + B \arctan\left(\frac{x}{3}\right) + c \]

where \( c \) is an arbitrary constant and \( A \) and \( B \) are constants to be found. \hspace{1cm} (4)

(b) Hence show that the mean value of \( f(x) \) over the interval \([0, 3]\) is

\[ \frac{1}{6} \ln 2 + \frac{1}{18} \pi \]

\hspace{1cm} (3)

(c) Use the answer to part (b) to find the mean value, over the interval \([0, 3]\), of

\[ f(x) + \ln k \]

where \( k \) is a positive constant, giving your answer in the form \( p + \frac{1}{6} \ln q \),

where \( p \) and \( q \) are constants and \( q \) is in terms of \( k \). \hspace{1cm} (2)
Question 6 continued

(Total for Question 6 is 9 marks)
Figure 2 shows the image of a gold pendant which has height 2 cm. The pendant is modelled by a solid of revolution of a curve $C$ about the $y$-axis. The curve $C$ has parametric equations

$$x = \cos\theta + \frac{1}{2}\sin 2\theta, \quad y = -(1 + \sin \theta) \quad 0 \leq \theta \leq 2\pi$$

(a) Show that a Cartesian equation of the curve $C$ is

$$x^2 = -(y^4 + 2y^3)$$

(4)

(b) Hence, using the model, find, in cm$^3$, the volume of the pendant.

(4)
Question 7 continued
(Total for Question 7 is 8 marks)
8. The line \( l_1 \) has equation \( \frac{x - 2}{4} = \frac{y - 4}{-2} = \frac{z + 6}{1} \)

The plane \( \Pi \) has equation \( x - 2y + z = 6 \)

The line \( l_2 \) is the reflection of the line \( l_1 \) in the plane \( \Pi \).

Find a vector equation of the line \( l_2 \) 

(7)
Question 8 continued
Question 8 continued

(Total for Question 8 is 7 marks)
9. A company plans to build a new fairground ride. The ride will consist of a capsule that will hold the passengers and the capsule will be attached to a tall tower. The capsule is to be released from rest from a point half way up the tower and then made to oscillate in a vertical line.

The vertical displacement, \( x \) metres, of the top of the capsule below its initial position at time \( t \) seconds is modelled by the differential equation,

\[
m\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + x = 200\cos t, \quad t \geq 0
\]

where \( m \) is the mass of the capsule including its passengers, in thousands of kilograms.

The maximum permissible weight for the capsule, including its passengers, is 30000 N.

Taking the value of \( g \) to be 10 \( \text{ms}^{-2} \) and assuming the capsule is at its maximum permissible weight,

(a) (i) explain why the value of \( m \) is 3

(ii) show that a particular solution to the differential equation is

\[
x = 40\sin t - 20\cos t
\]

(iii) hence find the general solution of the differential equation.

(8)

(b) Using the model, find, to the nearest metre, the vertical distance of the top of the capsule from its initial position, 9 seconds after it is released.

(4)
Question 9 continued
Question 9 continued
Question 9 continued

(Total for Question 9 is 12 marks)

TOTAL FOR PAPER IS 75 MARKS
Paper 1: Core Pure Mathematics

Mark Scheme

Question Scheme Marks AOs

1. \( \sum_{r=1}^{n} r(r+2) = \frac{1}{2}n(n+2)(n+1) \)

M1 3.1a

2. \( \sum_{r=1}^{n} (2r-3) = n^2 - 3n \)

M1 2.1

3. \( \sum_{r=1}^{n} (3r-2) = \frac{1}{2}n(3n-1) \)

M1 1.1b

A1 2.2a

Alternative by induction:

\( a \) and \( b \) are given such that \( ab = 2 \) and \( a + b = 1 \).

M1 3.1a

Assume true for \( n = k \) so

\( \sum_{r=1}^{k} r(r+2) = \frac{1}{2}k(k+2)(k+1) \)

M1 2.1

\( \sum_{r=1}^{k+1} r(r+2) = \sum_{r=1}^{k} r(r+2) + (k+1)(k+3) \)

M1 1.1b

So true for \( n = k+1 \)

A1 1.1b

(5 marks)
### Paper 1: Core Pure Mathematics 1 Mark Scheme

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<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>( \frac{1}{(r+1)(r+3)} = \frac{A}{(r+1)} + \frac{B}{(r+3)} \Rightarrow A = ..., B = ... )</td>
<td>M1 3.1a</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \sum_{r=1}^{n} \frac{1}{(r+1)(r+3)} = \frac{1}{2 \times 2} + \frac{1}{2 \times 4} + \frac{1}{2 \times 3} + ... + \frac{1}{2n} - \frac{1}{2(n+2)} + \frac{1}{2(n+1)} - \frac{1}{2(n+3)} )</td>
<td>M1 2.1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>= ( \frac{1}{4} + \frac{1}{6} - \frac{1}{2(n+2)} - \frac{1}{2(n+3)} )</td>
<td>A1 2.2a</td>
<td></td>
</tr>
<tr>
<td></td>
<td>= ( \frac{5(n+2)(n+3) - 6(n+3) - 6(n+2)}{12(n+2)(n+3)} )</td>
<td>M1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>= ( \frac{n(5n+13)}{12(n+2)(n+3)} )</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td><strong>(5)</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td><strong>Alternative by induction:</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( n = 1 \Rightarrow \frac{1}{8} = \frac{a+b}{12 \times 3 \times 4} ), ( n = 2 \Rightarrow \frac{1}{15} = \frac{1}{8} + \frac{1}{12 \times 4 \times 5} )</td>
<td>M1 3.1a</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( a+b = 18, \ 2a+b = 23 \Rightarrow a = ..., b = ... )</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Assume true for ( n = k ) so ( \sum_{r=1}^{k} \frac{1}{(r+1)(r+3)} = \frac{k(5k+13)}{12(k+2)(k+3)} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \sum_{r=1}^{k+1} \frac{1}{(r+1)(r+3)} = \frac{k(5k+13)}{12(k+2)(k+3)} + \frac{1}{(k+2)(k+4)} )</td>
<td>M1 2.1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \frac{k(5k+13)}{12(k+2)(k+3)} + \frac{1}{(k+2)(k+4)} = \frac{k(5k+13)(k+4)+12(k+3)}{12(k+2)(k+3)(k+4)} )</td>
<td>A1 2.2a</td>
<td></td>
</tr>
<tr>
<td></td>
<td>= ( \frac{5k^3 + 33k^2 + 52k + 12k + 36}{12(k+2)(k+3)(k+4)} = \frac{(k+1)(k+2)(5k+18)}{12(k+2)(k+3)(k+4)} )</td>
<td>M1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>= ( \frac{(k+1)(5(k+1)+13)}{12(k+1+2)(k+1+3)} )</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>So true for ( n = k+1 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>So ( \sum_{r=1}^{n} \frac{1}{(r+1)(r+3)} = \frac{n(5n+13)}{12(n+2)(n+3)} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(5)</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(5 marks)</strong></td>
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**Question 1 notes:**

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<th><strong>Main Scheme</strong></th>
<th><strong>M1:</strong> Valid attempt at partial fractions</th>
</tr>
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<tbody>
<tr>
<td><strong>M1:</strong></td>
<td>Starts the process of differences to identify the relevant fractions at the start and end</td>
</tr>
<tr>
<td><strong>A1:</strong></td>
<td>Correct fractions that do not cancel</td>
</tr>
<tr>
<td><strong>M1:</strong></td>
<td>Attempt common denominator</td>
</tr>
<tr>
<td><strong>A1:</strong></td>
<td>Correct answer</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Alternative by Induction:</strong></th>
<th><strong>M1:</strong> Uses $n = 1$ and $n = 2$ to identify values for $a$ and $b$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>M1:</strong></td>
<td>Starts the induction process by adding the $(k + 1)^{th}$ term to the sum of $k$ terms</td>
</tr>
<tr>
<td><strong>A1:</strong></td>
<td>Correct single fraction</td>
</tr>
<tr>
<td><strong>M1:</strong></td>
<td>Attempt to factorise the numerator</td>
</tr>
<tr>
<td><strong>A1:</strong></td>
<td>Correct answer and conclusion</td>
</tr>
</tbody>
</table>

When $n = 1$,

$$31 + 2(1) + 2 + 3 = 37,$$

so the statement is true for $n = 1$.

Assume true for $n = k$ so

$$31 + 2(1) + 2 + 3 = 37$$

is divisible by 17.

$$(k + 1)^{th} \text{ term} = 34 + 2(k + 1) + 2 + 3 = 37 + 2k$$

M1

$$(k + 1)^{th} \text{ term} + 17f(k + 1) = 37 + 2k + 17f(k + 1)$$

M1

$$= 37 + 2k + 17f(k) + 17f(k + 1)$$

A1

$$= 37 + 2k + 17f(k) + 17f(k + 1)$$

A1

If the statement is true for $n = k$, then it has been shown true for $n = k + 1$ and as it is true for $n = 1$, the statement is true for all positive integers $n$. A1

(6 marks)
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<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
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</thead>
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<tr>
<td>2</td>
<td>When ( n = 1 ), ( 2^{3n+1} + 3\left(5^{2n+1}\right) = 16 + 375 = 391 ) 391 = 17 \times 23 ) so the statement is true for ( n = 1 )</td>
<td>B1</td>
<td>2.2a</td>
</tr>
<tr>
<td></td>
<td>Assume true for ( n = k ) so ( 2^{3k+1} + 3\left(5^{2k+1}\right) ) is divisible by 17</td>
<td>M1</td>
<td>2.4</td>
</tr>
<tr>
<td></td>
<td>( f(k+1) - f(k) = 2^{3k+4} + 3\left(5^{2k+3}\right) - 2^{3k+1} - 3\left(5^{2k+1}\right) )</td>
<td>M1</td>
<td>2.1</td>
</tr>
<tr>
<td></td>
<td>( = 7 \times 2^{3k+1} + 7 \times 3\left(5^{2k+1}\right) + 17 \times 3\left(5^{2k+1}\right) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( = 7f(k) + 17 \times 3\left(5^{2k+1}\right) )</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>( f(k+1) = 8f(k) + 17 \times 3\left(5^{2k+1}\right) )</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>If the statement is true for ( n = k ) then it has been shown true for ( n = k + 1 ) and as it is true for ( n = 1 ), the statement is true for all positive integers ( n )</td>
<td>A1</td>
<td>2.4</td>
</tr>
</tbody>
</table>

(6 marks)

Notes:

- **B1:** Shows the statement is true for \( n = 1 \)
- **M1:** Assumes the statement is true for \( n = k \)
- **M1:** Attempts \( f(k+1) - f(k) \)
- **A1:** Correct expression in terms of \( f(k) \)
- **A1:** Correct expression in terms of \( f(k) \)
- **A1:** Obtains a correct expression for \( f(k+1) \)
- **A1:** Correct complete conclusion
3. \( z = 3 - 2i \) is also a root

\[(z - (3 + 2i))(z - (3 - 2i)) = \ldots \]

or

\( z^2 - 6z + 13 \)

Sum of roots = 6, Product of roots = 13 \( \Rightarrow \ldots \)

\( z^4 + az^3 + bz^2 + 6z + 65 = (z^2 - 6z + 13)(z^2 + cz + 5) \Rightarrow c = \ldots \)

\( z^2 + 2z + 5 = 0 \)

\( z^2 + 2z + 5 = 0 \Rightarrow z = \ldots \)

\( z = -1 \pm 2i \)

Notes:

**B1:** Identifies the complex conjugate as another root

**M1:** Uses the conjugate pair and a correct method to find a quadratic factor

**A1:** Correct quadratic

**M1:** Uses the given quartic and their quadratic to identify the value of \( c \)

**A1:** Correct 3TQ

**M1:** Solves their second quadratic

**A1:** Correct second conjugate pair

**B1:** First conjugate pair plotted correctly and labelled

**B1ft:** Second conjugate pair plotted correctly and labelled (Follow through their second conjugate pair)
<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>(4 + \cos 2\theta = \frac{9}{2} \Rightarrow \theta = \ldots)</td>
<td>M1</td>
<td>3.1a</td>
</tr>
<tr>
<td></td>
<td>(\theta = \frac{\pi}{6})</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>(\frac{1}{2} \int (4 + \cos 2\theta)^2 , d\theta = \frac{1}{2} \int (16 + 8\cos 2\theta + \cos^2 2\theta) , d\theta)</td>
<td>M1</td>
<td>3.1a</td>
</tr>
<tr>
<td></td>
<td>(\cos^2 2\theta = \frac{1}{2} + \frac{1}{2}\cos 4\theta \Rightarrow A = \frac{1}{2} \int (16 + 8\cos 2\theta + \frac{1}{2} + \frac{1}{2}\cos 4\theta) , d\theta)</td>
<td>M1</td>
<td>3.1a</td>
</tr>
<tr>
<td></td>
<td>(= \frac{1}{2} \left[16\theta + 4\sin 2\theta \cdot \frac{\sin 4\theta}{\frac{\theta}{8}} + \frac{\theta}{2}\right])</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>Using limits 0 and their (\frac{\pi}{6}): (\frac{1}{2} \left[\frac{33\pi}{12} + 2\sqrt{3} + \frac{\sqrt{3}}{16} - (0)\right])</td>
<td>M1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>Area of triangle = (\frac{1}{2} (r \cos \theta)(r \sin \theta) = \frac{1}{2} \cdot \frac{81}{4} \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2})</td>
<td>M1</td>
<td>3.1a</td>
</tr>
<tr>
<td></td>
<td>Area of (R = \frac{33\pi}{24} + \frac{33\sqrt{3}}{32} - \frac{81\sqrt{3}}{32})</td>
<td>M1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>(= \frac{11}{8} \pi - \frac{3\sqrt{3}}{2}\left(p = \frac{11}{8}, q = -\frac{3}{2}\right))</td>
<td>A1</td>
<td>1.1b</td>
</tr>
</tbody>
</table>

(9 marks)

**Notes:**
- **M1:** Realises the angle for \(A\) is required and attempts to find it
- **A1:** Correct angle
- **M1:** Uses a correct area formula and squares \(r\) to achieve a 3TQ integrand in \(\cos 2\theta\)
- **M1:** Use of the correct double angle identity on the integrand to achieve a suitable form for integration
- **A1:** Correct integration
- **M1:** Correct use of limits
- **M1:** Identifies the need to subtract the area of a triangle and so finds the area of the triangle
- **M1:** Complete method for the area of \(R\)
- **A1:** Correct final answer
### Question 5(a)

Pond contains \(1000 + 5t\) litres after \(t\) days

If \(x\) is the amount of pollutant in the pond after \(t\) days

Rate of pollutant out = \(20 \times \frac{x}{1000+5t}\) g per day

Rate of pollutant in = \(25 \times 2\) g = 50g per day

\[
\frac{dx}{dt} = 50 - \frac{4x}{200+t} \quad \text{M1 3.3}
\]

\[
(b)
\]

\[
I = e^{\int \frac{4}{200+t} dt} = (200+t)^4 \quad \Rightarrow \quad x(200+t)^4 = \int 50(200+t)^4 \, dt \quad \text{M1 3.1b}
\]

\[
x(200+t)^4 = 10(200+t)^3 + c \quad \text{A1 1.1b}
\]

\[
x = 0, \quad t = 0 \Rightarrow \quad c = -3.2 \times 10^{12} \quad \text{M1 3.4}
\]

\[
t = 8 \Rightarrow \quad x = 10(200+8) - \frac{3.2 \times 10^{12}}{(200+8)^3} \quad \text{M1 1.1b}
\]

\[
= 370g \quad \text{A1 2.2b}
\]

\[
(c)
\]

- The model should take into account the fact that the pollutant does not dissolve throughout the pond upon entry
- The rate of leaking could be made to vary with the volume of water in the pond

\[
(1) \quad \text{B1 3.5c}
\]

### Notes:

(a)

- **M1**: Forms an expression of the form \(1000 + kt\) for the volume of water in the pond at time \(t\)
- **M1**: Expresses the amount of pollutant out in terms of \(x\) and \(t\)
- **B1**: Correct interpretation for pollutant entering the pond
- **A1***: Puts all the components together to form the correct differential equation

(b)

- **M1**: Uses the model to find the integrating factor and attempts solution of their differential equation
- **A1**: Correct solution
- **M1**: Interprets the initial conditions to find the constant of integration
- **M1**: Uses their solution to the problem to find the amount of pollutant after 8 days
- **A1**: Correct number of grams

(c)

- **B1**: Suggests a suitable refinement to the model
<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>[ f(x) = \frac{x+2}{x^2+9} = \frac{x}{x^2+9} + \frac{2}{x^2+9} ]</td>
<td>B1</td>
<td>3.1a</td>
</tr>
<tr>
<td></td>
<td>[ \int \frac{x}{x^2+9} , dx = k \ln(x^2+9)(+c) ]</td>
<td>M1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>[ \int \frac{2}{x^2+9} , dx = k \arctan\left(\frac{x}{3}\right)(+c) ]</td>
<td>M1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>[ \int \frac{x+2}{x^3+9} , dx = \frac{1}{2} \ln(x^2+9) + \frac{2}{3} \arctan\left(\frac{x}{3}\right) + c ]</td>
<td>A1</td>
<td>1.1b</td>
</tr>
</tbody>
</table>

(4)

| (b) | \[ \int_0^3 f(x) \, dx = \left[ \frac{1}{2} \ln(x^2+9) + \frac{2}{3} \arctan\left(\frac{x}{3}\right) \right]_0^3 \] | M1 | 1.1b |
| | \[ = \frac{1}{2} \ln 18 + \frac{2}{3} \arctan\left(\frac{3}{3}\right) - \left(\frac{1}{2} \ln 9 + \frac{2}{3} \arctan(0)\right) \] | M1 | 1.1b |
| | \[ = \frac{1}{2} \ln 18 + \frac{2}{3} \arctan\left(\frac{3}{3}\right) \] | M1 | 2.1 |
| | Mean value = \[ \frac{1}{3-0} \left( \frac{1}{2} \ln 2 + \frac{\pi}{6} \right) \] | A1* | 2.2a |
| | \[ = \frac{1}{6} \ln 2 + \frac{1}{18} \pi \] | A1* | 2.2a |

(3)

| (c) | \[ \frac{1}{6} \ln 2 + \frac{1}{18} \pi + \ln k \] | M1 | 2.2a |
| | \[ = \frac{1}{6} \ln 2k^6 + \frac{1}{18} \pi \] | A1 | 1.1b |

(2)

(9 marks)

**Notes:**

(a)  
**B1:** Splits the fraction into two correct separate expressions  
**M1:** Recognises the required form for the first integration  
**M1:** Recognises the required form for the second integration  
**A1:** Both expressions integrated correctly and added together with constant of integration included

(b)  
**M1:** Uses limits correctly and combines logarithmic terms  
**M1:** Correctly applies the method for the mean value for their integration  
**A1*:** Correct work leading to the given answer

(c)  
**M1:** Realises that the effect of the transformation is to increase the mean value by \( \ln k \)  
**A1:** Combines ln’s correctly to obtain the correct expression
Question Scheme Marks AOs

7(a) \[ x = \cos \theta + \sin \theta \cos \theta = -y \cos \theta \] \[ \sin \theta = -y - 1 \] \[ \left( \frac{x}{-y} \right)^2 = 1 - (-y - 1)^2 \] \[ x^2 = -(y^4 + 2y^3)^* \] M1 2.1 M1 2.1 M1 2.1 A1* 1.1b

(b) \[ V = \pi \int x^2 \, dy = \pi \int -(y^4 + 2y^3) \, dy \] = \[ \pi \left[ -\left( \frac{y^5}{5} + \frac{y^4}{2} \right) \right] \] = \[ -\pi \left[ \left( \frac{0}{5} + \frac{0}{2} \right) - \left( \frac{-2}{5} + \frac{(-2)^4}{2} \right) \right] \] \[ = 1.6\pi \text{ cm}^3 \text{ or } 5.03 \text{ cm}^3 \] M1 3.4 A1 1.1b M1 3.4 A1 1.1b

Notes:
(a) M1: Obtains \( x \) in terms of \( y \) and \( \cos \theta \)
M1: Obtains an equation connecting \( y \) and \( \sin \theta \)
M1: Uses Pythagoras to obtain an equation in \( x \) and \( y \) only
A1*: Obtains printed answer

(b) M1: Uses the correct volume of revolution formula with the given expression
A1: Correct integration
M1: Correct use of correct limits
A1: Correct volume
<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>$2 + 4\lambda - 2(4 - 2\lambda) - 6 + \lambda = 6 \Rightarrow \lambda = ...$</td>
<td>M1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>$\lambda = 2 \Rightarrow \text{Required point is } (2 + 2(4), 4 + 2(-2), -6 + 2(1))$</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>$(10, 0, -4)$</td>
<td>M1</td>
<td>3.1a</td>
</tr>
<tr>
<td></td>
<td>$2 + t - 2(4 - 2t) - 6 + t = 6 \Rightarrow t = ...$</td>
<td>M1</td>
<td>3.1a</td>
</tr>
<tr>
<td></td>
<td>$t = 3 \text{so reflection of } (2, 4, -6) \text{is } (2 + 6(1), 4 + 6(-2), -6 + 6(1))$</td>
<td>M1</td>
<td>3.1a</td>
</tr>
<tr>
<td></td>
<td>$(8, -8, 0)$</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>$\begin{bmatrix} 10 \ 0 \ -4 \end{bmatrix} - \begin{bmatrix} 8 \ -8 \ 0 \end{bmatrix} = \begin{bmatrix} 2 \ 8 \ -4 \end{bmatrix}$</td>
<td>M1</td>
<td>3.1a</td>
</tr>
<tr>
<td></td>
<td>$\begin{bmatrix} 10 \ 0 \ -4 \end{bmatrix} + k \begin{bmatrix} 1 \ 4 \ -2 \end{bmatrix}$ or equivalent e.g. $\begin{bmatrix} r \end{bmatrix} - \begin{bmatrix} 10 \ 0 \ -4 \end{bmatrix} \times \begin{bmatrix} 1 \ 4 \ -2 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$</td>
<td>A1</td>
<td>2.5</td>
</tr>
</tbody>
</table>

**Notes:**

**M1:** Substitutes the parametric equation of the line into the equation of the plane and solves for $\lambda$

**A1:** Obtains the correct coordinates of the intersection of the line and the plane

**M1:** Substitutes the parametric form of the line perpendicular to the plane passing through $(2, 4, -6)$ into the equation of the plane to find $t$

**M1:** Find the reflection of $(2, 4, -6)$ in the plane

**A1:** Correct coordinates

**M1:** Determines the direction of $l$ by subtracting the appropriate vectors

**A1:** Correct vector equation using the correct notation

---

**Diagram:**

- Point $(8, -8, 0)$
- Point $(10, 0, -4)$
- Point $(2, 4, -6)$
- Plane $\pi$
(a)(i) Weight = mass $\times$ g $\Rightarrow m = \frac{30000}{g} = 3000$

But mass is in thousands of kg, so $m = 3$

(ii) $\frac{dx}{dt} = 40 \cos t + 20 \sin t$, $\frac{d^2x}{dt^2} = -40 \sin t + 20 \cos t$

$3(-40 \sin t + 20 \cos t) + 4(40 \cos t + 20 \sin t)$

$+ 40 \sin t - 20 \cos t = ...$

$= 200 \cos t \text{ so PI is } x = 40 \sin t - 20 \cos$

or

Let $x = a \cos t + b \sin t$

$\frac{dx}{dt} = -a \sin t + b \cos t$, $\frac{d^2x}{dt^2} = -a \cos t - b \sin t$

$4b - 2a = 200, -2b - 4a = 0 \Rightarrow a = ..., b = ...$

$x = 40 \sin t - 20 \cos$

(iii) $3 \lambda^2 + 4 \lambda + 1 = 0 \Rightarrow \lambda = -1, -\frac{1}{3}$

$x = Ae^{-t} + Be^{-\frac{t}{3}}$

$x = PI + CF$

$x = Ae^{-t} + Be^{-\frac{t}{3}} + 40 \sin t - 20 \cos t$

(b) $t = 0, x = 0 \Rightarrow A + B = 20$

$x = 0, \frac{dx}{dt} = -Ae^{-t} - \frac{1}{3}Be^{-\frac{t}{3}} + 40 \cos t + 20 \sin t = 0$

$\Rightarrow A + \frac{1}{3}B = 40$

$x = 50e^{-t} - 30e^{-\frac{t}{3}} + 40 \sin t - 20 \cos t$

$t = 9 \Rightarrow x = 33$

(12 marks)
### Question 9 notes:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)(i)</td>
<td>M1: Correct explanation that in the model, $m = 3$</td>
<td></td>
</tr>
</tbody>
</table>
| (ii) | M1: Differentiates the given PI twice  
      | M1: Substitutes into the given differential equation  
      | A1*: Reaches $200\cos t$ and makes a conclusion  
      | or  
      | M1: Uses the correct form for the PI and differentiates twice  
      | M1: Substitutes into the given differential equation and attempts to solve  
      | A1*: Correct PI |
| (iii) | M1: Uses the model to form and solve the auxiliary equation  
      | A1: Correct complementary function  
      | M1: Uses the correct notation for the general solution by combining PI and CF  
      | A1: Correct General Solution for the model |
| (b) | M1: Uses the initial conditions of the model, $t = 0$ at $x = 0$, to form an equation in $A$ and $B$  
     | M1: Uses $\frac{dx}{dt} = 0$ at $x = 0$ in the model to form an equation in $A$ and $B$  
     | A1: Correct PS  
     | A1: Obtains 33m using the assumptions made in the model |
You must have:
Mathematical Formulae and Statistical Tables, calculator

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions
• Use black ink or ball-point pen.
• If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
• Fill in the boxes at the top of this page with your name, centre number and candidate number.
• Answer all questions and ensure that your answers to parts of questions are clearly labelled.
• Answer the questions in the spaces provided – there may be more space than you need.
• You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
• Answers should be given to three significant figures unless otherwise stated.

Information
• A booklet ‘Mathematical Formulae and Statistical Tables’ is provided.
• There are 7 questions in this question paper. The total mark for this paper is 75.
• The marks for each question are shown in brackets – use this as a guide as to how much time to spend on each question.

Advice
• Read each question carefully before you start to answer it.
• Try to answer every question.
• Check your answers if you have time at the end.
Answer ALL questions. Write your answers in the spaces provided.

1. The roots of the equation

\[ x^3 - 8x^2 + 28x - 32 = 0 \]

are \( \alpha, \beta \) and \( \gamma \)

Without solving the equation, find the value of

(i) \( \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \)

(ii) \( (\alpha + 2)(\beta + 2)(\gamma + 2) \)

(iii) \( \alpha^2 + \beta^2 + \gamma^2 \)

(8)
Question 1 continued

(Total for Question 1 is 8 marks)
2. The plane \( \Pi_1 \) has vector equation

\[ r \cdot (3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 5 \]

(a) Find the perpendicular distance from the point \((6, 2, 12)\) to the plane \( \Pi_1 \) \((3)\)

The plane \( \Pi_2 \) has vector equation

\[ r = \lambda(2\mathbf{i} + \mathbf{j} + 5\mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} - 2\mathbf{k}) \]

where \( \lambda \) and \( \mu \) are scalar parameters.

(b) Show that the vector \(-\mathbf{i} - 3\mathbf{j} + \mathbf{k}\) is perpendicular to \( \Pi_2 \) \((2)\)

(c) Show that the acute angle between \( \Pi_1 \) and \( \Pi_2 \) is 52° to the nearest degree. \((3)\)
Question 2 continued

(Total for Question 2 is 8 marks)
3. (i) 

\[
\mathbf{M} = \begin{pmatrix}
2 & a & 4 \\
1 & -1 & -1 \\
-1 & 2 & -1
\end{pmatrix}
\]

where \(a\) is a constant.

(a) For which values of \(a\) does the matrix \(\mathbf{M}\) have an inverse? (2)

Given that \(\mathbf{M}\) is non-singular,

(b) find \(\mathbf{M}^{-1}\) in terms of \(a\) (4)

(ii) Prove by induction that for all positive integers \(n\),

\[
\begin{pmatrix}
3 & 0 \\
6 & 1
\end{pmatrix}^n = \begin{pmatrix}
3^n & 0 \\
3(3^n - 1) & 1
\end{pmatrix}
\] (6)
Question 3 continued

(Total for Question 3 is 12 marks)
4. A complex number $z$ has modulus 1 and argument $\theta$.

(a) Show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta, \quad n \in \mathbb{Z}^+$$

(b) Hence, show that

$$\cos^{4}\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$$
Question 4 continued

(Total for Question 4 is 7 marks)
5. 

\[ y = \sin x \sinh x \]

(a) Show that \[ \frac{d^4 y}{dx^4} = -4y \]

(b) Hence find the first three non-zero terms of the Maclaurin series for \( y \), giving each coefficient in its simplest form.

(c) Find an expression for the \( n \)th non-zero term of the Maclaurin series for \( y \).
Question 5 continued

(Total for Question 5 is 10 marks)
6. (a) (i) Show on an Argand diagram the locus of points given by the values of $z$ satisfying

$$|z - 4 - 3i| = 5$$

Taking the initial line as the positive real axis with the pole at the origin and given that

$$\theta \in [\alpha, \alpha + \pi], \text{ where } \alpha = -\arctan\left(\frac{4}{3}\right),$$

(ii) show that this locus of points can be represented by the polar curve with equation

$$r = 8\cos \theta + 6\sin \theta$$

The set of points $A$ is defined by

$$A = \left\{ z : 0 \leq \arg z \leq \frac{\pi}{3} \right\} \cap \left\{ z : |z - 4 - 3i| \leq 5 \right\}$$

(b) (i) Show, by shading on your Argand diagram, the set of points $A$.

(ii) Find the exact area of the region defined by $A$, giving your answer in simplest form.
Question 6 continued

(Total for Question 6 is 13 marks)
7. At the start of the year 2000, a survey began of the number of foxes and rabbits on an island.

At time $t$ years after the survey began, the number of foxes, $f$, and the number of rabbits, $r$, on the island are modelled by the differential equations

\[
\frac{df}{dt} = 0.2f + 0.1r \\
\frac{dr}{dt} = -0.2f + 0.4r
\]

(a) Show that \( \frac{d^2f}{dt^2} - 0.6 \frac{df}{dt} + 0.1f = 0 \) \( \hspace{1cm} (3) \)

(b) Find a general solution for the number of foxes on the island at time $t$ years. \( \hspace{1cm} (4) \)

(c) Hence find a general solution for the number of rabbits on the island at time $t$ years. \( \hspace{1cm} (3) \)

At the start of the year 2000 there were 6 foxes and 20 rabbits on the island.

(d) (i) According to this model, in which year are the rabbits predicted to die out?

(ii) According to this model, how many foxes will be on the island when the rabbits die out?

(iii) Use your answers to parts (i) and (ii) to comment on the model. \( \hspace{1cm} (7) \)
At the start of the year 2000 there were 6 foxes and 20 rabbits on the island.

Hence find a general solution for the number of rabbits on the island at time $t$.

Find a general solution for the number of foxes on the island at time $t$.

Show that

(i) According to this model, how many foxes will be on the island when the rabbits die out?

(ii) According to this model, how many foxes will be on the island when the rabbits die out?

$\frac{df}{dt} = 0.2f + 0.1r$

$\frac{dr}{dt} = -0.2r + 0.4t$

Question 7 continued
Question 7 continued
Paper 2: Core Pure Mathematics 2 Mark Scheme

Question Scheme Marks AOs

1(i)

\[ \alpha + \beta + \gamma = 8, \quad \alpha + \beta + \gamma = 28, \quad \alpha + \beta + \gamma = 32 \]

B1 3.1a

\[ \alpha + \beta + \gamma = 11 \]

M1 1.1b

\[ \alpha + \beta + \gamma = 7 \]

A1 ft 1.1b

(ii)

\[ \alpha + \beta + \gamma = 2 + 2 + 2 = 4 \]

M1 1.1b

\[ \alpha + \beta + \gamma = 28 + 2 + 28 + 8 = 64 \]

A1 1.1b

\[ \alpha + \beta + \gamma = 32 + 2 + 32 + 8 = 76 \]

(3)

Alternative:

\[ x^3 - 2x^2 + 2x - 2 = 0 \]

M1 1.1b

\[ x = 8, \quad x = 32, \quad x = 56 \]

A1 1.1b

\[ \alpha + \beta + \gamma = 2 + 2 + 2 = 6 \]

A1 1.1b

(iii)

\[ \alpha + \beta + \gamma = 2 + 2 + 2 = 6 \]

M1 3.1a

\[ \alpha + \beta + \gamma = 28 + 2 + 28 + 8 = 64 \]

A1 ft 1.1b

\[ \alpha + \beta + \gamma = 128 \]

(2)

(8 marks)

Notes:

(i)

B1:

Identifies the correct values for all 3 expressions (can score anywhere)

M1:

Uses a correct identity

A1 ft:

Correct value (follow through their 8, 28 and 32)

(ii)

M1:

Attempts to expand

A1:

Correct expansion

A1:

Correct value

Alternative:

M1:

Substitutes \( x - 2 \) for \( x \) in the given cubic

A1:

Calculates the correct constant term

A1:

Changes sign and so obtains the correct value

(iii)

M1:

Establishes the correct identity

A1 ft:

Correct value (follow through their 8, 28 and 32)
Paper 2: Core Pure Mathematics 2 Mark Scheme

<table>
<thead>
<tr>
<th>Question</th>
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<th>Marks</th>
<th>AOs</th>
</tr>
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<tbody>
<tr>
<td>1(i)</td>
<td>$\alpha + \beta + \gamma = 8$, $\alpha \beta + \beta \gamma + \gamma \alpha = 28$, $\alpha \beta \gamma = 32$</td>
<td>B1 3.1a</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta \gamma + \alpha \gamma + \alpha \beta}{\alpha \beta \gamma}$</td>
<td>M1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$= \frac{7}{8}$</td>
<td>A1ft 1.1b</td>
<td>(3)</td>
</tr>
<tr>
<td>(ii)</td>
<td>$(\alpha + 2)(\beta + 2)(\gamma + 2) = (\alpha \beta + 2 \alpha + 2 \beta + 4)(\gamma + 2)$</td>
<td>M1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$= \alpha \beta \gamma + 2(\alpha \beta + \alpha \gamma + \beta \gamma) + 4(\alpha + \beta + \gamma) + 8$</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$= 32 + 2(28) + 4(8) + 8 = 128$</td>
<td>A1 1.1b</td>
<td>(3)</td>
</tr>
</tbody>
</table>

**Alternative:**

$$(x - 2)^3 - 8(x - 2)^2 + 28(x - 2) - 32 = 0$$

$$= ... - 8 + ... - 32 + ... - 56 - 32 = -128$$

$\therefore (\alpha + 2)(\beta + 2)(\gamma + 2) = 128$ | A1 1.1b | (3) |

| (iii)    | $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha \beta + \alpha \gamma + \beta \gamma)$ | M1 3.1a | |
|          | $= 8^2 - 2(28) = 8$ | A1ft 1.1b | (2) |

(8 marks)

Notes:

(i)  
B1: Identifies the correct values for all 3 expressions (can score anywhere)  
M1: Uses a correct identity  
A1ft: Correct value (follow through their 8, 28 and 32)

(ii)  
M1: Attempts to expand  
A1: Correct expansion  
A1: Correct value

**Alternative:**  
M1: Substitutes $x - 2$ for $x$ in the given cubic  
A1: Calculates the correct constant term  
A1: Changes sign and so obtains the correct value

(iii)  
M1: Establishes the correct identity  
A1ft: Correct value (follow through their 8, 28 and 32)
<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>2(a)</strong></td>
<td>2(a) ( \begin{pmatrix} 3 \ -4 \ 2 \end{pmatrix} \cdot \begin{pmatrix} 6 \ 2 \ 12 \end{pmatrix} = 18 - 8 + 24 )</td>
<td>M1</td>
<td>3.1a</td>
</tr>
<tr>
<td></td>
<td>( d = \frac{18 - 8 + 24 - 5}{\sqrt{3^2 + 4^2 + 2^2}} )</td>
<td>M1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>= ( \sqrt{29} )</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td><strong>(3)</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(b)</strong></td>
<td>( \begin{pmatrix} -1 \ -3 \ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \ 1 \ 5 \end{pmatrix} = ... ) and ( \begin{pmatrix} -1 \ -3 \end{pmatrix} \cdot \begin{pmatrix} -1 \end{pmatrix} = ... )</td>
<td>M1</td>
<td>2.1</td>
</tr>
<tr>
<td></td>
<td>( \begin{pmatrix} -1 \ -3 \end{pmatrix} \cdot \begin{pmatrix} 2 \ 1 \end{pmatrix} = 0 ) and ( \begin{pmatrix} -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \end{pmatrix} = 0 )</td>
<td>A1</td>
<td>2.2a</td>
</tr>
<tr>
<td></td>
<td>( \therefore -i - 3j + k ) is perpendicular to ( \Pi_2 )</td>
<td></td>
<td><em>(2)</em></td>
</tr>
<tr>
<td><strong>(c)</strong></td>
<td>( \begin{pmatrix} -1 \ -3 \ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \ -4 \ 2 \end{pmatrix} = -3 + 12 + 2 )</td>
<td>M1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>( \sqrt{(-1)^2 + (-3)^2 + 1^2} \sqrt{(3)^2 + (-4)^2 + 2^2} \cos \theta = 11 )</td>
<td>M1</td>
<td>2.1</td>
</tr>
<tr>
<td></td>
<td>( \Rightarrow \cos \theta = \frac{11}{\sqrt{(-1)^2 + (-3)^2 + 1^2} \sqrt{(3)^2 + (-4)^2 + 2^2}} )</td>
<td></td>
<td><em>(3)</em></td>
</tr>
<tr>
<td></td>
<td>So angle between planes ( \theta = 52^\circ )</td>
<td>A1*</td>
<td>2.4</td>
</tr>
<tr>
<td><strong>(8 marks)</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Notes:**

(a)  
**M1:** Realises the need to and so attempts the scalar product between the normal and the position vector

**M1:** Correct method for the perpendicular distance

**A1:** Correct distance

(b)  
**M1:** Recognises the need to calculate the scalar product between the given vector and both direction vectors

**A1:** Obtains zero both times and makes a conclusion

(c)  
**M1:** Calculates the scalar product between the two normal vectors

**M1:** Applies the scalar product formula with their 11 to find a value for \( \cos \theta \)

**A1:** Identifies the correct angle by linking the angle between the normal and the angle between the planes
### Question Scheme Marks AOs

<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>3(i)(a)</td>
<td>$</td>
<td>M</td>
<td>= 2(1 + 2) - a(-1 + 1) + 4(2 - 1) = 0 \Rightarrow a = \ldots$</td>
</tr>
<tr>
<td></td>
<td>The matrix $M$ has an inverse when $a \neq -5$</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td>(b)</td>
<td>Minors : $\begin{vmatrix} 3 &amp; -2 &amp; 1 \ -a - 8 &amp; 2 &amp; a + 4 \ 4 - a &amp; -6 &amp; -2 - a \end{vmatrix}$</td>
<td>B1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>or $\begin{vmatrix} 3 &amp; 2 &amp; 1 \ a + 8 &amp; 2 &amp; -a - 4 \ 4 - a &amp; 6 &amp; -2 - a \end{vmatrix}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$M^{-1} = \frac{1}{</td>
<td>M</td>
<td>} \text{adj}(M)$</td>
</tr>
<tr>
<td></td>
<td>$M^{-1} = \frac{1}{2a + 10} \begin{pmatrix} 3 &amp; a + 8 &amp; 4 - a \ 2 &amp; 2 &amp; 6 \ 1 &amp; -a - 4 &amp; -2 - a \end{pmatrix}$</td>
<td>A1ft 1.1b</td>
<td></td>
</tr>
<tr>
<td>(ii)</td>
<td>When $n = 1$, $\text{lhs} = \begin{pmatrix} 3 &amp; 0 \ 6 &amp; 1 \end{pmatrix}$, $\text{rhs} = \begin{pmatrix} 3^1 &amp; 0 \ 3(3^1 - 1) &amp; 1 \end{pmatrix} = \begin{pmatrix} 3 &amp; 0 \ 6 &amp; 1 \end{pmatrix}$</td>
<td>B1 2.2a</td>
<td></td>
</tr>
<tr>
<td></td>
<td>So the statement is true for $n = 1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Assume true for $n = k$ so $\begin{pmatrix} 3 &amp; 0 \ 6 &amp; 1 \end{pmatrix}^k = \begin{pmatrix} 3^k &amp; 0 \ 3(3^k - 1) &amp; 1 \end{pmatrix}$</td>
<td>M1 2.4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\begin{pmatrix} 3 &amp; 0 \ 6 &amp; 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3^k &amp; 0 \ 3(3^k - 1) &amp; 1 \end{pmatrix} \begin{pmatrix} 3 &amp; 0 \ 6 &amp; 1 \end{pmatrix}$</td>
<td>M1 2.1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$= \begin{pmatrix} 3 \times 3^k &amp; 0 \ 3 \times 3(3^k - 1) + 6 &amp; 1 \end{pmatrix}$</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$= \begin{pmatrix} 3^{k+1} &amp; 0 \ 3(3^{k+1} - 1) &amp; 1 \end{pmatrix}$</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>If the statement is true for $n = k$ then it has been shown true for $n = k + 1$ and as it is true for $n = 1$, the statement is true for all positive integers $n$</td>
<td>A1 2.4</td>
<td></td>
</tr>
<tr>
<td>(6)</td>
<td>(12 marks)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
# Question 3 notes:

## (i)(a)

<table>
<thead>
<tr>
<th>M1:</th>
<th>Attempts determinant, equates to zero and attempts to solve for $a$ in order to establish the restriction for $a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1:</td>
<td>Provides the correct condition for $a$ if $M$ has an inverse</td>
</tr>
</tbody>
</table>

## (i)(b)

<table>
<thead>
<tr>
<th>B1:</th>
<th>A correct matrix of minors or cofactors</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1:</td>
<td>For a complete method for the inverse</td>
</tr>
<tr>
<td>A1ft:</td>
<td>Two correct rows following through their determinant</td>
</tr>
<tr>
<td>A1ft:</td>
<td>Fully correct inverse following through their determinant</td>
</tr>
</tbody>
</table>

## (ii)

<table>
<thead>
<tr>
<th>B1:</th>
<th>Shows the statement is true for $n = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1:</td>
<td>Assumes the statement is true for $n = k$</td>
</tr>
<tr>
<td>M1:</td>
<td>Attempts to multiply the correct matrices</td>
</tr>
<tr>
<td>A1:</td>
<td>Correct matrix in terms of $k$</td>
</tr>
<tr>
<td>A1:</td>
<td>Correct matrix in terms of $k + 1$</td>
</tr>
<tr>
<td>A1:</td>
<td>Correct complete conclusion</td>
</tr>
<tr>
<td>Question</td>
<td>Scheme</td>
</tr>
<tr>
<td>----------</td>
<td>--------</td>
</tr>
<tr>
<td>4(a)</td>
<td>$z^n + z^{-n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$</td>
</tr>
<tr>
<td></td>
<td>$= 2\cos n\theta*$</td>
</tr>
<tr>
<td>(b)</td>
<td>$(z + z^{-1})^4 = 16 \cos^4 \theta$</td>
</tr>
<tr>
<td></td>
<td>$(z + z^{-1})^4 = z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4}$</td>
</tr>
<tr>
<td></td>
<td>$= z^4 + z^{-4} + 4(z^2 + z^{-2}) + 6$</td>
</tr>
<tr>
<td></td>
<td>$= 2\cos 4\theta + 4(2\cos 2\theta) + 6$</td>
</tr>
<tr>
<td></td>
<td>$\cos^4 \theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)*$</td>
</tr>
</tbody>
</table>

(7 marks)

Notes:

(a)  
M1: Identifies the correct form for $z^n$ and $z^{-n}$ and adds to progress to the printed answer  
A1*: Achieves printed answer with no errors

(b)  
B1: Begins the argument by using the correct index with the result from part (a)  
M1: Realises the need to find the expansion of $(z + z^{-1})^4$  
A1: Terms correctly combined  
M1: Links the expansion with the result in part (a)  
A1*: Achieves printed answer with no errors
<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>5(a)</strong></td>
<td>$\frac{dy}{dx} = \sin x \cosh x + \cos x \sinh x$</td>
<td>M1</td>
<td>1.1a</td>
</tr>
<tr>
<td></td>
<td>$\frac{d^2y}{dx^2} = \cos x \cosh x + \sin x \sinh x + \cos x \cosh x - \sin x \sinh x$ (= 2 \cos x \cosh x)</td>
<td>M1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>$\frac{d^3y}{dx^3} = 2 \cos x \sinh x - 2 \sin x \cosh x$</td>
<td>M1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>$\frac{d^4y}{dx^4} = -4 \sinh x \sin x = -4y^*$</td>
<td>A1*</td>
<td>2.1</td>
</tr>
</tbody>
</table>
| **(b)** | \(\left(\frac{d^2y}{dx^2}\right)_0 = 2, \left(\frac{d^5y}{dx^5}\right)_0 = -8, \left(\frac{d^{10}y}{dx^{10}}\right)_0 = 32\) \(\begin{array}{l}
\text{Uses } y = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \ldots \text{ with their values} \\
= \frac{x^2}{2!}(2) + \frac{x^6}{6!}(-8) + \frac{x^{10}}{10!}(32) \\
= x^2 - \frac{x^6}{90} + \frac{x^{10}}{113400}
\end{array}\) | B1 | 3.1a |
| **(c)** | \(2(-4)^{y-1}\frac{x^{4n-2}}{(4n-2)!}\) | M1 A1 | 3.1a 2.2a |

**Notes:**

(a)  
M1: Realises the need to use the product rule and attempts first derivative  
M1: Realises the need to use a second application of the product rule and attempts the second derivative  
M1: Correct method for the third derivative  
A1*: Obtains the correct 4th derivative and links this back to \(y\)

(b)  
B1: Makes the connection with part (a) to establish the general pattern of derivatives and finds the correct non-zero values  
M1: Correct attempt at Maclaurin series with their values  
A1: Correct expression un-simplified  
A1: Correct expression and simplified

(c)  
M1: Generalising, dealing with signs, powers and factorials  
A1: Correct expression
<table>
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<tr>
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<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>6(a)(i)</td>
<td><img src="image1.png" alt="Image" /></td>
<td>M1</td>
<td>1.1b</td>
</tr>
<tr>
<td>(a)(ii)</td>
<td></td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td>(b)(i)</td>
<td><img src="image2.png" alt="Image" /></td>
<td>B1</td>
<td>1.1b</td>
</tr>
<tr>
<td>(b)(ii)</td>
<td></td>
<td>B1ft</td>
<td>1.1b</td>
</tr>
</tbody>
</table>

**6(a)(i)**

$$|z - 4 - 3i| = 5 \Rightarrow |x + iy - 4 - 3i| = 5 \Rightarrow (x - 4)^2 + (y - 3)^2 = ...$$

$$M1 \quad 2.1$$

$$M1: \quad \text{Realises the need to use the product rule and attempts first derivative}$$

$$\quad M1: \quad \text{Realises the need to use a second application of the product rule and attempts the second derivative}$$

$$\quad M1: \quad \text{Correct method for the third derivative}$$

$$\quad A1*: \quad \text{Obtains the correct 4th derivative and links this back to } y$$

**6(a)(ii)**

$$z^2 = 25 \text{ or any correct form}$$

$$A1 \quad 1.1b$$

$$r^2 \cos^2 \theta - 8r \cos \theta + 16 + r^2 \sin^2 \theta - 6r \sin \theta + 9 = 25$$

$$\Rightarrow r^2 - 8r \cos \theta - 6r \sin \theta = 0$$

$$\therefore r = 8 \cos \theta + 6 \sin \theta*$$

**6(b)(i)**

$$A = \frac{1}{2} \int r^2 \ d\theta = \frac{1}{2} \int (8 \cos \theta + 6 \sin \theta)^2 \ d\theta$$

$$= \frac{1}{2} \int (64 \cos^2 \theta + 96 \sin \theta \cos \theta + 36 \sin^2 \theta) \ d\theta$$

$$= \frac{1}{2} \int (32 \cos 2\theta + 96 \sin \theta \cos \theta + 18(1 - \cos 2\theta)) \ d\theta$$

$$= \frac{1}{2} \int (14 \cos 2\theta + 50 + 48 \sin 2\theta) \ d\theta$$

$$= \frac{1}{2} \left[ 7 \sin 2\theta + 50\theta - 24 \cos 2\theta \right]_0^\pi = \frac{1}{2} \left[ \left( \frac{7\sqrt{3}}{2} + \frac{50\pi}{3} + 12 \right) - (-24) \right]$$

$$= \frac{7\sqrt{3}}{4} + \frac{25\pi}{3} + 18$$

**6(b)(ii)**

$$= \frac{7\sqrt{3}}{4} + \frac{25\pi}{3} + 18$$

**Note:**

- **(a)(i)**: Realises the need to use the product rule and attempts first derivative
- **(a)(ii)**: Realises the need to use a second application of the product rule and attempts the second derivative
- **(b)(i)**: Correct method for the third derivative
- **(b)(ii)**: Obtains the correct 4th derivative and links this back to $y$
Candidates may take a geometric approach e.g. by finding sector + 2 triangles

\[
\text{Angle } ACB = \left( \frac{2\pi}{3} \right) \text{ so area sector } ACB = \frac{1}{2} (5)^2 \left( \frac{2\pi}{3} \right)
\]
\[
\text{Area of triangle } OCB = \frac{1}{2} \times 8 \times 3
\]
\[
\text{Sector area } ACB + \text{ triangle area } OCB = \frac{25\pi}{3} + 12
\]

Area of triangle OAC:

\[
\text{Angle } ACO = 2\pi - \frac{2\pi}{3} - \cos^{-1}\left( \frac{5^2 + 5^2 - 8^2}{2 \times 5 \times 5} \right)
\]
\[
\text{so area } OAC = \frac{1}{2} (5)^2 \sin \left( \frac{4\pi}{3} - \cos^{-1}\left( \frac{-7}{25} \right) \right)
\]
\[
= \frac{25}{2} \left( \sin \frac{4\pi}{3} \cos \cos^{-1}\left( \frac{-7}{25} \right) - \cos \frac{4\pi}{3} \sin \cos^{-1}\left( \frac{-7}{25} \right) \right)
\]
\[
= \frac{25}{2} \left( \frac{7\sqrt{3}}{50} + \frac{1}{2} \sqrt{1 - \left( \frac{7}{25} \right)^2} \right) = \frac{7\sqrt{3}}{4} + 6
\]
\[
\text{Total area } = \frac{25\pi}{3} + \frac{1}{2} \times 8 \times 3 + \frac{7\sqrt{3}}{4}
\]
\[
= \frac{7\sqrt{3}}{4} + \frac{25\pi}{3} + 18
\]

(13 marks)
(b)(ii) Alternative:

M1: Selects an appropriate method by finding angle $ACB$ and area of sector $ACB$ and finds area of triangle $OCB$ to make progress towards finding the required area
A1: Correct combined area of sector $ACB$ + triangle $OCB$
M1: Starts the process of finding the area of triangle $OAC$ by calculating angle $ACO$ and attempts area of triangle $OAC$
M1: Uses the addition formula to find the exact area of triangle $OAC$ and employs a full correct method to find the area of the shaded region
A1: Correct area
<table>
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</tr>
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<tbody>
<tr>
<td><strong>7(a)</strong></td>
<td>( r = 10 \frac{df}{dt} - 2f \Rightarrow \frac{dr}{dt} = 10 \frac{d^2 f}{dt^2} - 2 \frac{df}{dt} )</td>
<td>M1</td>
<td>2.1</td>
</tr>
<tr>
<td></td>
<td>( 10 \frac{d^2 f}{dt^2} - 2 \frac{df}{dt} = -0.2f + 0.4 \left( 10 \frac{df}{dt} - 2f \right) )</td>
<td>M1</td>
<td>2.1</td>
</tr>
<tr>
<td></td>
<td>( \frac{d^2 f}{dt^2} - 0.6 \frac{df}{dt} + 0.1f = 0^* )</td>
<td>A1*</td>
<td>1.1b</td>
</tr>
<tr>
<td><strong>(3)</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(b)</strong></td>
<td>( m^2 - 0.6m + 0.1 = 0 \Rightarrow m = \frac{0.6 \pm \sqrt{0.6^2 - 4 \times 0.1}}{2} )</td>
<td>M1</td>
<td>3.4</td>
</tr>
<tr>
<td></td>
<td>( m = 0.3 \pm 0.1i )</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>( f = e^{\alpha t} \left( A \cos \beta t + B \sin \beta t \right) )</td>
<td>M1</td>
<td>3.4</td>
</tr>
<tr>
<td></td>
<td>( f = e^{0.3t} \left( A \cos 0.1t + B \sin 0.1t \right) )</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td><strong>(4)</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(c)</strong></td>
<td>( \frac{df}{dt} = 0.3e^{0.3t} \left( A \cos 0.1t + B \sin 0.1t \right) + 0.1e^{0.3t} \left( B \cos 0.1t - A \sin 0.1t \right) )</td>
<td>M1</td>
<td>3.4</td>
</tr>
<tr>
<td></td>
<td>( r = 10 \frac{df}{dt} - 2f )</td>
<td>M1</td>
<td>3.4</td>
</tr>
<tr>
<td></td>
<td>( = e^{0.3t} \left( (3A + B) \cos 0.1t + (3B - A) \sin 0.1t \right) - 2e^{0.3t} \left( A \cos 0.1t + B \sin 0.1t \right) )</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>( r = e^{0.3t} \left( (A + B) \cos 0.1t + (B - A) \sin 0.1t \right) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(3)</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(d)(i)</strong></td>
<td>( t = 0, f = 6 \Rightarrow A = 6 )</td>
<td>M1</td>
<td>3.1b</td>
</tr>
<tr>
<td></td>
<td>( t = 0, r = 20 \Rightarrow B = 14 )</td>
<td>M1</td>
<td>3.3</td>
</tr>
<tr>
<td></td>
<td>( r = e^{0.3t} \left( 20 \cos 0.1t + 8 \sin 0.1t \right) = 0 )</td>
<td>M1</td>
<td>3.1b</td>
</tr>
<tr>
<td></td>
<td>( \tan 0.1t = -2.5 )</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>2019</td>
<td>A1</td>
<td>3.2a</td>
</tr>
<tr>
<td><strong>(d)(ii)</strong></td>
<td>3750 foxes</td>
<td>B1</td>
<td>3.4</td>
</tr>
<tr>
<td><strong>(d)(iii)</strong></td>
<td>e.g. the model predicts a large number of foxes are on the island when the rabbits have died out and this may not be sensible</td>
<td>B1</td>
<td>3.5a</td>
</tr>
</tbody>
</table>
Question 7 notes:

(a)  
M1: Attempts to differentiate the first equation with respect to \( t \)  
M1: Proceeds to the printed answer by substituting into the second equation  
A1*: Achieves the printed answer with no errors

(b)  
M1: Uses the model to form and solve the auxiliary equation  
A1: Correct values for \( m \)  
M1: Uses the model to form the CF  
A1: Correct CF

(c)  
M1: Differentiates the expression for the number of foxes  
M1: Uses this result to find an expression for the number of rabbits  
A1: Correct equation

(d)(i)  
M1: Realises the need to use the initial conditions in the model for the number of foxes  
M1: Realises the need to use the initial conditions in the model for the number of rabbits to find both unknown constants  
M1: Obtains an expression for \( r \) in terms of \( t \) and sets \( a = 0 \)  
A1: Rearranges and obtains a correct value for tan  
A1: Identifies the correct year

(d)(ii)  
B1: Correct number of foxes

(d)(iii)  
B1: Makes a suitable comment on the outcome of the model
Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions
• Use black ink or ball-point pen.
• If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
• Fill in the boxes at the top of this page with your name, centre number and candidate number.
• Answer all questions and ensure that your answers to parts of questions are clearly labelled.
• Answer the questions in the spaces provided – there may be more space than you need.
• You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
• Answers should be given to three significant figures unless otherwise stated.

Information
• A booklet ‘Mathematical Formulae and Statistical Tables’ is provided.
• There are 8 questions in this question paper. The total mark for this paper is 75.
• The marks for each question are shown in brackets – use this as a guide as to how much time to spend on each question.

Advice
• Read each question carefully before you start to answer it.
• Try to answer every question.
• Check your answers if you have time at the end.
Answer ALL questions. Write your answers in the spaces provided.

1. Use Simpson’s Rule with 6 intervals to estimate

$$\int_{1}^{4} \sqrt{1 + x^3} \, dx$$

(5 marks)
Question 1 continued

(Total for Question 1 is 5 marks)
2. Given $k$ is a constant and that

$$y = x^3 e^{kx}$$

use Leibnitz theorem to show that

$$\frac{d^n y}{dx^n} = k^{n-3} e^{kx} (k^3x^3 + 3nk^2x^2 + 3n(n-1)kx + n(n-1)(n-2))$$

(4)
Question 2 continued
3. A vibrating spring, fixed at one end, has an external force acting on it such that the centre of the spring moves in a straight line. At time $t$ seconds, $t \geq 0$, the displacement of the centre $C$ of the spring from a fixed point $O$ is $x$ micrometres.

The displacement of $C$ from $O$ is modelled by the differential equation

$$t^2 \frac{d^3x}{dt^2} - 2t \frac{dx}{dt} + (2 + t^2)x = t^4$$  \hspace{1cm} (I)

(a) Show that the transformation $x = tv$ transforms equation (I) into the equation

$$\frac{d^2v}{dt^2} + v = t$$  \hspace{1cm} (II)

(b) Hence find the general equation for the displacement of $C$ from $O$ at time $t$ seconds.

(c) (i) State what happens to the displacement of $C$ from $O$ as $t$ becomes large.

(ii) Comment on the model with reference to this long term behaviour.
The displacement of

(i) State what happens to the displacement of

(ii) Comment on the model with reference to this long term behaviour.

Hence find the general equation for the displacement of

Show that the transformation

transforms equation (I) into the equation

The displacement of the centre of the spring moves in a straight line. At time

is micrometres.

from a fixed point

is modelled by the differential equation

DO NOT WRITE IN THIS AREA

(Total for Question 3 is 14 marks)
4.

\[ \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + y = 0 \quad (I) \]

(a) Show that

\[ \frac{d^3y}{dx^3} = ax \frac{d^4y}{dx^4} + b \frac{d^3y}{dx^3} \]

where \( a \) and \( b \) are integers to be found. \( \quad (4) \)

(b) Hence find a series solution, in ascending powers of \( x \), as far as the term in \( x^5 \), of the differential equation \( (I) \) where \( y = 0 \) and \( \frac{dy}{dx} = 1 \) at \( x = 0 \). \( \quad (5) \)
Question 4 continued
The normal to the parabola $y^2 = 4ax$ at the point $P(ap^2, 2ap)$ passes through the parabola again at the point $Q(aq^2, 2aq)$.

The line $OP$ is perpendicular to the line $OQ$, where $O$ is the origin.

Prove that $p^2 = 2$
Question 5 continued

(Total for Question 5 is 9 marks)
6. A tetrahedron has vertices $A(1, 2, 1)$, $B(0, 1, 0)$, $C(2, 1, 3)$ and $D(10, 5, 5)$.

Find

(a) a Cartesian equation of the plane $ABC$. (3)

(b) the volume of the tetrahedron $ABCD$. (3)

The plane $II$ has equation $2x - 3y + 3 = 0$

The point $E$ lies on the line $AC$ and the point $F$ lies on the line $AD$.

Given that $II$ contains the point $B$, the point $E$ and the point $F$,

(c) find the value of $k$ such that $\overrightarrow{AE} = k\overrightarrow{AC}$. (3)

Given that $\overrightarrow{AF} = \frac{1}{9}\overrightarrow{AD}$

(d) show that the volume of the tetrahedron $ABCD$ is 45 times the volume of the tetrahedron $ABEF$. (2)
Given that contains the point П with equation 2Пx + 3 − y + 3 = 0.

The point П lies on the line \( \vec{AB} \) such that .

A tetrahedron has vertices A(1, 2, 1), B(0, 1, 0), C(2, 1, 3) and D(10, 5, 5).

Find (a) the vector \( \vec{AE} \), (b) the volume of the tetrahedron ABCD, and the point E such that 45 times the volume of the tetrahedron ABCD lies on the line .

Question 6 continued

(Total for Question 6 is 11 marks)
7. \( P \) and \( Q \) are two distinct points on the ellipse described by the equation \( x^2 + 4y^2 = 4 \)

The line \( l \) passes through the point \( P \) and the point \( Q \).

The tangent to the ellipse at \( P \) and the tangent to the ellipse at \( Q \) intersect at the point \( (r, s) \).

Show that an equation of the line \( l \) is

\[
4xy + rx = 4
\]

(8)
Question 7 continued

(Total for Question 7 is 8 marks)
Figure 1 shows the graph of the function \( h(x) \) with equation

\[
h(x) = 45 + 15 \sin x + 21 \sin \left( \frac{x}{2} \right) + 25 \cos \left( \frac{x}{2} \right) \quad x \in [0, 40]
\]

(a) Show that

\[
\frac{dh}{dx} = \left( t^2 - 6t - 17 \right) \left( 9t^2 + 4t - 3 \right) \quad \frac{2}{(1 + t^2)^2}
\]

where \( t = \tan \left( \frac{x}{4} \right) \).

(b) (i) Suggest a value of \( k \) that could be used for the graph of \( kh(x) \) to form a suitable model.

(ii) Why may such a model be suitable to predict the times when the tide heights are at their peaks, but not to predict the heights of these peaks?

(c) Use Figure 2 and the result of part (a) to estimate, to the nearest minute, the time of the highest tide height on the 4th January 2017.

Figure 2 shows a graph of predicted tide heights, in metres, for Portland harbour from 08:00 on the 3rd January 2017 to the end of the 4th January 2017.

The graph of \( kh(x) \), where \( k \) is a constant and \( x \) is the number of hours after 08:00 on 3rd of January, can be used to model the predicted tide heights, in metres, for this period of time.

Source: ‘Data taken on 29th December 2016 from http://www.ukho.gov.uk/easytide/EasyTide'
Question 8 continued
Question 8 continued

(Total for Question 8 is 15 marks)

TOTAL FOR PAPER IS 75 MARKS
### Paper 3A: Further Pure Mathematics 1 Mark Scheme

<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Step 0.5</td>
<td>B1 1.1b</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y'$</th>
<th>$y_0$</th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$y_3$</th>
<th>$y_4$</th>
<th>$y_5$</th>
<th>$y_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\sqrt{2}$</td>
<td>1</td>
<td>1.5</td>
<td>2</td>
<td>2.5</td>
<td>3</td>
<td>3.5</td>
<td>4</td>
</tr>
</tbody>
</table>

| $y'$ | 3   | $\sqrt{4.375}$ | $\sqrt{16.625}$ | $\sqrt{28}$ | $\sqrt{43.875}$ | $\sqrt{65}$ |

$y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6 = "77.23"$

\[
\int_1^4 \sqrt{1 + x^3} \, dx \approx \frac{0.5}{3} \times "77.23"
\]

= 12.9

(5 marks)

**Notes:**

- **B1:** Use of step length 0.5
- **M1:** Attempt to find $y$ values with at least 2 correct
- **M1:** Use of formula $"y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6"$ with correct coefficients
- **A1:** $\frac{0.5}{3}$x their 77.23
- **A1:** awrt 12.9
<table>
<thead>
<tr>
<th>Question</th>
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<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
</table>
| 2 | \( y = x^3 e^{kx} \) so \( u = x^3 \) and \[
\frac{du}{dx} = 3x^2 \quad \text{and} \quad \frac{d^2u}{dx^2} = 6x \quad \text{and} \quad \frac{d^3u}{dx^3} = 6 \quad \left( \text{and} \quad \frac{d^4u}{dx^4} = 0 \right) 
\] | M1 | 1.1b |
| | \( v = e^{kx} \) and \[
\frac{dv}{dx^n} = k^n e^{kx} \quad \text{and} \quad \frac{d^{n-1}v}{dx^{n-1}} = k^{n-1} e^{kx} \quad \text{and} \quad \frac{d^{n-2}v}{dx^{n-2}} = k^{n-2} e^{kx} 
\](and...) | M1 | 2.1 |
| | \[
\frac{d^n y}{dx^n} = x^3 k^n e^{kx} + n3x^2 k^{n-1} e^{kx} + \frac{n(n-1)}{2} 6xk^{n-2} e^{kx} + \frac{n(n-1)(n-2)}{3!} 6k^{n-3} e^{kx} 
\] and remaining terms disappear | M1 | 2.1 |
| | So \[
\frac{d^n y}{dx^n} = k^{n-1} e^{kx} \left( kx^3 + 3nk^2 x^2 + 3n(n-1)kx + n(n-1)(n-2) \right)^
\] | A1* | 1.1b |
| | (4 marks) | | |

**Notes:**

**M1:** Differentiate \( u = x^3 \) three times  
**M1:** Use \( u = e^{kx} \) and establish the form of the derivatives, with at least the three shown  
**M1:** Uses correct formula, with 2 and 3! (or 6) and with terms shown to disappear after the fourth term  
**A1*:** Correct solution leading to the given answer stated. No errors seen
<table>
<thead>
<tr>
<th>Question</th>
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<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>3(a)</strong></td>
<td>Use of $x = tv$ to give $\frac{dx}{dt} = v + t \frac{dv}{dt}$</td>
<td>M1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>Hence $\frac{d^2x}{dt^2} = \frac{dv}{dt} + \frac{dv}{dt} + t \frac{d^2v}{dt^2}$</td>
<td>M1</td>
<td>2.1</td>
</tr>
<tr>
<td></td>
<td>Uses $t^2$ (their 2nd derivative) $- 2t$ (their 1st derivative) $+ (2 + t^2)x = t^4$ and simplifies LHS</td>
<td>M1</td>
<td>2.1</td>
</tr>
<tr>
<td></td>
<td>$\left(t \frac{d^2v}{dt^2} + t^3v = t^4 \text{ leading to} \right) \frac{d^2v}{dt^2} + v = t \ast$</td>
<td>A1*</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>(5)</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(b)</strong></td>
<td>Solve $\lambda^2 + 1 = 0$ to give $\lambda^2 = -1$</td>
<td>M1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>$\nu = A \cos t + B \sin t$</td>
<td>A1ft</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>Particular Integral is $\nu = kt + l$</td>
<td>B1</td>
<td>2.2a</td>
</tr>
<tr>
<td></td>
<td>$\frac{dv}{dt} = k$ and $\frac{d^2v}{dt^2} = 0$ and solve $0 + kt + l = t$ to give $k = 1, l = 0$</td>
<td>M1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>Solution: $\nu = Acost + Bsin t + t$</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>Displacement of $C$ from $O$ is given by $x = tv = ...$</td>
<td>M1</td>
<td>3.4</td>
</tr>
<tr>
<td></td>
<td>$x = t(Acost + Bsin t + t)$</td>
<td>A1</td>
<td>2.2a</td>
</tr>
<tr>
<td></td>
<td>(7)</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(c)(i)</strong></td>
<td>For large $t$, the displacement gets very large (and positive)</td>
<td>B1</td>
<td>3.2a</td>
</tr>
<tr>
<td><strong>(ii)</strong></td>
<td>Model suggests midpoint of spring moving relative to fixed point has large displacement when $t$ is large, which is unrealistic. The spring may reach elastic limit / will break</td>
<td>B1</td>
<td>3.5a</td>
</tr>
<tr>
<td></td>
<td>(2)</td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>(14 marks)</td>
<td></td>
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</tbody>
</table>
**Question 3 notes:**

(a)  
M1: Uses product rule to obtain first derivative  
M1: Continues to differentiate again, with product rule and chain rule as appropriate, in order to establish the second derivative  
A1: Correct second derivative. Accept equivalent expressions  
M1: Shows clearly the substitution into the given equation in order to form the new equation and gathers like terms  
A1*: Fully correct solution leading to the given answer

(b)  
Accept variations on symbols for constants throughout  
M1: Form and solve a quadratic Auxiliary Equation  
A1ft: Correct form of the Complementary Function for their solutions to the AE  
B1: Deduces the correct form for the Particular Integral (note $v = mt^2 + kt + l$ is fine)  
M1: Differentiates their Particular Integral and substitutes their derivatives into the equation to find the constants ($m = 0$ if used)  
A1: Correct general solution for equation (II)  
M1: Links the solution to equation (II) to the solution of the model equation correctly to find the displacement equation  
A1: Deduces the correct general solution for the displacement

(c)(i)  
B1: States that for large $t$ the displacement is large o.e. Accept e.g. as $t \to \infty$, $x \to \infty$

(c)(ii)  
B1: Reflect on the context of the original problem. Accept ‘model unrealistic’ / ‘spring will break’
<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td><strong>4(a)</strong></td>
<td>(y'' = 2xy' - y \Rightarrow y''' = 2xy'' + 2y' - y')</td>
<td>M1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>(y''' = 2xy'' + y' \Rightarrow y'''' = 2xy''' + 2y'' + y'')</td>
<td>M1</td>
<td>2.1</td>
</tr>
<tr>
<td></td>
<td>(y''''' = 2xy'''' + 3y'' \Rightarrow y''''' = 2xy'''' + 5y'''')</td>
<td>A1</td>
<td>2.1</td>
</tr>
<tr>
<td>(b)</td>
<td>(x = 0, y = 0, y' = 1 \Rightarrow y''(0) = 0\pi) from equation</td>
<td>B1</td>
<td>2.2a</td>
</tr>
<tr>
<td></td>
<td>(y'''(0) = 2 \times 0 \times y'(0) + 1 = 1; \quad y''''(0) = 2 \times 0 \times 1 + 3 \times 0 = 0;)</td>
<td>M1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>(x = 0, y''(0) = 1, y'''(0) = 0 \Rightarrow y''''(0) = 5)</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>(y = y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + \frac{y'''(0)}{6}x^3 + \frac{y''''(0)}{24}x^4 + \frac{y'''''(0)}{120}x^5 + ...)</td>
<td>M1</td>
<td>2.5</td>
</tr>
<tr>
<td></td>
<td>Series solution: (y = x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + ...)</td>
<td>A1ft</td>
<td>1.1b</td>
</tr>
</tbody>
</table>

**Notes:**

(a)  
**M1:** Attempts to differentiate equation with use of the product rule  
**A1:** cao. Accept if terms all on one side  
**M1:** Continues the process of differentiating to progress towards the goal. Terms may be kept on one side, but an expression in the fourth derivative should be obtained  
**A1:** Completes the process to reach the fifth derivative and rearranges to the correct form to obtain the correct answer by correct solution only

(b)  
**B1:** Deduces the correct value for \(y''(0)\) from the information in the question  
**M1:** Finds the values of the derivatives at the given point  
**A1:** All correct  
**M1:** Correct mathematical language required with given denominators. Can be in factorial form  
**A1ft:** Correct series, must start \(y = \ldots\) Follow through the values of their derivatives at 0
### Question 5

\[ y^2 = 4ax \implies 2y \frac{dy}{dx} = 4a \]

\[ \frac{dy}{dx} = 2a \implies \text{Gradient of normal is } -\frac{y}{2a} = -p \]

Equation of normal is: \( y - 2ap = -p(x - ap^2) \)

Normal passes through \( Q(aq^2, 2aq) \) so \( 2aq + apq^2 = 2ap + ap^3 \)

\[
\text{Grad } OP \times \text{ Grad } OQ = -1 \implies \frac{2ap}{ap^2} = -1
\]

\[ q = \frac{-4}{p} \]

\[ 2a \left( \frac{-4}{p} \right) + ap \left( \frac{16}{p^2} \right) = 2ap + ap^3 \implies p^4 + 2p^2 - 8 = 0 \]

\[ (p^2 - 2)(p^2 + 4) = 0 \implies p^2 = ... \]

Hence (as \( p^2 + 4 \neq 0 \)), \( p^2 = 2 \) *

### Alternative 1

First three marks as above and then as follows

Solves \( y^2 = 4ax \) and their normal simultaneously to find, in terms of \( a \) and \( p \), either \( x_Q = ap^2 + 4a + \frac{4a}{p^2} \) or \( y_Q = -2ap - \frac{4a}{p} \)

Finds the second coordinate of \( Q \) in terms of \( a \) and \( p \)

\[ (p^2 - 2)(p^2 + 4) = 0 \implies p^2 = ... \]

Hence (as \( p^2 + 2 \neq 0 \)), \( p^2 = 2 \)*
<table>
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<th>Marks</th>
<th>AOs</th>
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<tr>
<td><strong>5</strong></td>
<td></td>
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<tr>
<td></td>
<td></td>
<td><strong>Alternative 2</strong></td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>First three marks as above and then as follows</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td><strong>M1</strong> 2.1</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td><strong>A1</strong> 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td><strong>M1</strong> 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Solves $y^2 = 4ax$ and their normal simultaneously to find, in terms of $a$ and $p$, either $x_Q = ap^2 + 4a + \frac{4a}{p^2}$ or $y_Q = -2ap - \frac{4a}{p}$</td>
<td><strong>M1</strong> 3.1a</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Forms a relationship between $p$ and $q$ from their first coordinate:</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td><strong>either</strong> $y_Q = 2a \left(-p - \frac{2}{p}\right)$ \Rightarrow $q = -p - \frac{2}{p}$</td>
<td><strong>M1</strong> 2.1</td>
</tr>
<tr>
<td></td>
<td></td>
<td><strong>or</strong> $x_Q = a \left(p + \frac{2}{p}\right)^2$ \Rightarrow $q = \pm \left(p + \frac{2}{p}\right)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$q = -p - \frac{2}{p}$</td>
<td><strong>A1</strong> 1.1b</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(if $x$ coordinate used the correct root must be clearly identified before this mark is awarded)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td><strong>Grad OP \times Grad OQ = -1 \Rightarrow \frac{2ap}{ap^2} \times \frac{2aq}{aq^2} = -1 \Rightarrow q = -\frac{4}{p}</strong></td>
<td><strong>M1</strong> 2.1</td>
</tr>
<tr>
<td></td>
<td></td>
<td><strong>Sets</strong> $q = -p - \frac{2}{p} = -\frac{4}{p}$ and solves to give $p^2 = ...$</td>
<td><strong>M1</strong> 1.1b</td>
</tr>
<tr>
<td></td>
<td></td>
<td><strong>Hence</strong> [ as q = p + \frac{2}{p} = -\frac{4}{p} \text{ gives no solution} ], $p^2 = 2$ (only)*</td>
<td>*<em>A1</em> 1.1b</td>
</tr>
<tr>
<td></td>
<td></td>
<td>[ (9) ]</td>
<td></td>
</tr>
</tbody>
</table>

**Notes:**

(a)

- **M1:** Begins proof by differentiating and using the perpendicularity condition at point $P$ in order to find the equation of the normal
- **A1:** Correct gradient of normal, $-p$ only
- **M1:** Use of $y - y_1 = m(x - x_1)$. Accept use of $y = mx + c$ and then substitute to find $c$
- **M1:** Substitute coordinates of $Q$ into their equation to find an equation relating $p$ and $q$
- **M1:** Use of $m_1m_2 = -1$ with $OP$ and $OQ$ to form a second equation relating $p$ and $q$
- **A1:** $q = -\frac{4}{p}$ only
- **M1:** Solves the simultaneous equations and cancels $a$ from their results to obtain a quadratic equation in $p^2$ only
- **M1:** Attempts to solve their quadratic in $p^2$. Usual rules
- **A1*: Correct solution leading to given answer stated. No errors seen
Question 5 notes continued:

**Alternative 1:**

**M1A1M1:** As main scheme

**M1:** Solves $y^2 = 4ax$ and their normal simultaneously to find one of the coordinates for $Q$ in terms of $a$ and $p$ as shown

**M1:** Finds the second coordinate of $Q$ in terms of $a$ and $p$

**A1:** Both coordinates correct in terms of $a$ and $p$

**M1:** Use of $m_1 m_2 = -1$ with $OP$ and $OQ$, i.e. $\frac{2ap}{ap^2}$ with their $y_Q$ and $x_Q$.

**M1:** Cancels the $a$'s, simplifies to a quadratic in $p^2$ and solves the quadratic. Usual rules

**A1:** Correct solution leading to the given answer stated. No errors seen.

**Alternative 2:**

**M1A1M1:** As main scheme

**M1:** Solves $y^2 = 4ax$ and their normal simultaneously to find one of the coordinates for $Q$ in terms of $a$ and $p$ as shown

**M1:** Uses their coordinate to form a relationship between $p$ and $q$. Allow $q = \left( p + \frac{2}{p} \right)$ for this mark

**A1:** For $q = -p - \frac{2}{p}$, If the $x$ coordinate was used to find $q$ then consideration of the negative root is needed for this mark. Allow for $q = \pm \left( p + \frac{2}{p} \right)$

**M1:** Use of $m_1 m_2 = -1$ with $OP$ and $OQ$ to form a second equation relating $p$ and $q$ only

**M1:** Equates expressions for $q$ and attempts to solve to give $p^2 = ....$

**A1:** Correct solution leading to the given answer stated. No errors seen. If $x$ coordinate used, invalid solution must be rejected.
6(a) \[ \mathbf{AB} \times \mathbf{AC} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2-1 \\ -1+2 \\ -1+1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \]

Hence \(-3x + y + 2z = 1\)  

(b) Volume of Tetrahedron = \(\frac{1}{6} |\mathbf{h} \cdot (\mathbf{AD})|\)  

\[
\mathbf{r.} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}
\]

Hence \(-3x + y + 2z = 1\)  

(c) \[\mathbf{AE} = k\mathbf{AC} \text{ so } E \text{ is } (1+k, 2-k, 1+2k)\]  

\[E\text{ lies on plane so } 2(1+k) - 3(2-k) + 3 = 0, \text{ leading to } k = \ldots\]  

Hence \(k = \frac{1}{5}\)  

(d) Volume \(\mathbf{ABEF} = \frac{1}{6} (\mathbf{AB} \times \mathbf{AE}) \cdot \mathbf{AF} = \frac{1}{6} (\mathbf{AB} \times \frac{1}{5} \mathbf{AC}) \cdot \frac{1}{9} \mathbf{AD}\)  

\[= \frac{1}{45} \left( \frac{1}{6} (\mathbf{AB} \times \mathbf{AC}) \cdot \mathbf{AD} \right) \text{ and hence result } *\]  

\(11\) marks
**Question 6 notes:**

(a)

**M1:** Attempting a suitable cross product. Accept use of unit vectors  
**M1:** Complete method that would lead to finding the Cartesian equation of plane  
**A1:** Accept any equivalent form

(b)

**M1:** Identifies suitable vectors and attempts to substitute into a correct formula. Accept use of unit vectors  
**M1:** Correct form of scalar triple product using their \( \mathbf{n} \) from part (a)  
**A1:** \( \frac{8}{3} \) or exact equivalent form

(c)

**M1:** Uses that \( E \) is on \( AC \) in order to find an expression for \( E \)  
**M1:** Uses that \( E \) is in the plane \( \Pi \) to form and solve an expression in \( k \)  
**A1:** \( \frac{1}{5} \) o.e. only

(d)

**M1:** Uses formula for volume of tetrahedron and substitutes for \( \mathbf{AE} \) and \( \mathbf{AF} \)  
**A1**: Deduces result: Use of \( \frac{1}{6} (\mathbf{AB} \times \mathbf{AC}) \cdot \mathbf{AD} \) is required and no errors seen in solution
<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>$x^2 + 4y^2 = 4 \Rightarrow 2x + 8y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = ...$</td>
<td>M1</td>
<td>3.1a</td>
</tr>
<tr>
<td></td>
<td>Equation of tangent at $P(x_1, y_1)$ is $(y - y_1) = -\frac{x}{4y_1}(x - x_1)$</td>
<td>M1</td>
<td>3.1a</td>
</tr>
<tr>
<td></td>
<td>$xx_1 + 4yy_1 = x_1^2 + 4y_1^2 = 4$ and at $Q(x_2, y_2)$: $xx_2 + 4yy_2 = 4$</td>
<td>A1</td>
<td>2.2a</td>
</tr>
<tr>
<td></td>
<td>Intersect at $(r, s)$ gives $rx_1 + 4sy_1 = 4$ and $rx_2 + 4sy_2 = 4$</td>
<td>B1</td>
<td>2.1</td>
</tr>
<tr>
<td></td>
<td>Uses their previous results to find the gradient of the line $l$</td>
<td>M1</td>
<td>3.1a</td>
</tr>
<tr>
<td></td>
<td>$\frac{y_2 - y_1}{x_2 - x_1} = -\frac{r}{4s}$</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>Equation of $l$ is $y - y_1 = -\frac{r}{4s}(x - x_1)$</td>
<td>M1</td>
<td>2.1</td>
</tr>
<tr>
<td></td>
<td>$4sy + rx = 4sy_1 + rx_1 = 4^*$</td>
<td>A1*</td>
<td>2.2a</td>
</tr>
</tbody>
</table>

(8 marks)

Notes:

**M1:** Attempts to solve the problem by using differentiation to obtain an expression for $\frac{dy}{dx}$

**M1:** Realise the need to form a general equation of the tangent at $(x_1, y_1)$. May use alternative variables

**A1:** Deduces $x_1^2 + 4y_1^2 = 4$ to obtain a correct equation and deduces a correct second equation

**B1:** Uses $(r, s)$ in both equations to form the two given equations or exact equivalents

**M1:** Uses their previous results to find the gradient of the line $l$

**A1:** $-\frac{r}{4s}$

**M1:** Formulates the line $l$ with their $-\frac{r}{4s}$. Use of $y - y_1 = m(x - x_1)$ or $y = mx + c$ with their gradient and an attempt to find $c$

**A1***: Correct solution leading to $4sy + rx = 4sy_1 + rx_1$ with deduction that this equals 4 as $(x_1, y_1)$ is on the ellipse. No errors seen
<table>
<thead>
<tr>
<th>Question</th>
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<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>8(a)</strong></td>
<td>h(x) = 45 + 15\sin x + 21\sin\left(\frac{x}{2}\right) + 25\cos\left(\frac{x}{2}\right)</td>
<td></td>
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<tr>
<td></td>
<td>dh \over dx = 15\cos x + 21 \over 2 \cos\left(\frac{x}{2}\right) - 25 \over 2 \sin\left(\frac{x}{2}\right)</td>
<td>M1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>dh \over dx = \ldots + \ldots \over 1 + t^2 - \ldots \over 1 + t^2</td>
<td>M1 1.1a</td>
<td></td>
</tr>
<tr>
<td></td>
<td>e.g. \frac{dh}{dx} = \ldots \left(\frac{1-t^2}{1+t^2}\right)^2 \ldots + \ldots \over 1 + \left(\frac{2t}{1-t^2}\right)^2 \ldots + \ldots</td>
<td>M1 3.1a</td>
<td></td>
</tr>
<tr>
<td></td>
<td>e.g. \frac{dh}{dx} = 15 \left(\frac{1-t^2}{1+t^2}\right)^2 \ldots + \ldots \over 1 + \left(\frac{2t}{1-t^2}\right)^2 \ldots + \ldots</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>\ldots = 15\left[4(1-t^2)^2 - 2(1+t^2)^2\right] + 21\left(1-t^2\right)(1+t^2) - 50t(1+t^2) \over 2(1+t^2)^2</td>
<td>M1 2.1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>\ldots = \ldots \over 2(1+t^2)^2 \ldots = \ldots \over 2(1+t^2)^2 \ldots</td>
<td>A1* 2.1</td>
<td></td>
</tr>
<tr>
<td></td>
<td><strong>8(a) Alternative</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>h(x) = \ldots + 21 \over 1 + t^2 + 25 \over 1 + t^2</td>
<td>M1 1.1a</td>
<td></td>
</tr>
<tr>
<td></td>
<td>= \ldots + 15 \over 1 + \left(\frac{2t}{1-t^2}\right) \over 1 + \left(\frac{1-t^2}{1+t^2}\right) \ldots + \ldots</td>
<td>M1 2.1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>h(x) = 45 + 15(4t(1-t^2)) + 42t(1+t^2) + 25(1-t^4) \over (1+t^2)^2</td>
<td>M1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>h(x) = 45 - 25t^4 + 18t^3 - 102t - 25 \over (1+t^2)^2</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>dh \over dx = \frac{dh}{dt} \times \frac{dt}{dx} = \frac{\left(u'\right)(1+t^2)^2 - \left(u'\right)(4t(1+t^2))}{(1+t^2)^4} \times \frac{1}{4}(1+t^2)</td>
<td>M1 3.1a</td>
<td></td>
</tr>
<tr>
<td></td>
<td>\ldots = 9t^4 - 50t^4 - 180t^2 - 50t + 51 \over 2(1+t^2)^2 = \ldots \over 2(1+t^2)^2 \ldots</td>
<td>A1* 2.1</td>
<td></td>
</tr>
</tbody>
</table>
Question | Scheme | Marks | AOs
---|---|---|---
8(b)(i) | Accept any value between $\frac{1}{40} = 0.025$ and $\frac{1}{60} \approx 0.167$ inclusive | B1 | 3.3
(ii) | Suitable for times since the graphs both oscillate bi-modally with about the same periodicity | B1 | 3.4
Not suitable for predicting heights since the heights of the peaks vary over time, but the graph of $h(x)$ has fixed peak height | B1 | 3.5b
8(c) | Solves at least one of the quadratics $t = \frac{6 \pm \sqrt{36 - 4 \times 1 \times 17}}{2} = 3 \pm \sqrt{26}$ or $t = \frac{-4 \pm \sqrt{16 - 4 \times 9 \times (-3)}}{18} = -2 \pm \frac{\sqrt{31}}{9}$ | M1 | 1.1b
Find corresponding $x$ values, $x = 4 \tan^{-1}(t)$ for at least one value of $t$ from the $9t^2 + 4t - 3$ factor | M1 | 1.1b
One correct value for these $x$ e.g. $x = aw - 2.797$ or $9.770, 1.510$ | A1 | 1.1b
Maximum peak height occurs at smallest positive value of $x$, from first graph, but the third of these peaks needed, So $t = 1.509\ldots + 8\pi = 26.642$ is the required time | M1 | 3.4
$x = 26.642$ corresponds to 26 hours and 39 minutes (nearest minute) after 08:00 on 3rd January (Allow if a different greatest peak height used) | M1 | 3.4
Time of greatest tide height is approximately 10:39 (am) (also allow 10:38 or 10:40) | A1 | 3.2a

(15 marks)

Notes:
(a) **M1**: Differentiates $h(x)$
**M1**: Applies $t$-substitution to both $\left(\frac{x}{2}\right)$ terms with their coefficients
**M1**: Forms a correct expression in $t$ for the $\cos x$ term, using double angle formula and $t$-substitution, or double ‘$t’-substitution
**A1**: Fully correct expression in $t$ for $\frac{dh}{dx}$
**M1**: Gets all terms over the correct common factor. Numerators must be appropriate for their terms
**A1**: Achieves the correct answer via expression with correct quartic numerator before factorisation
<table>
<thead>
<tr>
<th>Question 8 notes continued:</th>
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<tbody>
<tr>
<td><strong>Alternative:</strong></td>
</tr>
<tr>
<td>(a)</td>
</tr>
<tr>
<td><strong>M1:</strong> Applies $t$-substitution to both $\left(\frac{x}{2}\right)$ terms</td>
</tr>
<tr>
<td><strong>M1:</strong> Forms a correct expression in $t$ for the $\sin x$ term, using double angle formula and $t$-substitution, or double ‘$t$’-substitution</td>
</tr>
<tr>
<td><strong>M1:</strong> Gets all terms in $t$ over the correct common factor. Numerators must be appropriate for their terms. May include the constant term too</td>
</tr>
<tr>
<td><strong>A1:</strong> Fully correct expression in $t$ for $h(x)$</td>
</tr>
<tr>
<td><strong>M1:</strong> Differentiates, using both chain rule and quotient rule with their ‘$u$’</td>
</tr>
<tr>
<td><strong>A1</strong>: Achieves the correct answer via expression with correct quartic numerator before factorisation</td>
</tr>
<tr>
<td><strong>Note:</strong> The individual terms may be differentiated before putting over a common denominator. In this case score the third M for differentiating with chain rule and quotient rule, then return to the original scheme</td>
</tr>
</tbody>
</table>

| (b)(i) |
| **B1:** Any value between $\frac{1}{40}$ (e.g. taking $h(0)$ as reference point) or $\frac{1}{60}$ (taking lower peaks as reference) |
| **NB:** Taking high peak as reference gives $\frac{1}{50}$ |

| (b)(ii) |
| **B1:** Should mention both the bimodal nature and periodicity for the actual data match the graph of $h$ |
| **B1:** Mentions that the heights of peaks vary in each oscillation |

| (c) |
| **M1:** Solves (at least) one of the quadratic equations in the numerator |
| **M1:** Must be attempting to solve the quadratic factor from which the solution comes $9t^2 + 4t - 3$ and using $t = \tan\left(\frac{x}{4}\right)$ to find a corresponding value for $x$ |
| **A1:** At least one correct $x$ value from solving the requisite quadratic: awrt any of $-2.797$, $1.510$, $9.770$, $14.076$, $22.336$, $26.642$, $34.902$ or $39.208$ |
| **M1:** Uses graph of $h$ to pick out their $x = 26.642$ as the time corresponding to the third of the higher peaks, which is the highest of the peaks on 4th January on the tide height graph. As per scheme or allow if all times listed and correct one picked |
| **M1:** Finds the time for one of the values of $t$ corresponding to the highest peaks. E.g. $1.5096\ldots \sim 09:31$ (3rd January) or $14.076\ldots \sim 22:05$ (3rd January) or $26.642\ldots \sim 10:39$ (4th January) or $39.208\ldots \sim 23:13$ (4th January). (Only follow through on use of the smallest positive $t$ solution $+ 4k\pi$) |
| **A1:** Time of greatest tide height on 4th January is approximately 10:39. Also allow 10:38 or 10:40 |
Alternative:
(a) 
M1: Applies \( t \)-substitution to both terms

M1: Forms a correct expression in \( t \) for the sine term, using double angle formula and \( t \)-substitution, or double \( t \)-substitution

M1: Gets all terms in \( t \) over the correct common factor. Numerators must be appropriate for their terms.

A1: Fully correct expression in \( t \) for

M1: Differentiates, using both chain rule and quotient rule with their \( u \)'s

A1*: Achieves the correct answer via expression with correct quartic numerator before factorisation

Note: The individual terms may be differentiated before putting over a common denominator. In this case score the third M for differentiating with chain rule and quotient rule, then return to the original scheme

(b)(i)
B1: Any value between \( \frac{1}{40} \) (e.g. taking \( h(0) \) as reference point)

NB: Taking high peak as reference gives \( \frac{1}{50} \)

(b)(ii)
B1: Should mention both the bimodal nature and periodicity for the actual data match the graph of \( h \)

B1: Mentions that the heights of peaks vary in each oscillation

(c)
M1: Solves (at least) one of the quadratic equations in the numerator

M1: Must be attempting to solve the quadratic factor from which the solution comes

\( 9t^2 + 4t - 3 \) and using \( \tan 4 \theta \) to find a corresponding value for \( x \)

A1: At least one correct \( x \) value from solving the requisite quadratic: awrt any of \(-2.797, 1.510, 9.770, 14.076, 22.336, 26.642, 34.902, 39.208\)

M1: Uses graph of \( h \) to pick out their \( x = 26.642 \) as the time corresponding to the third of the higher peaks, which is the highest of the peaks on 4th January on the tide height graph.

As per scheme or allow if all times listed and correct one picked

M1: Finds the time for one of the values of \( t \) corresponding to the highest peaks. E.g. \( 1.5096... \approx 09:31 \) (3rd January) or \( 14.076... \approx 22:05 \) (3rd January) or \( 26.642... \approx 10:39 \) (4th January) or \( 39.208... \approx 23:13 \) (4th January). (Only follow through on use of the smallest positive \( k \) \( \pi \) +)

A1: Time of greatest tide height on 4th January is approximately 10:39. Also allow 10:38 or 10:40
Answer ALL questions. Write your answers in the spaces provided.

1. (i) Use the Euclidean algorithm to find the highest common factor of 602 and 161.

Show each step of the algorithm.

(ii) The digits which can be used in a security code are the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9.

Originally the code used consisted of two distinct odd digits, followed by three distinct even digits.

To enable more codes to be generated, a new system is devised. This uses two distinct even digits, followed by any three other distinct digits. No digits are repeated.

Find the increase in the number of possible codes which results from using the new system.
Find the increase in the number of possible codes which results from using the new system.

To enable more codes to be generated, a new system is devised. This uses two repeated distinct even digits, followed by any three other distinct digits. No digits are repeated.

Show each step of the algorithm.

(i) Use the Euclidean algorithm to find the highest common factor of 602 and 161.

1.

(Total for Question 1 is 7 marks)
2. A transformation from the $z$-plane to the $w$-plane is given by

$$w = z^2$$

(a) Show that the line with equation $\text{Im}(z) = 1$ in the $z$-plane is mapped to a parabola in the $w$-plane, giving an equation for this parabola.

(4)

(b) Sketch the parabola on an Argand diagram.

(2)
Question 2 continued

(Total for Question 2 is 6 marks)
3. The matrix $M$ is given by

$$M = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{pmatrix}$$

(a) Show that 4 is an eigenvalue of $M$, and find the other two eigenvalues. \hspace{1cm} (4)

(b) For each of the eigenvalues find a corresponding eigenvector. \hspace{1cm} (4)

(c) Find a matrix $P$ such that $P^{-1}MP$ is a diagonal matrix. \hspace{1cm} (2)
Question 3 continued
Question 3 continued

(Total for Question 3 is 10 marks)
4. (i) A group $G$ contains distinct elements $a$, $b$ and $e$ where $e$ is the identity element and the group operation is multiplication.

Given $a^3 b = ba$, prove $ab \neq ba$ \hspace{1cm} (4)

(ii) The set $H = \{1, 2, 4, 7, 8, 11, 13, 14\}$ forms a group under the operation of multiplication modulo 15

(a) Find the order of each element of $H$. \hspace{1cm} (3)

(b) Find three subgroups of $H$ each of order 4, and describe each of these subgroups. \hspace{1cm} (4)

The elements of another group $J$ are the matrices

$$\begin{pmatrix} \cos\left(\frac{k\pi}{4}\right) & \sin\left(\frac{k\pi}{4}\right) \\ -\sin\left(\frac{k\pi}{4}\right) & \cos\left(\frac{k\pi}{4}\right) \end{pmatrix}$$

where $k = 1, 2, 3, 4, 5, 6, 7, 8$ and the group operation is matrix multiplication.

(c) Determine whether $H$ and $J$ are isomorphic, giving a reason for your answer. \hspace{1cm} (2)
Question 4 continued
Question 4 continued
Question 4 continued

(Total for Question 4 is 13 marks)
5.

An engineering student makes a miniature arch as part of the design for a piece of coursework.

The cross-section of this arch is modelled by the curve with equation

\[ y = A - \frac{1}{2} \cosh 2x, \quad -\ln a \leq x \leq \ln a \]

where \( a > 1 \) and \( A \) is a positive constant. The curve begins and ends on the \( x \)-axis, as shown in Figure 1.

(a) Show that the length of this curve is \( k \left( a^2 - \frac{1}{a^2} \right) \), stating the value of the constant \( k \). \( \phantom{\text{(a)}} \) (5)

The length of the curved cross-section of the miniature arch is required to be 2 m long.

(b) Find the height of the arch, according to this model, giving your answer to 2 significant figures. \( \phantom{\text{(b)}} \) (4)

(c) Find also the width of the base of the arch giving your answer to 2 significant figures. \( \phantom{\text{(c)}} \) (1)

(d) Give the equation of another curve that could be used as a suitable model for the cross-section of an arch, with approximately the same height and width as you found using the first model.

(You do not need to consider the arc length of your curve) \( \phantom{\text{(d)}} \) (2)
Question 5 continued
Question 5 continued
Question 5 continued

(Total for Question 5 is 12 marks)
6. A curve has equation

$$|z + 6| = 2 |z - 6| \quad z \in \mathbb{C}$$

(a) Show that the curve is a circle with equation $x^2 + y^2 - 20x + 36 = 0$ (2)

(b) Sketch the curve on an Argand diagram. (2)

The line $l$ has equation $az^* + a^*z = 0$, where $a \in \mathbb{C}$ and $z \in \mathbb{C}$

Given that the line $l$ is a tangent to the curve and that $\arg a = \theta$

(c) find the possible values of $\tan \theta$ (5)
(b) Sketch the curve on an Argand diagram.

6. A curve has equation \(az + a^* = \frac{1}{2} - iz\) where \(z\) is a tangent to the curve and that \(\arg (z + 6) = 2\). Solve for \(z\) and \(\arg (z^2 + \frac{1}{2}z - 20)\).
Question 6 continued
Question 6 continued

(Total for Question 6 is 9 marks)
7.  

\[ I_n = \int_0^\pi \sin^n x \, dx, \quad n \geq 0 \]

(a) Prove that, for \( n \geq 2 \),

\[ nI_n = (n - 1)I_{n-2} \quad (4) \]

(b)  

Figure 2  

A designer is asked to produce a poster to completely cover the curved surface area of a solid cylinder which has diameter 1 m and height 0.7 m.  

He uses a large sheet of paper with height 0.7 m and width of \( \pi \) m.  

Figure 2 shows the first stage of the design, where the poster is divided into two sections by a curve.  

The curve is given by the equation  

\[ y = \sin^2(4x) - \sin^{10}(4x) \]

relative to axes taken along the bottom and left hand edge of the paper.  

The region of the poster below the curve is shaded and the region above the curve remains unshaded, as shown in Figure 2.  

Find the exact area of the poster which is shaded.  

(5)
Question 7 continued
Question 7 continued
8. A staircase has \( n \) steps. A tourist moves from the bottom (step zero) to the top (step \( n \)). At each move up the staircase she can go up either one step or two steps, and her overall climb up the staircase is a combination of such moves.

If \( u_n \) is the number of ways that the tourist can climb up a staircase with \( n \) steps,

(a) explain why \( u_n \) satisfies the recurrence relation

\[
u_n = u_{n-1} + u_{n-2}, \text{ with } u_1 = 1 \text{ and } u_2 = 2
\]

(b) Find the number of ways in which she can climb up a staircase when there are eight steps.

A staircase at a certain tourist attraction has 400 steps.

(c) Show that the number of ways in which she could climb up to the top of this staircase is given by

\[
\frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^{401} - \left( \frac{1 - \sqrt{5}}{2} \right)^{401}
\]
Show that the number of ways in which she could climb up to the top of this staircase.

A staircase at a certain tourist attraction has 400 steps.

(a) Explain why

(b) Find the number of ways in which she can climb up a staircase when there are 8 steps.

\[
\begin{align*}
\text{climb up the staircase is a combination of such moves.} \\
\text{At each move up the staircase she can go up either one step or two steps, and her overall} \\
n \text{steps. A tourist moves from the bottom (step zero) to the top (step n).} \\
\text{is the number of ways that the tourist can climb up a staircase with} \\
n \text{satisfies the recurrence relation} \\
\end{align*}
\]

\[
\begin{align*}
\text{} = \binom{n}{1} \times 1 + \binom{n}{2} \times 2 \\
= 1 + \binom{n}{2} \\
\end{align*}
\]
Question 8 continued
1(i) 
\[ 602 \times 3 + 161 = 119 \times M1 1.1b \]
\[ 161 + 119 = 42 \times 2 + 42 = 35 \times M1 1.1b \]
\[ 42 + 35 = 7 \times 5 = 7 \times A1 1.1b \]

(ii)
Number of codes under old system = 5 4 4 3 2 \( \times \times \times \times = B1 3.1b \)
Number of codes under new system = 4 3 7 6 5 \( \times \times \times \times \times = B1 3.1b \)
Subtracts first answer from second M1 1.1b
Increase in number of codes is 2040 A1 1.1b

(4) (7 marks)
Notes:
(i) M1: Attempts Euclid's algorithm – (there may be an arithmetic slip finding 119)
M1: Uses Euclid's algorithm a further two times with 161 and "their 119" and then with "their 119" and "their 42"
A1: This should be accurate with all the steps shown
(ii) B1: Correctly interprets the problem and uses the five odd digits and four even digits to form a correct product
B1: Interprets the new situation using the four even digits, then the seven digits which have not been used, to form a correct product
M1: Subtracts one answer from the other
A1: Correct answer
### Paper 4A: Further Pure Mathematics 2 Mark Scheme

<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1(i)</td>
<td>(602 = 3 \times 161 + 119)</td>
<td>M1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>(161 = 119 + 42, \ 119 = 2 \times 42 + 35)</td>
<td>M1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>(42 = 35 + 7, \ 35 = 5 \times 7, \ \text{hcf} = 7)</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td>(ii)</td>
<td>Number of codes under old system = (5 \times 4 \times 4 \times 3 \times 2) ((= 480))</td>
<td>B1</td>
<td>3.1b</td>
</tr>
<tr>
<td></td>
<td>Number of codes under new system = (4 \times 3 \times 7 \times 6 \times 5) ((= 2520))</td>
<td>B1</td>
<td>3.1b</td>
</tr>
<tr>
<td></td>
<td>Subtracts first answer from second</td>
<td>M1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>Increase in number of codes is 2040</td>
<td>A1</td>
<td>1.1b</td>
</tr>
</tbody>
</table>

(3 marks)

(7 marks)

**Notes:**

(i) **M1:** Attempts Euclid’s algorithm – (there may be an arithmetic slip finding 119)

**M1:** Uses Euclid’s algorithm a further two times with 161 and “their 119” and then with “their 119” and “their 42”

**A1:** This should be accurate with all the steps shown

(ii) **B1:** Correctly interprets the problem and uses the five odd digits and four even digits to form a correct product

**B1:** Interprets the new situation using the four even digits, then the seven digits which have not been used, to form a correct product

**M1:** Subtracts one answer from the other

**A1:** Correct answer
### Question Scheme Marks AOs

<table>
<thead>
<tr>
<th>Question</th>
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<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>2(a)</strong></td>
<td>Let ( z = x + i )</td>
<td>M1 2.1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( w = (x+i)^2 = (x^2 - 1) + 2xi )</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Let ( w = u + iv ), then ( u = (x^2 - 1) ) and ( v = 2x )</td>
<td>M1 2.1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \Rightarrow v^2 = 4(u+1) ), which therefore represents a parabola</td>
<td>A1ft 2.2a</td>
<td></td>
</tr>
<tr>
<td><strong>(b)</strong></td>
<td><img src="image" alt="Diagram" /></td>
<td>M1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>M1: Sketches a parabola with symmetry about the real axis</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>A1: Accurate sketch</td>
<td>A1 1.1b</td>
<td></td>
</tr>
</tbody>
</table>

#### Notes:

**(a)**
- **M1**: Translates the information that \( \text{Im}(z) = 1 \) into a cartesian form; e.g. \( z = x + i \)
- **A1**: Obtains a correct expression for \( w \)
- **M1**: Separates the real and imaginary parts and equates to \( u \) and \( v \) respectively
- **A1ft**: Obtains a quadratic equation and states that their quadratic equation represents a parabola

**.(b)**
- **M1**: Sketches a parabola with symmetry about the real axis
- **A1**: Accurate sketch
### Question Scheme Marks AOs

<table>
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<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>3(a)</td>
<td>Finds the characteristic equation $(2-\lambda)^2(4-\lambda)-(4-\lambda)=0$</td>
<td>M1 2.1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>So $(4-\lambda)(\lambda^2-4\lambda+3)=0$ so $\lambda = 4^*$</td>
<td>A1* 2.2a</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Solves quadratic equation to give $\lambda = 1$ and $\lambda = 3$</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(4)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(b)</td>
<td>Uses a correct method to find an eigenvector</td>
<td>M1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Obtains a vector parallel to one of $\begin{pmatrix} 0 \ 0 \ 1 \end{pmatrix}$ or $\begin{pmatrix} 1 \ 1 \ 1 \end{pmatrix}$ or $\begin{pmatrix} 3 \ -3 \ 1 \end{pmatrix}$</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Obtains two correct vectors</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Obtains all three correct vectors</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(4)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(c)</td>
<td>Uses their three vectors to form a matrix</td>
<td>M1 1.2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\begin{pmatrix} 0 &amp; 1 &amp; 3 \ 0 &amp; 1 &amp; -3 \ 1 &amp; 1 &amp; 1 \end{pmatrix}$ or other correct answer with columns in a different order</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(2)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Notes:

**(a)**
- **M1**: Attempts to find the characteristic equation (there may be one slip)
- **A1**: Deduces that $\lambda = 4$ is a solution by the method shown or by checking that $\lambda = 4$ satisfies the characteristic equation
- **M1**: Solves their quadratic equation
- **A1**: Obtains the two correct answers as shown above

***(b)**
- **M1**: Uses a correct method to find an eigenvector
- **A1**: Obtains one correct vector (may be a multiple of the given vectors)
- **A1**: Obtains two correct vectors (may be multiples of the given vectors)
- **A1**: Obtains all three correct vectors (may be multiples of the given vectors)

**(c)**
- **M1**: Forms a matrix with their vectors as columns
- **A1**: $\begin{pmatrix} 0 & 1 & 3 \\ 0 & 1 & -3 \\ 1 & 1 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & 3 \\ 1 & 0 & -3 \\ 1 & 1 & 1 \end{pmatrix}$ or $\begin{pmatrix} 3 & 1 & 0 \\ -3 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ or other correct alternative
<table>
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</tr>
</thead>
<tbody>
<tr>
<td>4(i)</td>
<td>If we assume ( ab = ba; ) as ( a^2b = ba ) then ( ab = a^2b )</td>
<td>M1 2.1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \text{So } a^{-1}abb^{-1} = a^{-1}a^2bb^{-1} )</td>
<td>M1 2.1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \text{So } e = a )</td>
<td>A1 2.2a</td>
<td></td>
</tr>
<tr>
<td></td>
<td>But this is a contradiction, as the elements ( e ) and ( a ) are distinct so ( ab \neq ba )</td>
<td>A1 2.4</td>
<td></td>
</tr>
<tr>
<td>(ii)(a)</td>
<td>2 has order 4 and 4 has order 2</td>
<td>M1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>7, 8 and 13 have order 4</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>11 and 14 have order 2 and 1 has order 1</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td>(ii)(b)</td>
<td>Finds the subgroup {1, 2, 4, 8} or the subgroup {1, 7, 4, 13}</td>
<td>M1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Finds both and refers to them as cyclic groups, or gives generator 2 and generator 7</td>
<td>A1 2.4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Finds {1, 4, 11, 14}</td>
<td>B1 2.2a</td>
<td></td>
</tr>
<tr>
<td></td>
<td>States each element has order 2 or refers to it as Klein Group</td>
<td>B1 2.5</td>
<td></td>
</tr>
<tr>
<td>(ii)(c)</td>
<td>( J ) has an element of order 8, ( (H ) does not) or ( J ) is a cyclic group ( (H ) is not) or other valid reason</td>
<td>M1 2.4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>They are not isomorphic</td>
<td>A1 2.2a</td>
<td></td>
</tr>
</tbody>
</table>

(13 marks)
### Question 4 notes:

#### (i)
- **M1:** Proof begins with assumption that $ab = ba$ and deduces that this implies $ab = a^2b$
- **M1:** A correct proof with working shown follows, and may be done in two stages
- **A1:** Concludes that assumption implies that $e = a$
- **A1:** Explains clearly that this is a contradiction, as the elements $e$ and $a$ are distinct so $ab \neq ba$

#### (ii)(a)
- **M1:** Obtains two correct orders (usually the two in the scheme)
- **A1:** Finds another three correctly
- **A1:** Finds the final three so that all eight are correct

#### (ii)(b)
- **M1:** Finds one of the cyclic subgroups
- **A1:** Finds both subgroups and explains that they are cyclic groups, or gives generators 2 and 7
- **B1:** Finds the non cyclic group
- **B1:** Uses correct terms that each element has order 2 or refers to it as Klein Group

#### (ii)(c)
- **M1:** Clearly explains how $J$ differs from $H$
- **A1:** Correct deduction
### Question 5(a)

\[ \frac{dy}{dx} = -\sinh 2x \]

So \[ S = \int \sqrt{1 + \sinh^2 2x} \, dx \]

\[ \therefore s = \int \cosh 2x \, dx \]

\[ = \left[ \frac{1}{2} \sinh 2x \right]_0^a \text{ or } [\sinh 2x]_0^a \]

\[ = \sinh 2\ln a = \frac{1}{2} \left[ e^{2\ln a} - e^{-2\ln a} \right] = \frac{1}{2} \left( a^2 - \frac{1}{a^2} \right) \quad (\text{so } k = \frac{1}{2}) \]

### (5)  

### (b)

\[ \frac{1}{2} \left( a^2 - \frac{1}{a^2} \right) = 2 \text{ so } a^4 - 4a^2 - 1 = 0 \]

\[ a^2 = 2 + \sqrt{5} \quad (\text{and } a = 2.06 \text{ (approx.)}) \]

When \( x = \ln a, y = 0 \) so \( A = \frac{1}{2} \cosh (2\ln a) \)

Height = \( A - 0.5 = \text{awt 0.62m} \)

### (4)  

### (c)

The width of the base = \( 2\ln a = 1.4m \)

### (1)  

### (d)

A parabola of the form \( y = 0.62 - 1.19x^2 \), or other symmetric curve with its equation e.g. 0.62cos(2.2x)

### (12 marks)

### Notes:

(a)  
**B1:** Starts explanation by finding the correct derivative  
**M1:** Uses their derivative in the formula for arc length  
**A1:** Uses suitable identity to simplify the integrand and to obtain the expression in scheme  
**M1:** Integrates and uses appropriate limits to find the required arc length  
**A1:** Uses the definition of sinh to complete the proof and identifies the value for \( k \)

(b)  
**M1:** Uses the formula obtained from the model and the length of the arch to create a quartic equation  
**M1:** Continues to use this model to obtain a quadratic and to obtain values for \( a \)  
**M1:** Attempts to find a value for \( A \) in order to find \( h \)  
**A1:** Finds a value for the height correct to 2sf (or accept exact answer)

(c)  
**B1:** Finds width to 2 sf i.e. 1.4m

(d)  
**M1:** Chooses or describes an even function with maximum point on the y axis  
**A1:** Gives suitable equation passing through \((0, 0.62)\) and \((0.7, 0)\) and \((-0.7, 0)\)
<table>
<thead>
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</tr>
</thead>
<tbody>
<tr>
<td><strong>6(a)</strong></td>
<td>$(x + 6)^2 + y^2 = 4[(x - 6)^2 + y^2]$</td>
<td>M1</td>
<td>2.1</td>
</tr>
<tr>
<td></td>
<td>$x^2 + y^2 - 20x + 36 = 0$ which is the equation of a circle*</td>
<td>A1*</td>
<td>2.2a</td>
</tr>
<tr>
<td><strong>(b)</strong></td>
<td><img src="image" alt="Graph of a circle" /></td>
<td>M1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td><img src="image" alt="Graph of a circle" /></td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td><strong>(c)</strong></td>
<td>Let $a = c + id$ and $a^* = c - id$ then $(c + id)(x - iy) + (c - id)(x + iy) = 0$</td>
<td>M1</td>
<td>3.1a</td>
</tr>
<tr>
<td></td>
<td>So $y = -\frac{c}{d}x$</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td><strong>(d)</strong></td>
<td><img src="image" alt="Graph of a circle and tangent" /></td>
<td>B1</td>
<td>3.1a</td>
</tr>
<tr>
<td></td>
<td>The gradients of the tangents (from geometry) are $\pm \frac{4}{3}$</td>
<td>M1</td>
<td>3.1a</td>
</tr>
<tr>
<td></td>
<td>So $-\frac{c}{d} = \pm \frac{4}{3}$ and $\frac{d}{c} = \mp \frac{3}{4}$</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>So $\tan \theta = \pm \frac{3}{4}$</td>
<td>(5)</td>
<td></td>
</tr>
</tbody>
</table>
Question 6 notes:

(a)
M1: Obtains an equation in terms of $x$ and $y$ using the given information
A1*: Expands and simplifies the algebra, collecting terms and obtains a circle equation correctly, deducing that this is a circle

(b)
M1: Draws a circle with centre at $(10, 0)$
A1: (Radius is 8) so circle does not cross the $y$ axis

(c)
M1: Attempts to convert line equation into a cartesian form
A1: Obtains a simplified line equation
B1: Uses geometry to deduce the gradients of the tangents
M1: Understands the connection between $\arg a$ and the gradient of the tangents and uses this connection
A1: Correct answers
### Question 7(a)

\[ I_n = \int_0^\pi \sin x \sin^{n-1} x \, dx \]

\[
= \left[ -\cos x \sin^{n-1} x \right]_0^\pi + \int_0^\pi \cos^2 x (n-1) \sin^{n-2} x \, dx 
\]

Obtains \(-0-\int_0^\pi (1-\sin^2 x)(n-1) \sin^{n-2} x \, dx\)

So \( I_n = (n-1)I_{n-2} - (n-1)I_1 \) and hence \( nI_n = (n-1)I_{n-2} \) \* \( \text{A1*} \)

(9 marks)

### Notes:

(a)

M1: Splits the integrand into the product shown and begins process of integration by parts (there may be sign errors)

A1: Correct work

M1: Uses limits on the first term and expresses \( \cos^2 \) term in terms of \( \sin^2 \)

A1*: Completes the proof collecting \( I_n \) terms correctly with all stages shown

(b)

uses \( I_n = \frac{(n-1)}{n} I_{n-2} \) to give \( I_{10} = \frac{9}{10} I_8 \) or \( I_2 = \frac{1}{2} I_0 \)

So \( I_{10} = \frac{9 \times 7 \times 5 \times 3 \times 1}{10 \times 8 \times 6 \times 4 \times 2} I_0 \)

\[ I_0 = \frac{\pi}{2} \]

B1 1.1b

Required area is \( 2(I_2 - I_{10}) = \), or \( 8 \times \frac{1}{2}(I_2 - I_{10}) = \)

\[ = 2 \left( \frac{\pi}{4} - \frac{63\pi}{512} \right) = \frac{65\pi}{256} \]

A1 1.1b

(5)

(9 marks)
<table>
<thead>
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<tbody>
<tr>
<td><strong>8(a)</strong></td>
<td>$u_1 = 1$ as there is only one way to go up one step</td>
<td>B1</td>
<td>2.4</td>
</tr>
<tr>
<td></td>
<td>$u_2 = 2$ as there are two ways: one step then one step or two steps</td>
<td>B1</td>
<td>2.4</td>
</tr>
<tr>
<td></td>
<td>If first move is one step then can climb the other $(n-1)$ steps in $u_{n-1}$ ways. If first move is two steps can climb the other $(n-2)$ steps in $u_{n-2}$ ways so $u_n = u_{n-1} + u_{n-2}$</td>
<td>B1</td>
<td>2.4</td>
</tr>
<tr>
<td>(3)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(b)</strong></td>
<td>Sequence begins 1, 2, 3, 5, 8, 13, 21, 34,... so 34 ways of climbing 8 steps</td>
<td>B1</td>
<td>1.1b</td>
</tr>
<tr>
<td>(1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(c)</strong></td>
<td>To find general term use $u_n = u_{n-1} + u_{n-2}$ gives $\lambda^2 = \lambda + 1$</td>
<td>M1</td>
<td>2.1</td>
</tr>
<tr>
<td></td>
<td>This has roots $\frac{1 \pm \sqrt{5}}{2}$</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>So general form is $A\left(\frac{1 + \sqrt{5}}{2}\right)^n + B\left(\frac{1 - \sqrt{5}}{2}\right)^n$</td>
<td>M1</td>
<td>2.2a</td>
</tr>
<tr>
<td></td>
<td>Uses initial conditions to find $A$ and $B$ reaching two equations in $A$ and $B$</td>
<td>M1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>Obtains $A = \left(\frac{1 + \sqrt{5}}{2}\sqrt{5}\right)$ and $B = -\left(\frac{1 - \sqrt{5}}{2}\sqrt{5}\right)$ and so when $n = 400$ obtains $\frac{1}{\sqrt{5}}\left[\left(\frac{1 + \sqrt{5}}{2}\right)^{401} - \left(\frac{1 - \sqrt{5}}{2}\right)^{401}\right]$</td>
<td>A1*</td>
<td>1.1b</td>
</tr>
<tr>
<td>(5)</td>
<td></td>
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</tr>
</tbody>
</table>

**Notes:**

(a)  
**B1:** Need to see explanation for $u_1 = 1$
**B1:** Need to see explanation for $u_2 = 2$ with the two ways spelled out
**B1:** Need to see the first move can be one step or can be two steps and clear explanation of the iterative expression as in the scheme

(b)  
**B1:** The answer is enough for this mark

(c)  
**M1:** Obtains this characteristic equation
**A1:** Solves quadratic – giving exact answers
**M1:** Obtains a general form
**M1:** Use initial conditions to obtains two equations which should be $A\left(1 + \sqrt{5}\right) + B\left(1 - \sqrt{5}\right) = 2$
o.e. and $A\left(3 + \sqrt{5}\right) + B\left(3 - \sqrt{5}\right) = 4$ but allow slips here
**A1*:** Must see exact correct values for $A$ and $B$ and conclusion given for $n = 400$
(a)

\[ u = \text{as there is only one way to go up one step} \]

\[ B_1 = 2.4 \]

\[ u = \text{as there are two ways: one step then one step or two steps} \]

\[ B_1 = 2.4 \]

If first move is one step then can climb the other \((n - 1)\) steps in \(u^{n-1}\) ways. If first move is two steps can climb the other \((n - 2)\) steps in \(2u^{n-2}\) ways so

\[ 12nuu = \]

\[ B_1 = 2.4 \]

(b)

Sequence begins 1, 2, 3, 5, 8, 13, 21, 34, … so 34 ways of climbing 8 steps \(B_1 = 1.1b\)

(c)

To find general term use

\[ 2^{12} \text{ gives } 1 \]

\[ uu = \]

\[ M_1 = 2.1 \]

This has roots

\[ 152 \]

\[ A_1 = 1.1b \]

So general form is

\[ 152nn = \]

\[ AB \]

\[ M_1 = 2.2a \]

Uses initial conditions to find

\[ A \]

and

\[ B \]

reaching two equations in

\[ A \]

and

\[ B \]

M1 1.1b

Obtains

\[ 152A = \]

\[ \frac{1}{2} \]

\[ \frac{1}{2} \]

\[ 152B = \]

\[ \frac{1}{2} \]

\[ \frac{1}{2} \]

and so when \(n = 400\) obtains

\[ 401225 = \]

\[ \frac{1}{2} \]

\[ \frac{1}{2} \]

\[ \frac{1}{2} \]

\[ *A1* = 1.1b \]
1. Bacteria are randomly distributed in a river at a rate of 5 per litre of water. A new factory opens and a scientist claims it is polluting the river with bacteria. He takes a sample of 0.5 litres of water from the river near the factory and finds that it contains 7 bacteria. Stating your hypotheses clearly test, at the 5% level of significance, whether there is evidence that the level of pollution has increased.
2. A call centre routes incoming telephone calls to agents who have specialist knowledge to deal with the call. The probability of a caller, chosen at random, being connected to the wrong agent is \( p \).

The probability of at least 1 call in 5 consecutive calls being connected to the wrong agent is 0.049.

The call centre receives 1000 calls each day.

(a) Find the mean and variance of the number of wrongly connected calls a day. (7)

(b) Use a Poisson approximation to find, to 3 decimal places, the probability that more than 6 calls each day are connected to the wrong agent. (2)

(c) Explain why the approximation used in part (b) is valid. (2)

The probability that more than 6 calls each day are connected to the wrong agent using the binomial distribution is 0.8711 to 4 decimal places.

(d) Comment on the accuracy of your answer in part (b). (1)
Question 2 continued
Question 2 continued

(Total for Question 2 is 12 marks)
3. Bags of £1 coins are paid into a bank. Each bag contains 20 coins.

The bank manager believes that 5% of the £1 coins paid into the bank are fakes. He decides to use the distribution $X \sim B(20, 0.05)$ to model the random variable $X$, the number of fake £1 coins in each bag.

The bank manager checks a random sample of 150 bags of £1 coins and records the number of fake coins found in each bag. His results are summarised in Table 1. He then calculates some of the expected frequencies, correct to 1 decimal place.

<table>
<thead>
<tr>
<th>Number of fake coins in each bag</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4 or more</th>
</tr>
</thead>
<tbody>
<tr>
<td>Observed frequency</td>
<td>43</td>
<td>62</td>
<td>26</td>
<td>13</td>
<td>6</td>
</tr>
<tr>
<td>Expected frequency</td>
<td>53.8</td>
<td>56.6</td>
<td>8.9</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1

(a) Carry out a hypothesis test, at the 5% significance level, to see if the data supports the bank manager’s statistical model. State your hypotheses clearly.

The assistant manager thinks that a binomial distribution is a good model but suggests that the proportion of fake coins is higher than 5%. She calculates the actual proportion of fake coins in the sample and uses this value to carry out a new hypothesis test on the data. Her expected frequencies are shown in Table 2.

<table>
<thead>
<tr>
<th>Number of fake coins in each bag</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4 or more</th>
</tr>
</thead>
<tbody>
<tr>
<td>Observed frequency</td>
<td>43</td>
<td>62</td>
<td>26</td>
<td>13</td>
<td>6</td>
</tr>
<tr>
<td>Expected frequency</td>
<td>44.5</td>
<td>55.7</td>
<td>33.2</td>
<td>12.5</td>
<td>4.1</td>
</tr>
</tbody>
</table>

Table 2

(b) Explain why there are 2 degrees of freedom in this case.

(c) Given that she obtains a $\chi^2$ test statistic of 2.67, test the assistant manager’s hypothesis that the binomial distribution is a good model for the number of fake coins in each bag. Use a 5% level of significance and state your hypotheses clearly.
Question 3 continued
4. A random sample of 100 observations is taken from a Poisson distribution with mean 2.3

Estimate the probability that the mean of the sample is greater than 2.5

(4)
Estimate the probability that the mean of the sample is greater than 2.5

4.
A random sample of 100 observations is taken from a Poisson distribution with mean 2.3

(Total for Question 4 is 4 marks)
5. The probability of Richard winning a prize in a game at the fair is 0.15

Richard plays a number of games.

(a) Find the probability of Richard winning his second prize on his 8th game. (2)

(b) State two assumptions that have to be made, for the model used in part (a) to be valid. (2)

Mary plays the same game, but has a different probability of winning a prize. She plays until she has won \( r \) prizes. The random variable \( G \) represents the total number of games Mary plays.

(c) Given that the mean and standard deviation of \( G \) are 18 and 6 respectively, determine whether Richard or Mary has the greater probability of winning a prize in a game. (4)
Question 5 continued

(Total for Question 5 is 8 marks)
6. The probability generating function of the discrete random variable $X$ is given by

$$G_X(t) = k(3 + t + 2t^2)^2$$

(a) Show that $k = \frac{1}{36}$  

(b) Find $P(X = 3)$  

(c) Show that $\text{Var}(X) = \frac{29}{18}$  

(d) Find the probability generating function of $2X + 1$
Question 6 continued
Question 6 continued

(Total for Question 6 is 14 marks)
7. Sam and Tessa are testing a spinner to see if the probability, $p$, of it landing on red is less than $\frac{1}{5}$. They both use a 10% significance level.

Sam decides to spin the spinner 20 times and record the number of times it lands on red.

(a) Find the critical region for Sam’s test. (2)

(b) Write down the size of Sam’s test. (1)

Tessa decides to spin the spinner until it lands on red and she records the number of spins.

(c) Find the critical region for Tessa’s test. (6)

(d) Find the size of Tessa’s test. (1)

(e) (i) Show that the power function for Sam’s test is given by

$$(1 - p)^{19}(1 + 19p)$$

(ii) Find the power function for Tessa’s test. (4)

(f) With reference to parts (b), (d) and (e), state, giving your reasons, whether you would recommend Sam’s test or Tessa’s test when $p = 0.15$ (4)
Question 7 continued
Question 7 continued
Question 7 continued
Ho: \( \lambda = 5 \) (\( \lambda = 2.5 \))

H1: \( \lambda > 5 \) (\( \lambda > 2.5 \))

\( X \sim \text{Po}(2.5) \)

**Method 1:**

\[
P(X = 7) = 0.1088
\]

\[
P(X = 6) = 0.042
\]

7 is in critical region

Reject Ho. There is evidence at the 5% significance level that the level of pollution has increased.

or

There is evidence to support the scientists claim is justified

**Notes:**

B1: Both hypotheses correct using \( \lambda \) or \( \mu \) and 5 or 2.5

B1: Realising that the model \( \text{Po}(2.5) \) is to be used. This may be stated or used

M1: Using or writing

\[
1 - P(X \leq 5) < 0.05
\]

awrt 0.0142

A1: awrt 0.0142 or CR

X \( \geq 6 \) or \( X > 5 \)

M1: A fully correct solution and drawing a correct inference in context
### Paper 3B/4B: Further Statistics 1 Mark Schemes

<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1</strong></td>
<td>$H_0 : \lambda = 5 \ (\lambda = 2.5)$ \ $H_1 : \lambda &gt; 5 \ (\lambda &gt; 2.5)$</td>
<td>B1 2.5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$X \sim \text{Po} (2.5)$</td>
<td>B1 3.3</td>
<td></td>
</tr>
<tr>
<td></td>
<td><strong>Method 1:</strong></td>
<td><strong>Method 2:</strong></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$P(X \geq 7) = 1 - P(X \leq 6)$</td>
<td>$P(X \geq 5) = 0.1088$ \ $P(X \geq 6) = 0.042$</td>
<td>M1 1.1b</td>
</tr>
<tr>
<td></td>
<td>$= 1 - 0.9858$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$= 0.0142$</td>
<td>CR $X \geq 6$</td>
<td>A1 1.1b</td>
</tr>
<tr>
<td></td>
<td>$0.0142 &lt; 0.05$ \ $7 \geq 6 \ or \ 7$ is in critical region \ or \ 7 is significant</td>
<td>Reject $H_0$. There is evidence at the 5% significance level that the level of pollution has increased. \ or \ There is evidence to support the scientists claim is justified</td>
<td>A1cso 2.2b</td>
</tr>
</tbody>
</table>

### Notes:

**B1:** Both hypotheses correct using $\lambda$ or $\mu$ and 5 or 2.5

**B1:** Realising that the model Po(2.5) is to be used. This may be stated or used

**M1:** Using or writing \ $1 - P(X \leq 6)$ or $1 - P(X < 7)$ \ a correct CR or $P(X \geq 5) = \text{awrt} \ 0.109$ and $P(X \geq 6) = \text{awrt} \ 0.042$

**A1:** awrt 0.0142 or CR $X \geq 6$ or $X > 5$

**M1:** A fully correct solution and drawing a correct inference in context
<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
</tr>
</thead>
</table>
| 2(a)   | $P(X \geq 1) = 1 - P(X = 0)$  
$1 - P(X = 0) = 0.049$  
$P(X = 0) = 0.951$  
$x^2 = 0.951$  
$x = 0.99$  
$p = 0.01$  
$X \sim B(1000, 0.01)$  
Mean = $np = 10$  
Variance = $np(1 - p) = 9.9$ |
|        | B1     | 3.1b |
|        | B1     | 1.1b |
|        | M1     | 3.1b |
|        | A1     | 1.1b |
|        | M1     | 3.3  |
|        | A1ft   | 1.1b |
|        | A1ft   | 1.1b |
|        | (7)    |      |
| (b)    | $X \sim Po(“10”)$ then require: $P(X > 6) = 1 - P(X \leq 6)$  
$= 1 - 0.1301$  
$= 0.870$ |
|        | M1     | 3.4  |
|        | A1     | 1.1b |
|        | (2)    |      |
| (c)    | The approximation is valid as: the number of calls is large  
The probability of connecting to the wrong agent is small |
|        | B1     | 2.4  |
|        | B1     | 2.4  |
| (d)    | The answer is accurate to 2 decimal place |
|        | B1     | 3.2b |
|        | (1)    |      |
|        | (12 marks) |      |

**Notes:**

(a)  
B1: Realising that the $P$ (at least 1 call) = $1 - P(X = 0)$  
B1: Calculating $P(X = 0) = 0.951$  
M1: Forming the equation $x^2 = \text{their } 0.951$ may be implied by $p = 0.01$  
A1: 0.01 only  
M1: Realising the need to use the model $B(1000, 0.01)$ This may be stated or used  
A1: Mean = 10 or ft their $p$ but only if $0 < p < 1$  
A1: Var = 9.9 or ft their $p$ but only if $0 < p < 1$

(b)  
M1: Using the model $Po(“their 10”)$ (this may be written or used) and $1 - P(X \leq 6)$  
A1: awrt 0.870 Award M1 A1 for awrt 0.870 with no incorrect working

(c)  
B1: Explaining why approximation is valid - need the context of number and calls  
B1: Need the context connecting, wrong agent

(d)  
B1: Evaluating the accuracy of their answer in (b). Allow 2 significant figures
<table>
<thead>
<tr>
<th>Question</th>
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<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>3(a)</strong></td>
<td>Expected value for (2 = 150 \times P(X = 2)) &amp; M1 &amp; 3.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>&amp; (= 28.3015\ldots) &amp; A1 &amp; 1.1b</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Expected value for 4 or more (= 150 - (53.8 + 56.6 + 28.3 + 8.9)) &amp; A1ft &amp; 1.1b</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>&amp; (= 2.4) &amp; &amp;</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(H_0: \text{Bin}(20, 0.05)) is a suitable model &amp; B1 &amp; 2.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(H_1: \text{Bin}(20, 0.05)) is not a suitable model &amp; &amp; &amp;</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Combining last two groups (\geq 3) &amp; M1 &amp; 2.1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Observed frequency</strong> &amp; 19 &amp; &amp;</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Expected frequency</strong> &amp; 11.3 &amp; &amp;</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\nu = 4 - 1 = 3) &amp; B1 &amp; 1.1b</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical value, (\chi^2 (0.05) = 7.815) &amp; B1 &amp; 1.1a</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Test statistic (= \frac{(43 - 53.8)^2}{53.8} + \frac{(62 - 56.6)^2}{56.6} + \ldots) &amp; M1 &amp; 1.1b</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>&amp; (= 8.117) &amp; A1 &amp; 1.1b</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>In critical region, sufficient evidence to reject (H_0), accept (H_1) &amp; A1 &amp; 3.5a</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Significant evidence at 5% level to reject the manager’s model &amp; &amp; &amp;</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(10)</strong> &amp; &amp; &amp;</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(b)</strong></td>
<td>(\nu = 4 - 2 = 2) &amp; &amp;</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 classes due to pooling &amp; B1 &amp; 2.4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 restrictions (equal total and mean/proportion) &amp; B1 &amp; 2.4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(2)</strong> &amp; &amp; &amp;</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(c)</strong></td>
<td>(H_0: \text{Binomial distribution is a good model}) &amp; B1 &amp; 3.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(H_1: \text{Binomial distribution is not a good model}) &amp; &amp; &amp;</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical value, (\chi^2 (0.05) = 5.991) &amp; B1 &amp; 3.5a</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Test statistic is not in critical region, insufficient evidence to reject (H_0) &amp; &amp; &amp;</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>There is evidence that the Binomial distribution is a good model &amp; &amp; &amp;</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(2)</strong> &amp; &amp; &amp;</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(14 marks)</strong> &amp; &amp; &amp;</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Question 3 notes:

(a)  
M1: Using the binomial model $150 \times p^2 \times (1-p)^{18}$ may be implied by 28.3  
A1: awrt 28.3  
A1: awrt 2.4 or ft their “28.3”  
B1: Both hypotheses correct using the correct notation or written out in full  
M1: For recognising the need to combine groups  
B1: Number of degrees of freedom = 3 may be implied by a correct CV  
B1: awrt 7.82  
M1: Attempting to find $\sum \frac{(O_i - E_i)^2}{E_i}$ or $\sum \frac{O_i^2}{E_i} - N$ may be implied by awrt 8.12  
A1: awrt 8.12  
A1: Evaluating the outcome of a model by drawing a correct inference in context

(b)  
B1: Explaining why there are 4 classes  
B1: Explanation of why 2 is subtracted

(c)  
B1: Correct hypotheses for the refined model  
B1: The CV awrt 5.99 and drawing the correct inference for the refined model
<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>Po(2.3) $n = 100 \mu = 2.3 \sigma^2 = 2.3$</td>
<td>M1</td>
<td>3.1a</td>
</tr>
<tr>
<td></td>
<td>CLT $\Rightarrow \bar{X} \approx N\left(2.3, \frac{2.3}{100}\right)$</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>$P(\bar{X} &gt; 2.5) = P\left(Z &gt; \frac{2.5-2.3}{\sqrt{0.023}}\right)$</td>
<td>M1</td>
<td>3.4</td>
</tr>
<tr>
<td></td>
<td>$= P(Z &gt; 1.318..)$</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>$= 0.09632…$</td>
<td>(4)</td>
<td></td>
</tr>
</tbody>
</table>

### Notes:

**M1:** For realising the need to use the CLT to set $\bar{X} \approx$ normal with correct mean
May be implied by using the correct normal distribution

**A1:** For fully correct normal stated or used

**M1:** Use of the normal model to find $P(\bar{X} > 2.5)$. Can be awarded for
or awrt 1.32

**A1:** awrt 0.0963
<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>5(a)</td>
<td>( \binom{7}{1} \times 0.15^2 \times (0.85)^6 )</td>
<td>M1</td>
<td>3.3</td>
</tr>
<tr>
<td></td>
<td>= 0.05940... = awrt 0.0594</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>(2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(b)</td>
<td>The model is only valid if:</td>
<td>B1</td>
<td>3.5b</td>
</tr>
<tr>
<td></td>
<td>the games (trials) are independent</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>the probability of winning a prize, 0.15, is constant for each game</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(c)</td>
<td>18 = ( \frac{r}{p} ) and 6² = ( \frac{r(1-p)}{p^2} )</td>
<td>M1</td>
<td>3.1b</td>
</tr>
<tr>
<td></td>
<td>Solving: 2p = 1 - p</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>( p = \frac{1}{3} ) (&gt; 0.15) so Mary has the greater chance of winning a prize</td>
<td>A1</td>
<td>3.2a</td>
</tr>
<tr>
<td></td>
<td>(4)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(8 marks)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Notes:**

5(a)
- **M1:** For selecting an appropriate model negative binomial or B(7, 0.15) with an extra success in 8th trial e.g.
  \[ \binom{7}{1} \times 0.15 \times (0.85)^6 \times 0.15 \]
  Allow \[ \binom{7}{1} \times 0.85 \times (0.15)^6 \times 0.85 \] may be implied by awrt 0.0594
- **A1:** awrt 0.0594

(b)
- **B1:** Stating the first assumption that games are independent
- **B1:** Stating the second assumption that the probability remains constant

(c)
- **M1:** Forming an equation for the mean or for the standard deviation
- **A1:** Both equations correct
- **M1:** Solving the 2 equations leading to 2p = 1 - p
- **A1:** For \( p = \frac{1}{3} \) followed by a correct deduction
<table>
<thead>
<tr>
<th>Question</th>
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<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
</table>
| **6(a)** | $G_X(I) = 1$ gives  
  $k \times 6^2 = 1$ so $k = \frac{1}{36} \,*$ | M1 2.1 | A1*cs0 1.1b |
| **(b)** | $P(X = 3) = \text{coefficient of } t^3$ so $G_X(t) = k \left(\ldots + 4t^3 \ldots \right)$ | M1 1.1b |  |
| | $\left[ P(X = 3) = \frac{1}{9} \right] \,$ | A1 1.1b | |
| **(c)** | $G'_X(t) = 2k \left(3 + t + 2t^2\right) \times (1 + 4t)$ | M1 2.1 |  |
| | $E(X) = G'_X(I) = 2k \left(3 + 1 + 2\right) \times (1 + 4)$ | M1 1.1b |  |
| | $= \frac{5}{3}$ | A1 1.1b |  |
| | $G'_X(t) = 2k \left[\left(3 + t + 2t^2\right) \times 4 + \left(1 + 4t\right)^2\right]$ | M1 1.1b |  |
| | $G''_X(I) = 2k \left[6 \times 4 + 5^2\right]$ | M1 1.1b |  |
| | $\{= \frac{49}{18}\}$ | A1 1.1b |  |
| | $\text{Var}(X) = G''_X(I) + G'_X(I) - \left[ G'_X(I) \right]^2 = \frac{49}{18} + \frac{5}{3} - \frac{25}{9}$ | M1 2.1 | |
| | $= \frac{29}{18} \,*$ | A1*cs0 1.1b | |
| **(d)** | $G_{2X+1}(t) = \frac{t}{36} \left(3 + t^2 + 2 \left(t^3\right)^2\right)^2$  
  $[\times t \text{ or sub } t^2 \text{ for } t]$ | M1 3.1a | |
| | $= G_{2X+1}(t) = \frac{t}{36} \left(3 + t^2 + 2t^4\right)^2$ | A1 1.1b |  |
| |  | (2) | (14 marks) |

**Notes:**

(a)  
**M1:** Stating $G_X(I) = 1$  
**A1*:** Fully correct proof with no errors cs0

(b)  
**M1:** Attempting to find the coefficient of $t^3$. May be implied by obtaining $\frac{1}{9}$ or awrt 0.11  
**A1:** $\frac{1}{9}$, allow awrt 0.111
Question 6 notes continued:

(c) 
M1: Attempting to find $G_X(t)$. Allow Chain rule or multiplying out the brackets and differentiating
M1: Substituting $t = 1$ into $G_X(t)$
A1: $\frac{5}{3}$, allow awrt 1.67
M1: Attempting to find $G'_X(t)$
A1: $2k\left[(3+t+2t^2)\times 4+(1+4t)^2\right]$ or $k(48t^2+24t+26)$ o.e.
A1: $2k\{6 \times 4 + 5^2\}$ o.e.
M1: Using $G'^*_X(1) + G'^*_X(1) - \left[G'^*_X(1)\right]^2$ to find the Variance
A1*: $\frac{29}{18}$ cso

(d) 
M1: Realising the need to $\times t$ or sub $t^2$ for $t$
A1: $\frac{t}{36}(3 + t^2 + 2t^4)$, or $\frac{t}{36}(9 + 6t^2 + 13t^4 + 4t^6 + 4t^3)$ o.e.
### Question 7(a)

**Scheme**

\[ X \sim B(20, 0.2) \text{ and seek } c \text{ such that } P(X \leq c) < 0.10 \]

\[ [P(X \leq 1) = 0.0692] \quad \text{CR is } X \leq 1 \]

**Marks**

M1 3.3

**AOs**

A1 1.1b

(2)

### (b)

**Size = 0.0692**

B1ft 1.2

(1)

### (c)

\[ Y = \text{no. of spins until red obtained} \text{ so } Y \sim \text{Geo}(0.2) \]

\[ \mu = \frac{1}{p} \text{ so if } p < 0.2 \text{ then mean is larger} \text{ so seek } d \text{ so that } P(Y \geq d) < 0.10 \]

\[ P(Y \geq d) = (0.8)^{d-1} \]

\[ (0.8)^{d-1} < 0.10 \quad \Rightarrow \quad d - 1 > \frac{\log(0.1)}{\log(0.8)} \]

\[ d > 11.3.. \]

**CR is** \( Y \geq 12 \)

A1 1.1b

(6)

### (d)

**Size = \[ 0.8^{11} = 0.085899\ldots \] = 0.0859**

B1 1.1b

(1)

### (e)(i)

**Power = P(reject Ho when it is false) = P(X \leq 1 \mid X \sim B(20, p))**

\[ = (1-p)^{20} + 20(1-p)^{19} p \]

\[ = (1-p)^{19}(1+19p) \]

A1* cso 1.1b

(4)

### (ii)

**Power = (1-p)\^{11}**

B1 1.1b

(4)

### (f)

**Sam’s test has smaller P(Type I error) (or size) so is better**

B1 2.2a

**Power of Sam’s test = 0.1755…**

B1 1.1b

**Power of Tessa’s test = 0.85^{11} = 0.1673…**

B1 1.1b

**So for \( p = 0.15 \) Sam’s test is recommended**

B1 2.2b

(4)
**Question 7 notes:**

(a)  
M1: Realising the need to use the model $B(20,0.2)$ with method for finding the CR or implied by a correct CR  
A1: $X \leq 1$ or $X < 2$

(b)  
B1: awrt 0.0692

(c)  
M1: Realising that the model $Geo(0.2)$ is needed. This may be written or used  
M1: Realising the key step that they need to find $P(Y \geq d) < 0.10$  
M1: Using the model $(0.8)^{d-1}$  
M1: Using the model $(0.8)^{d-1} < 0.10$ and finding a method to solve leading to a value/range of values for $d$  
A1: For $d > 11.3..$  
A1: For $Y \geq 12$ or $Y > 11$ (a correct inference)

(d)  
B1ft: awrt 0.0692. ft their answer to part (c)

(e)(i)  
M1: Using $B(20, p)$ and realizing they need to find $P(X \leq 1)$ o.e. This may be used or written  
M1: Using $P(X = 0) + P(X = 1)$  
A1*: Fully correct proof (no errors) e so

(ii)  
B1: For $(1 - p)^{11}$

(f)  
B1: Making a deduction about the tests using the answers to part(b) and (d)  
B1: awrt 0.0176  
B1: awrt 0.167  
B1: A correct inference about which test is recommended
Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions
- Use black ink or ball-point pen.
- If pencil is used for diagrams.sketches.graphs it must be dark (HB or B).
- Fill in the boxes at the top of this page with your name, centre number and candidate number.
- Answer all questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – there may be more space than you need.
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information
- A booklet ‘Mathematical Formulae and Statistical Tables’ is provided.
- There are 7 questions in this question paper. The total mark for this paper is 75.
- The marks for each question are shown in brackets – use this as a guide as to how much time to spend on each question.

Advice
- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
Answer ALL questions. Write your answers in the spaces provided.

1. The three independent random variables $A$, $B$ and $C$ each have a continuous uniform distribution over the interval $[0, 5]$.

(a) Find the probability that $A$, $B$ and $C$ are all greater than 3

The random variable $Y$ represents the maximum value of $A$, $B$ and $C$. The cumulative distribution function of $Y$ is

$$F(y) = \begin{cases} 0 & y < 0 \\ \frac{y^3}{125} & 0 \leq y \leq 5 \\ 1 & y > 5 \end{cases}$$

(b) Using algebraic integration, show that $\text{Var} (Y) = 0.9375$

(c) Find the mode of $Y$, giving a reason for your answer.

(d) Describe the skewness of the distribution of $Y$. Give a reason for your answer.

(e) Find the value of $k$ such that $P(k < Y < 2k) = 0.189$
Question 1 continued
Question 1 continued
2. A researcher claims that, at a river bend, the water gradually gets deeper as the distance from the inner bank increases. He measures the distance from the inner bank, \( b \) cm, and the depth of a river, \( s \) cm, at 7 positions. The results are shown in the table below.

<table>
<thead>
<tr>
<th>Position</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance from inner bank ( b ) cm</td>
<td>100</td>
<td>200</td>
<td>300</td>
<td>400</td>
<td>500</td>
<td>600</td>
<td>700</td>
</tr>
<tr>
<td>Depth ( s ) cm</td>
<td>60</td>
<td>75</td>
<td>85</td>
<td>76</td>
<td>110</td>
<td>120</td>
<td>104</td>
</tr>
</tbody>
</table>

The Spearman’s rank correlation coefficient between \( b \) and \( s \) is \( \frac{6}{7} \).

(a) Stating your hypotheses clearly, test whether or not the data provides support for the researcher’s claim. Use a 1% level of significance.

(b) Without re-calculating the correlation coefficient, explain how the Spearman’s rank correlation coefficient would change if

(i) the depth for G is 109 instead of 104

(ii) an extra value H with distance from the inner bank of 800 cm and depth 130 cm is included.

(c) Describe how you would find the correlation with many tied ranks.
(c) Describe how you would find the correlation with many tied ranks.

The researcher decided to collect extra data and found that there were now many tied ranks.

(ii) an extra value $H$ with distance from the inner bank of 800 cm and depth 130 cm

(i) the depth for $G$ is 109 instead of 104

(b) Without re-calculating the correlation coefficient, explain how the Spearman’s rank

Stating your hypotheses clearly, test whether or not the data provides support for the

A researcher claims that, at a river bend, the water gradually gets deeper as the distance

the depth of a river, $s$ cm, at 7 positions. The results are shown in the table below.

<table>
<thead>
<tr>
<th>Position</th>
<th>Distance from inner bank (cm)</th>
<th>Depth (s) cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100</td>
<td>200</td>
</tr>
<tr>
<td>2</td>
<td>200</td>
<td>300</td>
</tr>
<tr>
<td>3</td>
<td>300</td>
<td>400</td>
</tr>
<tr>
<td>4</td>
<td>400</td>
<td>500</td>
</tr>
<tr>
<td>5</td>
<td>500</td>
<td>600</td>
</tr>
<tr>
<td>6</td>
<td>600</td>
<td>700</td>
</tr>
<tr>
<td>7</td>
<td>700</td>
<td>800</td>
</tr>
</tbody>
</table>

$\rho$ is included.

(3)

(4)

DO NOT WRITE IN THIS AREA

(Total for Question 2 is 9 marks)
3. A nutritionist studied the levels of cholesterol, $X \text{ mg/cm}^3$, of male students at a large college. She assumed that $X$ was distributed $N(\mu, \sigma^2)$ and examined a random sample of 25 male students. Using this sample she obtained unbiased estimates of $\mu$ and $\sigma^2$ as $\hat{\mu}$ and $\hat{\sigma}^2$.

A 95% confidence interval for $\mu$ was found to be (1.128, 2.232)

(a) Show that $\hat{\sigma}^2 = 1.79$ (correct to 3 significant figures) \hspace{1cm} (4)

(b) Obtain a 95% confidence interval for $\sigma^2$ \hspace{1cm} (3)
Question 3 continued

(Total for Question 3 is 7 marks)
4. The times, \( x \) seconds, taken by the competitors in the 100m freestyle events at a school swimming gala are recorded. The following statistics are obtained from the data.

<table>
<thead>
<tr>
<th>No. of competitors</th>
<th>Sample mean ( \bar{x} )</th>
<th>( \sum x^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Girls</td>
<td>8</td>
<td>83.1</td>
</tr>
<tr>
<td>Boys</td>
<td>7</td>
<td>88.9</td>
</tr>
</tbody>
</table>

Following the gala, a mother claims that girls are faster swimmers than boys. Assuming that the times taken by the competitors are two independent random samples from normal distributions,

(a) test, at the 10% level of significance, whether or not the variances of the two distributions are the same. State your hypotheses clearly.

(b) Stating your hypotheses clearly, test the mother’s claim. Use a 5% level of significance.
Question 4 continued
Question 4 continued
Question 4 continued

(Total for Question 4 is 13 marks)
5. Scaffolding poles come in two sizes, long and short. The length \( L \) of a long pole has the normal distribution \( N(19.6, 0.6^2) \). The length \( S \) of a short pole has the normal distribution \( N(4.8, 0.3^2) \). The random variables \( L \) and \( S \) are independent.

A long pole and a short pole are selected at random.

(a) Find the probability that the length of the long pole is more than 4 times the length of the short pole. Show your working clearly. \( \text{(6)} \)

Four short poles are selected at random and placed end to end in a row. The random variable \( T \) represents the length of the row.

(b) Find the distribution of \( T \). \( \text{(3)} \)

(c) Find \( P(|L - T| < 0.2) \) \( \text{(4)} \)
Question 5 continued
Question 5 continued
Question 5 continued

(Total for Question 5 is 13 marks)
6. A random sample of 10 female pigs was taken. The number of piglets, \( x \), born to each female pig and their average weight at birth, \( m \text{ kg} \), was recorded. The results were as follows:

<table>
<thead>
<tr>
<th>Number of piglets, ( x )</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average weight at birth, ( m \text{ kg} )</td>
<td>1.50</td>
<td>1.20</td>
<td>1.40</td>
<td>1.40</td>
<td>1.23</td>
<td>1.30</td>
<td>1.20</td>
<td>1.15</td>
<td>1.25</td>
<td>1.15</td>
</tr>
</tbody>
</table>

(You may use \( S_{xx} = 82.5 \) and \( S_{xm} = 0.12756 \) and \( S_{mm} = -2.29 \))

(a) Find the equation of the regression line of \( m \) on \( x \) in the form \( m = a + bx \) as a model for these results.  

(b) Show that the residual sum of squares (RSS) is 0.064 to 3 decimal places. 

(c) Calculate the residual values. 

(d) Write down the outlier. 

(e) (i) Comment on the validity of ignoring this outlier. 

(ii) Ignoring the outlier, produce another model. 

(iii) Use this model to estimate the average weight at birth if \( x = 15 \) 

(iv) Comment, giving a reason, on the reliability of your estimate.
Question 6 continued
Question 6 continued
7. Over a period of time, researchers took 10 blood samples from one patient with a blood
disease. For each sample, they measured the levels of serum magnesium, $s$ mg/dl, in the
blood and the corresponding level of the disease protein, $d$ mg/dl. One of the researchers
coded the data for each sample using $x = 10s$ and $y = 10(d - 9)$ but spilt ink over his work.

The following summary statistics and unfinished scatter diagram are the only remaining
information.

\[
\sum d^2 = 1081.74 \quad S_{ds} = 59.524
\]

and

\[
\sum y = 64 \quad S_{ss} = 2658.9
\]

(a) Use the formula for $S_{ss}$ to show that $S_{ss} = 26.589$ \hspace{1cm} (3)

(b) Find the value of the product moment correlation coefficient between $s$ and $d$. \hspace{1cm} (4)

(c) With reference to the unfinished scatter diagram, comment on your result in part (b). \hspace{1cm} (1)
Question 7 continued
Question 7 continued
1(a) \[ P(A > 3) = \frac{2}{5} \] B1 1.1b

\[ \frac{3}{28} \begin{bmatrix} 5 & 125 \end{bmatrix} \] M1 A1 1.1a 1.1b

(b) \[ f(y) = \frac{125}{35} \] M1 2.1

\[ E(y) = \int_{0}^{5} \frac{35}{3} y^{2} dy = \frac{500}{3} \] 15= \begin{bmatrix} 0 \end{bmatrix} M1 1.1b

\[ \text{Var}(y) = \int_{0}^{3} \left( y^{2} - \frac{500}{3} \right) dy = 0.9375 \] A1 cso 1.1b

(c) Mode = 5 B1 1.2

Or reason based on \( df \) \( y > 0 \)

(d) From a sketch or mode > mean therefore it has negative skew B1ft 2.4

(e) \[ k = 0.189125 \] M1 3.1a

\[ k = 0.189125 \] A1 1.1b

\[ k = 1.5 \] A1 1.1b

(13 marks)
<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1(a)</strong></td>
<td>[ P(A &gt; 3) = \frac{2}{5} ]</td>
<td>B1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>( \left( \frac{2}{5} \right)^3 = \frac{8}{125} )</td>
<td>M1</td>
<td>1.1a</td>
</tr>
<tr>
<td></td>
<td></td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td><strong>(b)</strong></td>
<td>[ f(y) = \frac{3y^2}{125} ]</td>
<td>M1</td>
<td>2.1</td>
</tr>
<tr>
<td></td>
<td>[ E(Y) = \int_0^5 \frac{3y^3}{125} , dy = \left[ \frac{3y^4}{500} \right]_0^5 = \left[ \frac{15}{4} \right] ]</td>
<td>M1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>[ \text{Var}(Y) = \int_0^5 \left( \frac{3y^4}{125} \right) , dy - \left( \frac{15}{4} \right)^2 ]</td>
<td>M1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>= 0.9375*</td>
<td>A1*</td>
<td>1.1b</td>
</tr>
<tr>
<td><strong>(c)</strong></td>
<td>Mode = 5</td>
<td>B1</td>
<td>1.2</td>
</tr>
<tr>
<td></td>
<td>Or reason based on ( \frac{df(y)}{dy} &gt; 0 )</td>
<td>B1</td>
<td>2.4</td>
</tr>
<tr>
<td><strong>(d)</strong></td>
<td>From a sketch or mode &gt; mean therefore it has negative skew</td>
<td>B1ft</td>
<td>2.4</td>
</tr>
<tr>
<td><strong>(e)</strong></td>
<td>( \left( \frac{2k}{125} \right)^3 - \frac{k^3}{125} = 0.189 )</td>
<td>M1</td>
<td>3.1a</td>
</tr>
<tr>
<td></td>
<td>[ \frac{7k^3}{125} = 0.189 ]</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>( k = 1.5 )</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>(13 marks)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Question 1 notes:

#### (a)
- **B1:** \( \frac{2}{5} \) o.e. may be implied by a correct answer
- **M1:** \( \left( \frac{2}{5} \right)^n \) may be implied by a correct answer
- **A1:** \( \frac{8}{125} \) o.e.

#### (b)
- **M1:** Realising that firstly need to find pdf \( f(y) \) and attempt to differentiate \( F(y) \)
- **M1:** Continuing the argument with an attempt to integrate \( y \times \text{"their } f(y)\text{"} \)
  \( y^n \to y^{n+1} \)
- **M1:** Integrating \( y^2 \times \text{"their } f(y)\text{"} \) - \( \text{["their } E(Y)^2\text{"]} y^n \to y^{n+1} \)
- **A1:** Complete correct solution no errors

#### (c)
- **B1:** 5 only
- **B1:** Explain their reason by either an accurate sketch or \( \frac{df(y)}{dy} > 0 \) therefore an increasing function o.e.

#### (d)
- **B1ft:** Explaining the reason for their answer. Follow through their part(b) or mean from(d) and mode from(c). A correct sketch of “their \( f(y)\)” – may be seen anywhere in question or ft their mean and mode plus a correct conclusion
- **NB:** Watch for gaming. A student who writes both negative skew with a reason and positive skew with a reason. Please send these to your Team Leader

#### (e)
- **M1:** Attempting to translate the problem into an equation using \( 2k \) and \( k \). Allow if the brackets are missing e.g. \( \frac{2k^3}{125} - \frac{k^3}{125} \). No need for the 0.189
- **A1:** A correct equation in any form
- **A1:** A correct answer only
<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>2(a)</td>
<td>$H_0 : \rho = 0, H_1 : \rho &gt; 0$</td>
<td>B1 2.5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Critical value at 1% level is 0.8929</td>
<td>B1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$r_s &lt; 0.8929$ so not significant evidence to reject $H_0$.</td>
<td>M1 2.1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>The researcher’s claim is not correct (at 1% level) <strong>or</strong> insufficient evidence for researcher’s claim <strong>or</strong> there is insufficient evidence that water gets deeper further from inner bank <strong>or</strong> no (positive) correlation between depth of water and distance from inner bank</td>
<td>A1ft 2.2b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(4)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(b)(i)</td>
<td>The <strong>ranks will remain the same</strong> therefore there will be <strong>no change</strong> to the spearman’s rank correlation coefficient</td>
<td>B1 2.4</td>
<td></td>
</tr>
<tr>
<td>(ii)</td>
<td>Spearman’s rank correlation coefficient will <strong>increase</strong> since</td>
<td>B1 2.2a</td>
<td></td>
</tr>
<tr>
<td></td>
<td>The <strong>ranks are the same</strong> for both distance and depth therefore $d = 0$ however, $n$ has increased <strong>or</strong> the new position follows the pattern that large $b$ is associated with large $s$ and so $r_s$ will increase</td>
<td>B1 2.4</td>
<td></td>
</tr>
<tr>
<td>(c)</td>
<td>The mean of the tied ranks is given to each…</td>
<td>B1 2.4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>… then use PMCC</td>
<td>B1 2.4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(9 marks)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Notes:**

(a)  
B1: Both hypotheses correct written using the notation $\rho$  
B1: awrt 0.893  
M1: Drawing a correct inference using their answer to part(a) and their CV  
A1ft: Drawing a correct inference in context using their answer to part(a) and their CV

(b)(i)  
B1: Stating **no change** and an explanation including **ranks remain unchanged** o.e. and **no change** o.e.

(b)(ii)  
B1: Interpreted the outcome of adding a point as **increased** oe  
B1: Explaining why. Need to mention the **ranks are the same for both oe** and **$n$ has increased** oe

(c)  
B1: Explaining that the mean of the values for the tied ranks is given to both values  
B1: Explaining that the PMCC must be used
### Question Scheme Marks AOs

#### 3(a)
95% CI for \( \mu \) uses \( t \) value of \( 2.064 \)

\[
\frac{\hat{\sigma}}{\sqrt{25}} \times "t \ value" = \frac{1}{2} (2.232-1.128) \quad \text{or} \\
\frac{1}{2} (2.232+1.128) + "t \ value" \times \frac{\hat{\sigma}}{\sqrt{25}} = 2.232 \quad \text{(oe)}
\]

\[
\hat{\sigma} = \frac{2.76}{"t \ value"} \quad \text{or} \quad 1.3372...
\]

\[
\sigma^2 = 1.788...[=1.79 \ (3sf)] \ *
\]

M1 2.1

B1 3.3

A1 1.1b

#### (4)

#### (b)
12.401, \( \frac{24 \times 1.79}{\sigma^2} \) < 39.364

\[
1.09 < \sigma^2 < 3.46
\]

A1 1.1b

A1 1.1a

(3)

### Notes:

**Notes:**

(a)

**B1:** Realising that the \( t \)-distribution must be used as a model and finding the correct value awrt 2.06

**M1:** Using the correct formula with a \( t \)-value, 

\[
\frac{\hat{\sigma}}{\sqrt{25}} \times "t \ value" = \frac{1}{2} (2.232-1.128) \quad \text{or} \\
\frac{1}{2} (2.232+1.128) + "t \ value" \times \frac{\hat{\sigma}}{\sqrt{25}} = 2.232 \quad \text{or} \\
\frac{1}{2} (2.232+1.128) - "t \ value" \times \frac{\hat{\sigma}}{\sqrt{25}} = 1.128
\]

**M1:** Rearranging one of these formula accurately to find a value of \( \hat{\sigma} \)

**A1cso:** A correct solution only using awrt 1.79

(b)

**B1:** awrt 12.4 or 39.4 May be implied by a correct confidence interval

**M1:** \( \frac{24 \times 1.79}{\sigma^2} \) May be implied by a correct confidence interval

**A1:** awrt 1.09 and awrt 3.46
<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>4(a)</td>
<td>H₀: $\sigma_G^2 = \sigma_B^2$, H₁: $\sigma_G^2 \neq \sigma_B^2$,</td>
<td>B1 2.5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$s_B^2 = \frac{1}{4}(56130 - 7 \times 88.9^2) = \frac{807.53}{6} = 134.6$</td>
<td>M1 2.1</td>
<td>A1 1.1b</td>
</tr>
<tr>
<td></td>
<td>$s_G^2 = \frac{1}{4}(55746 - 8 \times 83.1^2) = \frac{501.12}{7} = 71.58$</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{s_B^2}{s_G^2} = 1.880...$</td>
<td>M1 3.4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Critical value $F_{6, 7} = 3.87$</td>
<td>B1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Not significant, variances can be treated as the same</td>
<td>A1 ft 2.2b</td>
<td>(7)</td>
</tr>
<tr>
<td>(b)</td>
<td>H₀: $\mu_B = \mu_G$, H₁: $\mu_B &gt; \mu_G$</td>
<td>B1 2.5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Pooled estimate of variance $s^2 = \frac{6 \times 1346 + 7 \times 71.58}{13} = 100.6653...$</td>
<td>M1 3.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Test statistic $t = \frac{88.9 - 83.1}{s \sqrt{\frac{1}{7} + \frac{1}{8}}}$ = awrt 1.12</td>
<td>M1 1.1b</td>
<td>A1 1.1b</td>
</tr>
<tr>
<td></td>
<td>Critical value $t_{13}(5%) = 1.771$</td>
<td>B1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Insufficient evidence to support mother’s claim</td>
<td>A1 ft 2.2b</td>
<td>(6)</td>
</tr>
</tbody>
</table>

**Notes:**

(a)
   - **B1:** Both hypotheses correct using the notation $\sigma^2$. Allow $\sigma$ rather than $\sigma^2$
   - **M1:** Using a correct Method for either $s_B^2$ or $s_G^2$; May be implied by a correct value
   - **A1:** awrt 135
   - **A1:** awrt 71.6

   - **M1:** Using the F-distribution as the model e.g. $\frac{s_B^2}{s_G^2}$
   - **B1:** awrt 3.87

   - **A1 ft:** Drawing a correct inference following through their CV and value for $\frac{s_B^2}{s_G^2}$

(b)
   - **B1:** Both hypotheses correct using the notation $\mu$
   - **M1:** For realising the need to find the pooled estimate for the test require from a correct interpretation of the question

   - **M1:** Correct method for test statistic $t = \frac{88.9 - 83.1}{s \sqrt{\frac{1}{7} + \frac{1}{8}}}$; May be implied by a correct

   - **A1:** awrt 1.12
   - **B1:** awrt 1.77

   - **A1 ft:** Drawing a correct inference following through their CV and test statistic
### Question Scheme Marks AOs

<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>5(a)</strong></td>
<td>Let ( X = L - 4S ) then ( E(X) = 19.6 - 4\times 4.8 )</td>
<td>M1 2.3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( = 0.4 )</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \text{Var}(X) = \text{Var}(L) + 4^2 \text{Var}(S) = 0.6^2 + 16 \times 0.3^2 )</td>
<td>M1 2.1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( = 1.8 )</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( P(X &gt; 0) = [P(Z &gt; \frac{0 - 0.4}{\sqrt{1.8}}) = -0.298\ldots\ldots)] )</td>
<td>M1 2.1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( = 0.617202\ldots ) awrt ( 0.617 )</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td><strong>(b)</strong></td>
<td>( T = S_1 + S_2 + S_3 + S_4 ) (May be implied by 0.36)</td>
<td>M1 3.3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( T \sim N(19.2, 0.36) ) ( E(T) = 19.2 )</td>
<td>B1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \text{Var}(T) = 0.36 ) or ( 0.6^2 )</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td><strong>(c)</strong></td>
<td>Let ( Y = L - T ) ( E(Y) = E(L) - E(T) = [ 0.4] )</td>
<td>M1 3.3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \text{Var}(Y) = \text{Var}(L) + \text{Var}(T) = [ 0.72] )</td>
<td>M1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Require ( P( -0.2 &lt; Y &lt; 0.2) )</td>
<td>M1 3.1a</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( = 0.16708\ldots ) awrt ( 0.167 )</td>
<td>A1 1.1b</td>
<td></td>
</tr>
</tbody>
</table>

### Notes:

(a)  
M1: Selecting and using an appropriate model i.e. \( \pm (L - 4S) \). May be implied by 0.4  
A1: 0.4 oe  
M1: For realising the need to use \( \text{Var}(L) + 4^2 \text{Var}(S) \). Allow use of 0.6 for \( \text{Var}(L) \) instead of \( 0.6^2 \) and/or 0.3 for \( \text{Var}(S) \) instead of \( 0.3^2 \) may be implied by 1.8  
A1: 1.8 only  
M1: For realising \( P(X > 0) \) is required and an attempt to find it e.g. \( \frac{0 - 0.4}{\sqrt{\text{Var}(X)}} \) but do not allow a negative \( \text{Var}(X) \)  
A1: awrt 0.617  

(b)  
M1: Selecting and using an appropriate model i.e \( \sum S_1 + S_2 + S_3 + S_4 \) : may be implied by 0.36  
B1: 19.2 only  
A1: 0.36  

(c)  
M1: Setting up and using the model \( Y = L - T \). May be implied by \( E(Y) = E(L) - E(T) \)  
M1: Using \( \text{Var}(Y) = \text{Var}(L) + \text{Var}(T) \)  
M1: Dealing with the modulus and realising they need to find \( P( -0.2 < Y < 0.2) \)  
A1: awrt 0.167
### Question 6(a)

Scheme:

\[ b = \frac{S_{ym}}{S_{xx}} = -0.0277576 \]

\[ a = \bar{m} - bx = 1.278 + 0.0277576 \times 8.5 = 1.5139 \]

\[ m = 1.5139 - 0.02775...x \]

(2)

Marks: M1 3.3

AOs: A1 1.1b

### Question 6(b)

\[ RSS = 0.12756 - \frac{(-2.29)^2}{82.5} \]

\[ = 0.06399* \]

Marks: M1 1.1b

AOs: A1* 1.1b

(2)

### Question 6(c)

<table>
<thead>
<tr>
<th>(x)</th>
<th>(m)</th>
<th>(m = a + bx)</th>
<th>(\varepsilon)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1.50</td>
<td>1.4029</td>
<td>+0.0971</td>
</tr>
<tr>
<td>5</td>
<td>1.20</td>
<td>1.3752</td>
<td>-0.1752</td>
</tr>
<tr>
<td>6</td>
<td>1.40</td>
<td>1.3474</td>
<td>+0.0526</td>
</tr>
<tr>
<td>7</td>
<td>1.40</td>
<td>1.3196</td>
<td>+0.0804</td>
</tr>
<tr>
<td>8</td>
<td>1.23</td>
<td>1.2919</td>
<td>-0.0619</td>
</tr>
<tr>
<td>9</td>
<td>1.30</td>
<td>1.2641</td>
<td>+0.0359</td>
</tr>
<tr>
<td>10</td>
<td>1.20</td>
<td>1.2364</td>
<td>-0.0364</td>
</tr>
<tr>
<td>11</td>
<td>1.15</td>
<td>1.2086</td>
<td>-0.0586</td>
</tr>
<tr>
<td>12</td>
<td>1.25</td>
<td>1.1808</td>
<td>+0.0692</td>
</tr>
<tr>
<td>13</td>
<td>1.15</td>
<td>1.1531</td>
<td>-0.0031</td>
</tr>
</tbody>
</table>

Marks: M1 3.4

AOs: A1 1.1b

(2)

### Question 6(d)

The point (5, 1.2) is an outlier

Marks: B1ft 2.2b

AOs: (1)

### Question 6(e)(i)

It is a valid piece of data so should be used

or

It does not follow the pattern according to the residuals so may contain an error making the result invalid so should be removed

(1)

### Question 6(e)(ii)

\[ a = m - bx = 1.28667 + 0.03765 \times 8.5 = 1.6213 \]

\[ m = 1.6213 - 0.03765x \]

(2)

Marks: M1 3.3

AOs: A1 1.1b

### Question 6(e)(iii)

\[ m = 1.6213 - 0.03765 \times 15 \]

\[ = 1.056 \text{ or awrt } 1.06 \]

Marks: B1ft 3.4

### Question 6(e)(iv)

The model is only reliable if the values are limited to those in the given range so probably not reliable

(5)

(12 marks)
**Question 6 notes:**

6(a)

<table>
<thead>
<tr>
<th>M1:</th>
<th>Realising the need to use $b = \frac{S_{sm}}{S_{xx}}$ and $a = \bar{m} - b\bar{x}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1:</td>
<td>$m = \text{awrt 1.51} - (\text{awrt 0.0278})x$. Award M1A1 for correct equation</td>
</tr>
</tbody>
</table>

(b)

<table>
<thead>
<tr>
<th>M1:</th>
<th>Using $\frac{S_{sm} - (S_{sm})^2}{S_{xx}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1*:</td>
<td>awrt 0.064</td>
</tr>
</tbody>
</table>

(c)

<table>
<thead>
<tr>
<th>M1:</th>
<th>Using the model in part (a) i.e. $m - (\text{awrt 1.5139} - \text{awrt 0.02775})x$ implied by a correct value</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1:</td>
<td>All correct.</td>
</tr>
<tr>
<td>A1:</td>
<td>Award M1A1 for a list of correct residuals</td>
</tr>
</tbody>
</table>

(d)

| B1: | Inferring from the residuals that the outlier is (5, 1.2) ft their residuals. |

(e)(i)

| B1: | Explaining why the outlier should be removed or not. |

(ii)

<table>
<thead>
<tr>
<th>M1:</th>
<th>Removing the outlier and refining the model by finding a new regression line.</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1:</td>
<td>$m = \text{awrt 1.62} - (\text{awrt 0.0377})x$</td>
</tr>
</tbody>
</table>

(iii)  

| B1ft: | using their model in e(i) with $x = 15$. awrt 1.06 or ft their e(ii) |

(iv)

| B1: | Realising the limitations of the model by stating it is **not reliable** and giving the reason why i.e. extrapolation/out of range o.e. |
### Question

<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>7(a)</td>
<td>[ S_{xx} = \sum (10s)^2 - \left( \frac{\sum 10s)^2}{10} \right) ]</td>
<td>M1</td>
<td>2.1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>M1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>2658.9 = 100 S_{xx}</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>[ S_{xx} = 26.589 ] *</td>
<td>A1* cso</td>
<td>1.1b</td>
</tr>
<tr>
<td>(b)</td>
<td>[ 64 = \sum 10(d_i - 9) ]</td>
<td>M1</td>
<td>3.1a</td>
</tr>
<tr>
<td></td>
<td>[ 64 = 10 \sum d_i - 900 ]</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>[ \sum d_i = 96.4 ]</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>[ S_{dd} = 1081.74 - \left( \frac{\text{&quot;96.4&quot;}^2}{10} \right) ]</td>
<td>M1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>[ = 152.444 ]</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>[ r = 0.935 ]</td>
<td>A1 ft</td>
<td>1.1b</td>
</tr>
<tr>
<td>(c)</td>
<td>Linear correlation is significant but scatter diagram suggests a non-linear relationship between the level of serum magnesium, and the level of the disease protein</td>
<td>B1</td>
<td>3.5a</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Notes:**

(a)

**M1:** Attempting to use \[ S_{xx} = \sum x^2 - \left( \frac{\sum x}{10} \right)^2 \] with \( x = 10s \)

**M1:** Substituting in 2658.9 and dealing with the 10 correctly

**A1*:** cso A complete solution with no errors leading to 26.589 only

(b)

**M1:** Realising that either \( 64 = 10 \sum (d_i - 9) \) or \( 64 = 10 \sum d_i - 900 \) o.e. must be used. May be implied by seeing 96.4

**A1:** 96.4 only

**M1:** Attempting to use \( S_{dd} = \sum d^2 - \left( \frac{\sum d}{10} \right)^2 \) may be implied by 0.935

**A1 ft:** awrt 0.935 ft “their 96.4”

(c)

**B1:** A correct comment comparing their value of \( r \) and the scatter diagram in context
Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions
- Use black ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- Fill in the boxes at the top of this page with your name, centre number and candidate number.
- Answer all questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided — there may be more space than you need.
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information
- A booklet ‘Mathematical Formulae and Statistical Tables’ is provided.
- There are 8 questions in this question paper. The total mark for this paper is 75.
- The marks for each question are shown in brackets — use this as a guide as to how much time to spend on each question.

Advice
- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
Answer ALL questions. Write your answers in the spaces provided.

Unless otherwise indicated, whenever a numerical value of $g$ is required, take $g = 9.8\text{m s}^{-2}$ and give your answer to either 2 significant figures or 3 significant figures.

1. A particle $P$ of mass 0.5 kg is moving with velocity $(4i + j)\text{m s}^{-1}$ when it receives an impulse $(2i - j)\text{N s}$.

Show that the kinetic energy gained by $P$ as a result of the impulse is 12 J.  

(6)
Question 1 continued

(Total for Question 1 is 6 marks)
2. A parcel of mass 5 kg is projected with speed 8 m s\(^{-1}\) up a line of greatest slope of a fixed rough inclined ramp.

The ramp is inclined at angle \(\alpha\) to the horizontal, where \(\sin \alpha = \frac{1}{7}\).

The parcel is projected from the point \(A\) on the ramp and comes to instantaneous rest at the point \(B\) on the ramp, where \(AB = 14\) m.

The coefficient of friction between the parcel and the ramp is \(\mu\).

In a model of the parcel’s motion, the parcel is treated as a particle.

(a) Use the work-energy principle to find the value of \(\mu\).  

\[ \text{(5 marks)} \]

(b) Suggest one way in which the model could be refined to make it more realistic.  

\[ \text{(1 mark)} \]
Question 2 continued

(Total for Question 2 is 6 marks)
3. A particle of mass \( m \) kg lies on a smooth horizontal surface.

Initially the particle is at rest at a point \( O \) between two fixed parallel vertical walls.

The point \( O \) is equidistant from the two walls and the walls are 4 m apart.

At time \( t = 0 \) the particle is projected from \( O \) with speed \( u \) m s\(^{-1}\) in a direction perpendicular to the walls.

The coefficient of restitution between the particle and each wall is \( \frac{3}{4} \).

The magnitude of the impulse on the particle due to the first impact with a wall is \( \lambda mu \) Ns.

(a) Find the value of \( \lambda \). \hfill (3)

The particle returns to \( O \), having bounced off each wall once, at time \( t = 7 \) seconds.

(b) Find the value of \( u \). \hfill (5)
At time $t = 7$ seconds.

DO NOT WRITE IN THIS AREA

(Total for Question 3 is 8 marks)
Figure 1 represents the plan view of part of a horizontal floor, where $AB$ and $BC$ are perpendicular vertical walls.

The floor and the walls are modelled as smooth.

A ball is projected along the floor towards $AB$ with speed $u \text{ m s}^{-1}$ on a path at an angle of $60^\circ$ to $AB$. The ball hits $AB$ and then hits $BC$.

The ball is modelled as a particle.

The coefficient of restitution between the ball and wall $AB$ is $\frac{1}{\sqrt{3}}$.

The coefficient of restitution between the ball and wall $BC$ is $\frac{2}{\sqrt{5}}$.

(a) Show that, using this model, the final kinetic energy of the ball is 35% of the initial kinetic energy of the ball. \hspace{1cm} (8)

(b) In reality the floor and the walls may not be smooth. What effect will the model have had on the calculation of the percentage of kinetic energy remaining? \hspace{1cm} (1)
Question 4 continued
Question 4 continued
5. A car of mass 600 kg is moving along a straight horizontal road.

At the instant when the speed of the car is \( v \) m\(s^{-1} \), the resistance to the motion of the car is modelled as a force of magnitude \((200 + 2v)\) N.

The engine of the car is working at a constant rate of 12 kW.

(a) Find the acceleration of the car at the instant when \( v = 20 \) \(m\text{s}^{-1}\) \(\text{.} \) (4)

Later on the car is moving up a straight road inclined at an angle \( \theta \) to the horizontal, where \( \sin \theta = \frac{1}{14} \).

At the instant when the speed of the car is \( v \) m\(s^{-1} \), the resistance to the motion of the car from non-gravitational forces is modelled as a force of magnitude \((200 + 2v)\) N.

The engine is again working at a constant rate of 12 kW.

At the instant when the car has speed \( w \) m\(s^{-1} \), the car is decelerating at 0.05 m\(s^{-2} \).

(b) Find the value of \( w \). (5)
Later on the car is moving up a straight road inclined at an angle $\theta$ to the horizontal.

The engine of the car is working at a constant rate of 12 kW.

At the instant when the speed of the car is $v$ m s$^{-1}$, the car is decelerating at 0.05 m s$^{-1}$, the resistance to the motion of the car is $F$ N.

The mass of the car is 600 kg. Model the resistance to the motion of the car as a force of magnitude $(200 + 2v)$ N.

At the instant when the speed of the car is $v$ m s$^{-1}$, the resistance to the motion of the car is $F$ N.

(b) Find the value of $v$ to the horizontal.

(Total for Question 5 is 9 marks)
6. [In this question \( \mathbf{i} \) and \( \mathbf{j} \) are perpendicular unit vectors in a horizontal plane.]

A smooth uniform sphere \( A \) has mass \( 2m \) kg and another smooth uniform sphere \( B \), with the same radius as \( A \), has mass \( 3m \) kg.

The spheres are moving on a smooth horizontal plane when they collide obliquely.

Immediately before the collision the velocity of \( A \) is \( (3\mathbf{i} + 3\mathbf{j}) \) m s\(^{-1} \) and the velocity of \( B \) is \( (-5\mathbf{i} + 2\mathbf{j}) \) m s\(^{-1} \).

At the instant of collision, the line joining the centres of the spheres is parallel to \( \mathbf{i} \).

The coefficient of restitution between the spheres is \( \frac{1}{4} \).

(a) Find the velocity of \( B \) immediately after the collision. \hspace{1cm} (7)

(b) Find, to the nearest degree, the size of the angle through which the direction of motion of \( B \) is deflected as a result of the collision. \hspace{1cm} (2)
Question 6 continued
Question 6 continued

(Total for Question 6 is 9 marks)
7. A particle $P$ of mass $m$ is attached to one end of a light elastic string of natural length $a$ and modulus of elasticity $3mg$.

The other end of the string is attached to a fixed point $O$ on a ceiling.

The particle hangs freely in equilibrium at a distance $d$ vertically below $O$.

(a) Show that $d = \frac{4}{3} a$.  \hspace{2cm} (3)

The point $A$ is vertically below $O$ such that $OA = 2a$.

The particle is held at rest at $A$, then released and first comes to instantaneous rest at the point $B$.

(b) Find, in terms of $g$, the acceleration of $P$ immediately after it is released from rest.  \hspace{2cm} (3)

(c) Find, in terms of $g$ and $a$, the maximum speed attained by $P$ as it moves from $A$ to $B$.  \hspace{2cm} (5)

(d) Find, in terms of $a$, the distance $OB$.  \hspace{2cm} (3)
Question 7 continued
8. A particle $P$ of mass $2m$ and a particle $Q$ of mass $5m$ are moving along the same straight line on a smooth horizontal plane.

They are moving in opposite directions towards each other and collide directly.

Immediately before the collision the speed of $P$ is $2u$ and the speed of $Q$ is $u$.

The direction of motion of $Q$ is reversed by the collision.

The coefficient of restitution between $P$ and $Q$ is $e$.

(a) Find the range of possible values of $e$.

Given that $e = \frac{1}{3}$

(b) show that the kinetic energy lost in the collision is $\frac{40mu^2}{7}$.

(c) Without doing any further calculation, state how the amount of kinetic energy lost in the collision would change if $e > \frac{1}{3}$.
Question 8 continued
Question 8 continued

(Total for Question 8 is 14 marks)

TOTAL FOR PAPER IS 75 MARKS
Use Impulse-momentum principle $M_1$ 2.1

\[ \vec{v}_i - \vec{v}_f = \vec{F} \Delta t \]

A1 1.1b

\[
\begin{align*}
\vec{v}_i &= (4, 0) \\
\vec{v}_f &= (-4, 0)
\end{align*}
\]

A1 1.1b

\[
\begin{align*}
\Delta \vec{v} &= (8, 0) \\
\Delta \vec{v} &= 8 \hat{i} \\
\Delta \vec{v} &= 8 \text{ (m/s)}
\end{align*}
\]

A1 1.1b

\[
\begin{align*}
\frac{1}{2} m \Delta v^2 &= \frac{1}{2} \times 2 \times 8^2 \\
\frac{1}{2} m \Delta v^2 &= 64
\end{align*}
\]

A1 1.1b

\[
64 \text{ J}
\]

A1* 1.1b

Notes:

- **M1**: Difference of terms & dimensionally correct
- **A1**: Correct unsimplified equation
- **A1**: Complete justification of given answer
### Paper 3C/4C: Further Mechanics 1 Mark Scheme

<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Use Impulse-momentum principle</td>
<td>M1 2.1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$2\mathbf{i} - \mathbf{j} = 0.5\mathbf{v} - 0.5(4\mathbf{i} + \mathbf{j})$</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{2}\mathbf{v} = 4\mathbf{i} - \frac{1}{2}\mathbf{j}$,  $\mathbf{v} = 8\mathbf{i} - \mathbf{j}$ (m s$^{-1}$)</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Use of $KE = \frac{1}{2}m</td>
<td>\mathbf{v}</td>
<td>^2 - \frac{1}{2}m</td>
</tr>
<tr>
<td></td>
<td>$= \frac{1}{2} \times 0.5 \times {(64+1) - (16+1)}$</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$= \frac{1}{4} \times 48 = 12$ (J)</td>
<td>A1* 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(6 marks)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Notes:**

- **M1:** Difference of terms & dimensionally correct
- **A1:** Correct unsimplified equation
- **A1:** cao
- **M1:** Must be a difference of two terms
- Must be dimensionally correct
- **A1:** Correct unsimplified equation
- **A1*:** Complete justification of given answer
<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>2(a)</td>
<td>[ R = 5g \cos \alpha \left( = 5g \times \frac{4\sqrt{3}}{7} = 48.497\ldots \right) ]</td>
<td>M1</td>
<td>3.4</td>
</tr>
<tr>
<td></td>
<td>Force due to friction = ( \mu \times 5g \cos \alpha )</td>
<td>M1</td>
<td>3.4</td>
</tr>
<tr>
<td></td>
<td>Work-Energy equation</td>
<td>M1</td>
<td>3.4</td>
</tr>
<tr>
<td></td>
<td>[ \frac{1}{2} 	imes 5 \times 64 = 5 \times 9.8 \times 14 \sin \alpha + 14 \mu R ]</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>( \mu = 0.0913 \text{ or } 0.091 )</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td>(b)</td>
<td>Appropriate refinement</td>
<td>B1</td>
<td>3.5c</td>
</tr>
</tbody>
</table>

**Notes:**

(a)

**M1:** Condone sin/cos confusion

**M1:** Use of \( \mu \times \) their \( R \)

**M1:** Must be using work-energy. Requires all terms

Condone sin/cos confusion, sign errors and their \( R \)

**A1:** Correct in \( \theta \) and \( \mu R \)

**A1:** Accept 0.0913 or 0.091

(b)

**B1:** e.g.

- do not model the parcel as a particle and therefore take air resistance into account
- take into account the dimensions/uniformity of the parcel
### Question Scheme Marks AOs

#### 3(a) Use NEL to find the speed of the particle after the first impact

\[ e\mu = \frac{3}{4}u \frac{\pi}{2} \]

**Impulse**

\[ \lambda \mu = mv - \mu = \pm \left[ \frac{3}{4}mu - (-\mu) \right] \]

\[ \lambda = \frac{7}{4} \]

- **M1** 3.1b
- **A1** 1.1b

(3)

#### (b) Use NEL to find the speed of the particle after the second impact

\[ \frac{3}{4} \times \frac{3}{4}u = \frac{9}{16}u \]

**Use of** \( s = vt \) **to find total time**

\[ 7 = \frac{2}{u} + \frac{4}{3} + \frac{2}{9u} \left( = \frac{2}{u} + \frac{16}{3u} + \frac{32}{9u} \right) \]

- **M1** 3.1b
- **A1** 1.1b

Solve for \( u \):

\[ 63u = 18 + 48 + 32 \]

\[ u = \frac{98}{63} = \frac{14}{9} (= 1.5) \]

- **M1** 1.1b
- **A1** 1.1b

(5)

### Notes:

(a) **B1:** Using Newton's experimental law as a model to find the speed after the first impact

**M1:** Must be a difference of two terms, taking account of the change in direction of motion

**A1:** cao

(b) **B1:** Using NEL as a model to find the speed after the second impact

**M1:** Needs to be used for at least one stage of the journey

**A1:** Ur equivalent

**M1:** Solve their linear equation for \( u \)

**A1:** Accept 1.56 or better
### Question 4(a)

<table>
<thead>
<tr>
<th>Scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td>Complete strategy to find the kinetic energy after the second impact</td>
</tr>
<tr>
<td>Parallel to $AB$ after collision: $u \cos 60^\circ$</td>
</tr>
<tr>
<td>Perpendicular to $AB$ after collision: $\frac{1}{\sqrt{3}}u \sin 60^\circ$</td>
</tr>
<tr>
<td>Components of velocity after first impact: $\frac{u}{2}, \frac{u}{2}$</td>
</tr>
<tr>
<td>Parallel to $BC$ after collision: $\frac{u}{2} \left( u \times \frac{1}{\sqrt{3}} \sin 60^\circ \right)$</td>
</tr>
<tr>
<td>Perpendicular to $BC$ after collision: $\frac{2}{\sqrt{5}} \times \frac{u}{2} \left( \frac{1}{\sqrt{10}}u \right)$</td>
</tr>
<tr>
<td>Components of velocity after second impact: $\frac{u}{2}, \frac{u}{\sqrt{10}}$</td>
</tr>
<tr>
<td>Final KE = $\frac{1}{2}m \left( \frac{u^2}{4} + \frac{u^2}{10} \right) = \frac{mu^2}{2} \times \frac{7}{20}$</td>
</tr>
<tr>
<td>Fraction of initial KE = $\frac{\frac{7}{20}}{\frac{7}{20}} = \frac{7}{20} = 35%$</td>
</tr>
<tr>
<td>(8)</td>
</tr>
</tbody>
</table>

**Notes:**

(a)
- **M1:** Use of CLM parallel to the wall. Condone sin/cos confusion
- **M1:** Use NEL as a model to find the speed perpendicular to the wall. Condone sin/cos confusion
- **A1:** Both components correct with trig substituted (seen or implied)
- **M1:** Use of CLM parallel to the wall. Condone sin/cos confusion
- **M1:** Use NEL as a model to find the speed perpendicular to the wall. Condone sin/cos confusion
- **A1:** Both components correct with trig substituted (seen or implied)
- **M1:** Correct expression for total KE using their components after 2nd collision
- **A1:** Obtain given answer with sufficient working to justify it

(b)
- **B1:** Clear explanation of how the modelling assumption has affected the outcome

---

**Notes:**

(a)
- **M1:** Use of CLM parallel to the wall. Condone sin/cos confusion
- **M1:** Use NEL as a model to find the speed perpendicular to the wall. Condone sin/cos confusion
- **A1:** Both components correct with trig substituted (seen or implied)
- **M1:** Use of CLM parallel to the wall. Condone sin/cos confusion
- **M1:** Use NEL as a model to find the speed perpendicular to the wall. Condone sin/cos confusion
- **A1:** Both components correct with trig substituted (seen or implied)
- **M1:** Correct expression for total KE using their components after 2nd collision
- **A1:** Obtain given answer with sufficient working to justify it

(b)
- **B1:** Clear explanation of how the modelling assumption has affected the outcome

---

**Notes:**

(a)
- **M1:** Use of CLM parallel to the wall. Condone sin/cos confusion
- **M1:** Use NEL as a model to find the speed perpendicular to the wall. Condone sin/cos confusion
- **A1:** Both components correct with trig substituted (seen or implied)
- **M1:** Use of CLM parallel to the wall. Condone sin/cos confusion
- **M1:** Use NEL as a model to find the speed perpendicular to the wall. Condone sin/cos confusion
- **A1:** Both components correct with trig substituted (seen or implied)
- **M1:** Correct expression for total KE using their components after 2nd collision
- **A1:** Obtain given answer with sufficient working to justify it

(b)
- **B1:** Clear explanation of how the modelling assumption has affected the outcome
<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>5(a)</td>
<td>Use of ( P = Fv ) : ( F = \frac{12000}{20} )</td>
<td>B1</td>
<td>3.3</td>
</tr>
<tr>
<td></td>
<td>Equation of motion: ( F - (200 + 2v) = 600a )</td>
<td>M1</td>
<td>3.4</td>
</tr>
<tr>
<td></td>
<td>( 600 - 240 = 600a )</td>
<td>A1ft</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>( 360 = 600a, a = 0.6 \text{ (m s}^{-2}\text{)} )</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>(4)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(b)</td>
<td>Equation of motion:</td>
<td>M1</td>
<td>3.3</td>
</tr>
<tr>
<td></td>
<td>( \frac{12000}{w} - (200 + 2w) - 600g \sin \theta = -600 \times 0.05 )</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>3 term quadratic and solve: ( 2w^2 + 590w - 12000 = 0 )</td>
<td>M1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>( w = \frac{-590 + \sqrt{590^2 + 96000}}{4} = 19.1 \text{ (m s}^{-1}\text{)} )</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>(5)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Notes:**

(a)  
**B1:** 600 or equivalent  
**M1:** Use the model to form the equation of motion  
Must include all terms. Condone sign errors  
**A1ft:** Correct for their \( F \)  
**A1:** cao

(b)  
**M1:** Use the model to form the equation of motion  
All terms needed. Condone sign errors and sin/cos confusion  
**A1:** All correct A1A1  
One error A1A0  
**M1:** Dependent on the preceding M1. Use the equation of motion to form a 3-term quadratic in \( w \) only  
**A1:** Accept 19. Do not accept more than 3 s.f.
<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>6(a)</td>
<td><img src="image" alt="Diagram" /></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Overall strategy to find $\mathbf{v}_A$</td>
<td>M1</td>
<td>3.1a</td>
</tr>
<tr>
<td></td>
<td>Velocity of $A$ perpendicular to loc after collision = $3 \mathbf{j}$ (m s$^{-1}$)</td>
<td>B1</td>
<td>3.4</td>
</tr>
<tr>
<td></td>
<td>CLM parallel to loc</td>
<td>M1</td>
<td>3.1a</td>
</tr>
<tr>
<td></td>
<td>$2m \times 3 - 3m \times 5 = 3mw - 2mv$ $(-9 = 3w - 2v)$</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>Correct use of impact law</td>
<td>M1</td>
<td>3.1a</td>
</tr>
<tr>
<td></td>
<td>$v + w = \frac{1}{4}(3 + 5)$ $(= 2)$</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>Solve for $w$</td>
<td>A1ft</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>$3w - 2v = -9$ [2v + 2w = 4]</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\mathbf{v}_B = -\mathbf{i} + 2\mathbf{j}$ (m s$^{-1}$),</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(7)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(b)</td>
<td>$\cos \theta = \frac{(-5\mathbf{i} + 2\mathbf{j})(-\mathbf{i} + 2\mathbf{j})}{\sqrt{29}\sqrt{5}}$</td>
<td>M1</td>
<td>3.1a</td>
</tr>
<tr>
<td></td>
<td>$\theta = 41.63...^\circ = 42^\circ$ (nearest degree)</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>Alternative method: $\tan^{-1} 2 - \tan^{-1} \frac{2}{5} = 41.63...^\circ = 42^\circ$ (nearest degree)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(2)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Notes:**

(a)  
**M1:** Correct overall strategy to form sufficient equations and solve for $\mathbf{v}_A$  
**B1:** Use the model to find the component of $\mathbf{v}_A$ perpendicular to the loc  
**M1:** Use CLM to form equation in $v$ and $w$. Need all 4 terms, dimensionally correct  
**A1:** Correct unsimplified  
**A1:** Must be used the right way round  
**A1:** Correct unsimplified  
**A1ft:** $\mathbf{v}_B$ correct. Follow their $2\mathbf{j}$

(b)  
**M1:** Complete method for finding the required angle. Follow their $\mathbf{v}_B$  
**A1:** cao
<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>7(a)</td>
<td>In equilibrium ⇒ no resultant vertical force</td>
<td>M1</td>
<td>2.1</td>
</tr>
<tr>
<td></td>
<td>$3mgx = mg$</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>$x = \frac{a}{3}$, $d = \frac{4}{3}a$</td>
<td>A1*</td>
<td>2.2a</td>
</tr>
<tr>
<td>(b)</td>
<td>Equation of motion:</td>
<td>M1</td>
<td>3.1a</td>
</tr>
<tr>
<td></td>
<td>$\frac{3mga}{a} - mg = m\ddot{x}$</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>$\ddot{x} = 2g$</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td>(c)</td>
<td>Max speed at equilibrium position</td>
<td>B1</td>
<td>3.1a</td>
</tr>
<tr>
<td></td>
<td>Work energy &amp; use of $E_{PE} = \frac{1}{2}x^2$</td>
<td>M1</td>
<td>3.1a</td>
</tr>
<tr>
<td></td>
<td>$\frac{3mga^2}{2a} = \frac{3mg}{2a} \left(\frac{a}{3}\right)^2 + \frac{1}{2}mv^2 + mg\frac{2a}{3}$</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{2}v^2 = ga\left(\frac{3}{2} - \frac{1}{6} - \frac{2}{3}\right) = \frac{2}{3}ga$, $v = \sqrt{\frac{4ga}{3}}$</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td>(d)</td>
<td>At max ht. KE = 0. EPE lost = GPE gained</td>
<td>M1</td>
<td>3.1a</td>
</tr>
<tr>
<td></td>
<td>$\frac{3mga^2}{2a} = mgh$</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>$OB = \frac{a}{2}$</td>
<td>A1</td>
<td>1.1b</td>
</tr>
</tbody>
</table>

(14 marks)
(a) Use \( T = \frac{\lambda x}{a} \) to form equation for equilibrium.

- M1: Correct unsimplified equation
- A1*: Requires sufficient working to justify given answer plus a 'statement' that the required result has been achieved

(b) Use \( T = \frac{\lambda x}{a} \) to form equation of motion.

- M1: Need all 3 terms. Condone sign errors
- A1: Correct unsimplified equation
- A1: cao

(c) Seen or implied

- M1: Form work-energy equation. All 4 terms needed
- Condone sign errors
- A1: Correct unsimplified equation
- A1: One error in the equation

(d) Form energy equation

- M1: Correct unsimplified equation
- A1: cao
Question 8(a)

Complete overall strategy to find $v$

Use of CLM

$2m \times 2u - 5m \times u = 5m \times v - 2m \times w$, \((-u = 5v - 2w)\)

Use of Impact law:

$v + w = e(2u + u)$

Solve for $v$:

$-u = 5v - 2w$

$6eu = 2v + 2w$

$7v = u(6e - 1)$ \(v = \frac{u}{7}(6e - 1)\)

Direction of $Q$ reversed: $v > 0$

$\Rightarrow 1 \geq e > \frac{1}{6}$

(b)

$e = \frac{1}{3}$ \(\Rightarrow v = \frac{u}{7}, \quad w = \frac{6u}{7}\)

Equation for KE lost

$\frac{1}{2} \times 2m \left(4u^2 - \frac{36u^2}{49}\right) + \frac{1}{2} \times 5m \left(u^2 - \frac{u^2}{49}\right)$

$\frac{1}{2} \left(\frac{40mu^2}{7}\right)$

Increase $e \Rightarrow$ more elastic $\Rightarrow$ less energy lost

(14 marks)
### Question 8 notes:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td></td>
</tr>
<tr>
<td><strong>M1:</strong></td>
<td>Complete strategy to form sufficient equations in ( v ) and ( w ) and solve for ( v )</td>
</tr>
</tbody>
</table>
| **M1:** | Use CLM to form equation in \( v \) and \( w \)  
|       | Needs all 4 terms & dimensionally correct |
| **A1:** | Correct unsimplified equation |
| **M1:** | Use NEL as a model to form a second equation in \( v \) and \( w \). Must be used the right way round |
| **A1:** | Correct unsimplified equation |
| **A1:** | for \( v \) or \( 7v \) correct |
| **M1:** | Use the model to form a correct inequality for their \( v \) |
| **A1:** | Both limits required |

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(b)</td>
<td></td>
</tr>
<tr>
<td><strong>B1:</strong></td>
<td>Or equivalent statements</td>
</tr>
<tr>
<td><strong>M1:</strong></td>
<td>Terms of correct structure combined correctly</td>
</tr>
</tbody>
</table>
| **A1:** | Fully correct unsimplified \( A1A1 \)  
|       | One error on unsimplified expression \( A1A0 \) |
| **A1:** | cso. plus a 'statement' that the required result has been achieved |

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(c)</td>
<td></td>
</tr>
<tr>
<td><strong>B1:</strong></td>
<td>&quot;less energy lost&quot; or equivalent</td>
</tr>
</tbody>
</table>
Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions
• Use **black** ink or ball-point pen.
• If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
• **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
• Answer all questions and ensure that your answers to parts of questions are clearly labelled.
• Answer the questions in the spaces provided – **there may be more space than you need**.
• You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
• Answers should be given to three significant figures unless otherwise stated.

Information
• A booklet ‘Mathematical Formulae and Statistical Tables’ is provided.
• There are 7 questions in this question paper. The total mark for this paper is 75.
• The marks for each question are shown in brackets – use this as a guide as to how much time to spend on each question.

Advice
• Read each question carefully before you start to answer it.
• Try to answer every question.
• Check your answers if you have time at the end.
1. A flag pole is 15 m long.

The flag pole is non-uniform so that, at a distance $x$ metres from its base, the mass per unit length of the flag pole, $m \text{ kg m}^{-1}$ is given by the formula $m = 10 \left(1 - \frac{x}{25}\right)$.

The flag pole is modelled as a rod.

(a) Show that the mass of the flag pole is 105 kg.

(b) Find the distance of the centre of mass of the flag pole from its base.
Question 1 continued

(Total for Question 1 is 7 marks)
A hollow right circular cone, of base diameter $4a$ and height $4a$ is fixed with its axis vertical and vertex $V$ downwards, as shown in Figure 1.

A particle of mass $m$ moves in a horizontal circle with centre $C$ on the rough inner surface of the cone with constant angular speed $\omega$.

The height of $C$ above $V$ is $3a$.

The coefficient of friction between the particle and the inner surface of the cone is $\frac{1}{4}$.

Find, in terms of $a$ and $g$, the greatest possible value of $\omega$. 

\[ \text{(8)} \]
A uniform solid cylinder has radius $2a$ and height $h$ ($h > a$).

A solid hemisphere of radius $a$ is removed from the cylinder to form the vessel $V$. The plane face of the hemisphere coincides with the upper plane face of the cylinder. The centre $O$ of the hemisphere is also the centre of the upper plane face of the cylinder, as shown in Figure 2.

(a) Show that the centre of mass of $V$ is $\frac{3(8h^2 - a^2)}{8(6h - a)}$ from $O$. 

The vessel $V$ is placed on a rough plane which is inclined at an angle $\phi$ to the horizontal. The lower plane circular face of $V$ is in contact with the inclined plane.

Given that $h = 5a$, the plane is sufficiently rough to prevent $V$ from slipping and $V$ is on the point of toppling.

(b) find, to three significant figures, the size of the angle $\phi$. 

---

**Figure 2**

A uniform solid cylinder has radius $2a$ and height $h$ ($h > a$). A solid hemisphere of radius $a$ is removed from the cylinder to form the vessel $V$. The plane face of the hemisphere coincides with the upper plane face of the cylinder. The centre $O$ of the hemisphere is also the centre of the upper plane face of the cylinder, as shown in Figure 2.

(a) Show that the centre of mass of $V$ is $\frac{3(8h^2 - a^2)}{8(6h - a)}$ from $O$. 

The vessel $V$ is placed on a rough plane which is inclined at an angle $\phi$ to the horizontal. The lower plane circular face of $V$ is in contact with the inclined plane.

Given that $h = 5a$, the plane is sufficiently rough to prevent $V$ from slipping and $V$ is on the point of toppling.

(b) find, to three significant figures, the size of the angle $\phi$. 

---
Question 3 continued
Question 3 continued
Question 3 continued

(Total for Question 3 is 9 marks)
4. A car of mass 500 kg moves along a straight horizontal road.

The engine of the car produces a constant driving force of 1800 N.

The car accelerates from rest from the fixed point $O$ at time $t = 0$ and at time $t$ seconds the car is $x$ metres from $O$, moving with speed $v$ m s$^{-1}$.

When the speed of the car is $v$ m s$^{-1}$, the resistance to the motion of the car has magnitude $2v^2$ N.

At time $T$ seconds, the car is at the point $A$, moving with speed $10$ m s$^{-1}$.

(a) Show that $T = \frac{25}{6} \ln 2$  

(b) Show that the distance from $O$ to $A$ is $125 \ln \frac{9}{8}$ m. 

(Total for Question 4 is 11 marks)
Question 4 continued

(Total for Question 4 is 11 marks)
A shop sign is modelled as a uniform rectangular lamina $ABCD$ with a semicircular lamina removed.

The semicircle has radius $a$, $BC = 4a$ and $CD = 2a$.

The centre of the semicircle is at the point $E$ on $AD$ such that $AE = d$, as shown in Figure 3.

(a) Show that the centre of mass of the sign is $\frac{44a}{3(16 - \pi)}$ from $AD$.

(b) Find, in terms of $W$ and $\pi$, the tension in the rope attached at $B$.

The rope attached at $B$ breaks and the sign hangs freely in equilibrium suspended from $A$, with $AD$ at an angle $\alpha$ to the downward vertical.

Given that $\tan \alpha = \frac{11}{18}$

(c) find $d$ in terms of $a$ and $\pi$. 
Question 5 continued

(Total for Question 5 is 12 marks)
6. A small bead $B$ of mass $m$ is threaded on a circular hoop.

The hoop has centre $O$ and radius $a$ and is fixed in a vertical plane.

The bead is projected with speed $\sqrt{\frac{7}{2} ga}$ from the lowest point of the hoop.

The hoop is modelled as being smooth.

When the angle between $OB$ and the downward vertical is $\theta$, the speed of $B$ is $v$.

(a) Show that $v^2 = ga \left( \frac{3}{2} + 2 \cos \theta \right)$

(b) Find the size of $\theta$ at the instant when the contact force between $B$ and the hoop is first zero.

(c) Give a reason why your answer to part (b) is not likely to be the actual value of $\theta$.

(d) Find the magnitude and direction of the acceleration of $B$ at the instant when $B$ is first at instantaneous rest.
Question 6 continued

(Total for Question 6 is 14 marks)
7. Two points $A$ and $B$ are $6$ m apart on a smooth horizontal surface.

A light elastic string of natural length $2$ m and modulus of elasticity $20$ N, has one end attached to the point $A$.

A second light elastic string of natural length $2$ m and modulus of elasticity $50$ N, has one end attached to the point $B$.

A particle $P$ of mass $3.5$ kg is attached to the free end of each string.

The particle $P$ is held at the point on $AB$ which is $2$ m from $B$ and then released from rest. In the subsequent motion both strings remain taut.

(a) Show that $P$ moves with simple harmonic motion about its equilibrium position. 

(b) Find the maximum speed of $P$. 

(c) Find the length of time within each oscillation for which $P$ is closer to $A$ than to $B$. 

(7) 

(2) 

(5)
Question 7 continued
Question 7 continued

(Total for Question 7 is 14 marks)

TOTAL FOR PAPER IS 75 MARKS
Total mass = 15 \int_{0}^{10} 25 \, dx = 105 \text{ kg}

Taking moments about the base:

\int_{0}^{10} 25 \, dx = 675 \text{ m}

\Rightarrow 105 \cdot d = 675

\Rightarrow d = 6.43 \text{ m}

Notes:

(a) M1: Use integration (usual rules)
A1: Correct integration
A1*: Use limits and show sufficient working to justify given answer

(b) M1: Use the model to find the moment about the base (usual rules for integration)
A1: Correct integration
M1: Use the model to complete the moments equation
Require 105 and their 675 used correctly
A1: 6.43 or better
### Paper 4F: Further Mechanics 2 Mark Scheme

<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1(a)</strong></td>
<td>Total mass = [ \int_{0}^{15} 10 \left( 1 - \frac{x}{25} \right) , dx ]</td>
<td>M1 2.1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[ = \left[ 10x - \frac{x^2}{5} \right]_{0}^{15} ]</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[ = 150 - \frac{225}{5} = 105 \text{ (kg)} ] *</td>
<td>A1* 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(3)</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(b)</strong></td>
<td>Taking moments about the base: [ \int_{0}^{15} 10x \left( 1 - \frac{x}{25} \right) , dx ]</td>
<td>M1 3.4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[ = \left[ 5x^2 - \frac{2}{15} x^3 \right]_{0}^{15} \left( = 675 \right) ]</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[ \Rightarrow 105d = 675 ]</td>
<td>M1 3.4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[ d = 6.43 \text{ (m)} ] 6 ( \frac{3}{7} ) (m)</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(4)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Notes:**

**(a)**
- **M1:** Use integration (usual rules)
- **A1:** Correct integration
- **A1:** Use limits and show sufficient working to justify given answer

**(b)**
- **M1:** Use the model to find the moment about the base (usual rules for integration)
- **A1:** Correct integration
- **M1:** Use the model to complete the moments equation
  - Require 105 and their 675 used correctly
- **A1:** 6.43 or better
<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td><img src="image" alt="Diagram" /></td>
<td></td>
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</tr>
</tbody>
</table>

1. **Complete overall strategy**
   - M1 3.1b

2. **Resolve vertically**
   - M1 3.3
   
   \[ mg + F \cos \theta = R \sin \theta \]
   - A1 1.1b

3. **Horizontal equation of motion**
   - M1 3.3
   
   \[ m r \omega^2 = R \cos \theta + F \sin \theta \]
   - A1 1.1b

4. **Use of limiting friction since maximum \( \omega \)**
   - M1 3.3

5. **Substitute for trig ratios:**
   - M1 1.1b

   \[
   \frac{3 a \omega^2}{2 g} = \frac{9}{2}
   \]

6. **Maximum \( \omega = \sqrt{\frac{3g}{a}} \)**
   - A1 1.1b

(8 marks)

**Notes:**

- **M1:** Overall strategy to form equation in \( \omega \) only e.g. consider vertical and horizontal motion and limiting friction
- **M1:** Needs all 3 terms. Condone sign errors and sin/cos confusion
- **A1:** Correct unsimplified equation
- **M1:** Needs all 3 terms. Condone sign errors and sin/cos confusion
- **A1:** Correct unsimplified equation
- **M1:** Seen or implied
- **M1:** Substitute to achieve equation in \( a, \omega \) and \( g \) only
- **A1:** Or equivalent exact form
### Question 3(a)

<table>
<thead>
<tr>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>mass</td>
<td>c of m from O</td>
<td></td>
</tr>
<tr>
<td>cylinder</td>
<td>(4\pi a^2 h)</td>
<td>(\frac{h}{2})</td>
</tr>
<tr>
<td>hemisphere</td>
<td>(\frac{2}{3}\pi a^3)</td>
<td>(\frac{3a}{8})</td>
</tr>
<tr>
<td>(V)</td>
<td>(4\pi a^2 h - \frac{2}{3} \pi a^3)</td>
<td>(d)</td>
</tr>
</tbody>
</table>

**Mass ratios**
- B1 1.2

**Correct distances**
- B1 1.2

**Moments about a diameter through \(O\)**
- M1 2.1

\[
4\pi a^2 h \cdot \frac{h}{2} - \frac{2}{3} \pi a^3 \times \frac{3}{8} a = 2\pi a^2 \left(2h - \frac{1}{3} a\right) \times d
\]

- A1 1.1b

\[
d = \frac{h^2 - \frac{a^2}{8}}{2h - \frac{a}{3}} = \frac{3\left(8h^2 - a^2\right)}{8\left(6h - a\right)} * \]

- A1* 2.2a

(5)

### Question 3(b)

\(h = 5a \Rightarrow d = 2.573...a\)

- B1 1.1b

**About to topple so c of m above tipping point**
- M1 2.2a

\[
\Rightarrow \tan \phi = \frac{2a}{5a - 2.573a}
\]

- A1ft 1.1b

\[
\phi = 39.5^\circ \text{ or } 0.689 \text{ rads}
\]

- A1 1.1b

(4)

(9 marks)
### Question 3 notes:

**Question Scheme Marks AOs**

<table>
<thead>
<tr>
<th>(a)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>B1:</td>
<td>Correct mass ratios</td>
</tr>
<tr>
<td>B1:</td>
<td>Correct distances</td>
</tr>
<tr>
<td>M1:</td>
<td>All three terms &amp; dimensionally correct. Could use a parallel axis but final answer must be for the distance from O</td>
</tr>
<tr>
<td>A1:</td>
<td>Correct unsimplified equation</td>
</tr>
<tr>
<td>A1*:</td>
<td>Deduce the given answer. Their working must make it clear how they reached their answer</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(b)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>B1:</td>
<td>Distance of com from base</td>
</tr>
<tr>
<td>M1:</td>
<td>Condone tan the wrong way up</td>
</tr>
<tr>
<td>A1ft:</td>
<td>Correct unsimplified expression for trig ratio for ( \phi ) following their ( d )</td>
</tr>
<tr>
<td>A1:</td>
<td>39.5° or 0.689 rads</td>
</tr>
</tbody>
</table>
### Question 3

#### (a)

**Equation of motion:** \( 1800 - 2v^2 = 500a \) (when seen)  

- **Scheme:**  
  - Select form for \( a \): \( = \frac{500}{dr} \)  
  - \( \int \frac{2}{500} dr = \int \frac{1}{900 - v^2} dv = \frac{1}{60} \int \left( \frac{1}{30 + v} + \frac{1}{30 - v} \right) dv \)  
  - \( \frac{t}{250} = \frac{1}{60} \ln(30 + v) - \frac{1}{60} \ln(30 - v) \) \( + C \)  
  - \( T = \frac{25}{6} \ln\left(\frac{30 + 10}{30 - 10}\right) = \frac{25}{6} \ln 2 \) \( * \)  

- **Marks:** 2.1  
- **AOs:** 2.1  
- **Notes:**  
  - B1: All three terms & dimensionally correct  
  - M1: Use of correct form for acceleration to give equation in \( v, t \) only  
  - M1: Separate variables and integrate  
  - A1: Condone missing \( C \)  
  - M1: Use boundary conditions correctly  
  - A1*: Show sufficient working to justify given answer and a 'statement' that the required result has been achieved

#### (b)

**Equation of motion:** \( 500v \frac{dv}{dx} = 1800 - 2v^2 \)  

- **Scheme:**  
  - \( \int \frac{500v}{1800 - 2v^2} dv = \int 1 dx \)  
  - \( -125 \ln(1800 - 2v^2) = x + C \)  

- **Marks:** 2.1  
- **AOs:** 2.2a  
- **Notes:**  
  - M1: Correct form of acceleration in the equation of motion to give equation in \( v, x \) only  
  - M1: Separate variables and integrate  
  - A1: Condone missing \( C \)  
  - M1: Extract and use boundary conditions  
  - A1*: Show sufficient working to justify given answer and a 'statement' that the required result has been achieved
Question 5 notes:

(a) B1: Correct mass ratios
A1: Need all three terms, must be dimensionally correct
A1*: Show sufficient working to justify the given answer and a 'statement' that the required result has been achieved

(b) M1: Could also take moments about B or about the c.o.m. and use.
A1: cso

(c) M1: All terms and dimensionally correct
A1: Correct unsimplified equation
A1: Or equivalent
M1: Condone tan the wrong way up
A1: Equation in a and d; follow through on their v
cao

<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>5(a)</td>
<td></td>
<td>12</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Mass</th>
<th>From AD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangle</td>
<td>$8a^2$</td>
<td>$a$</td>
</tr>
<tr>
<td>Semicircle</td>
<td>$\frac{1}{2}\pi a^2$</td>
<td>$\frac{4a}{3\pi}$</td>
</tr>
<tr>
<td>Sign</td>
<td>$a^2\left(8-\frac{\pi}{2}\right)$</td>
<td>$h$</td>
</tr>
</tbody>
</table>

Mass ratios

Moments about $AD$

$$a^2\left(8-\frac{\pi}{2}\right)h = 8a^2 \times a - \frac{1}{2}\pi a^2 \times \frac{4a}{3\pi} = 8a^3 - \frac{2}{3}a^3 = \frac{22}{3}a^3$$

$$\Rightarrow h = \frac{22}{3}a \div \left(8-\frac{\pi}{2}\right) = \frac{44a}{3(16-\pi)}$$

(4)

(b) Moments about $A$

$$2aT = \frac{44a}{3(16-\pi)}W$$

$$T = \frac{hW}{2a} = \frac{22W}{3(16-\pi)}$$

(2)

(c) Take moments about $AB$ to find distance of com from $AB$

$$8a^2 \times 2a - \frac{1}{2}\pi a^2 \times d = \left(8-\frac{1}{2}\pi\right) a^2 \times v$$

$$v = \frac{32a - \pi d}{16-\pi}$$

Correct trig for the given angle

$$\tan \alpha = \frac{11}{18} = \frac{h}{v} = \frac{44a}{3(32a - \pi d)}$$

$$d = \frac{8a}{\pi}$$

(6)

(12 marks)
### Question 5 notes:

#### (a)
- **B1:** Correct mass ratios
- **M1:** Need all three terms, must be dimensionally correct
- **A1:** Correct unsimplified equation
- **A1***: Show sufficient working to justify the given answer and a 'statement' that the required result has been achieved

#### (b)
- **M1:** Could also take moments about B or about the c.o.m. and use
- **A1:** $\text{cso}$

#### (c)
- **M1:** All terms and dimensionally correct
- **A1:** Correct unsimplified equation
- **A1:** Or equivalent
- **M1:** Condone tan the wrong way up
- **A1:** Equation in a and d; follow through on their v
- **A1:** $\text{cao}$
<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>6(a)</td>
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<tr>
<td></td>
<td><img src="https://via.placeholder.com/150" alt="Diagram" /></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Conservation of energy</td>
<td>M1</td>
<td>2.1</td>
<td></td>
</tr>
</tbody>
</table>
| \[
\frac{1}{2}mv^2 + mga(1 - \cos \theta) = \frac{1}{2}m\left(\frac{7}{2}ga\right)
\] | A1 | 1.1b | |
| \[
v^2 = ga\left(\frac{3}{2} + 2\cos \theta\right)^* \]
| A1* | 2.2a | |

(b) Resolve parallel to OB and use \( \frac{mv^2}{a} \)

\[
R - mg \cos \theta = \frac{mv^2}{a}
\]

A1 | 1.1b |

Use \( R = 0 \) \( g \cos \theta = -\frac{v^2}{a} \)

M1 | 3.1b |

Solve for \( \theta \) \( \Rightarrow g \cos \theta = -g\left(\frac{3}{2} + 2\cos \theta\right) \)

M1 | 1.1b |

\[
\theta = 120^\circ
\]

A1 | 1.1b |

(5)

(c) Any appropriate comment e.g. the hoop is unlikely to be smooth

B1 | 3.5b |

(1)
At rest \( v = 0 \)

\[ \Rightarrow \cos \theta = -\frac{3}{4} \]

Acceleration is tangential

Magnitude \( g \cos (\theta - 90) = 6.48 \text{ m s}^{-2} \) or \( \frac{\sqrt{7}}{4} g \)

At \( \cos^{-1} \left( -\frac{3}{4} \right) - 90 = 48.6^\circ \) to the downward vertical

(5)
7(a)  

In equilibrium $T_A = T_B$,  

\[ 10e = 25(2 - e) \]

\[ e = \frac{10}{7} \]

Equation of motion for $P$ when distance $x$ from equilibrium position towards $B$:  

\[ 3.5\ddot{x} = T_B - T_A = \frac{50(2 - e - x)}{2} - \frac{20(e + x)}{2} \]

\[ = 50 \left( \frac{4}{7} - x \right) - 20 \left( \frac{10}{7} + x \right) \]

\[ = \frac{50}{2} \left( \frac{4}{7} - x \right) - 20 \left( \frac{10}{7} + x \right) \]

\[ \Rightarrow 3.5\ddot{x} = -35x, \quad \ddot{x} = -10x \]

and hence SHM about the equilibrium position  

(7)  

(b)  

Amplitude = 2 - \frac{10}{7} = \frac{4}{7}  

Use of max speed = $a\omega$  

\[ = \frac{4}{7} \sqrt{10} = 1.81 \text{ (m s}^{-1}) \]

(3)
<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>7(e)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Nearer to $A$ than to $B$: $x &lt; -\frac{3}{7}$</td>
<td>B1</td>
<td>3.1a</td>
</tr>
<tr>
<td></td>
<td>Solve for $\sqrt{10t} \cos \sqrt{10t} = -\frac{3}{4}$, $\sqrt{10t} = 2.418$</td>
<td>M1</td>
<td>3.1a</td>
</tr>
<tr>
<td></td>
<td>Length of time: $\frac{2}{\sqrt{10}}(\pi - 2.418...)$</td>
<td>M1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>$0.457$ (seconds)</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td>Alternative: $\frac{3.864 - 2.419}{\sqrt{10}} = 0.457$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Alternative: $x = \frac{4}{7}\sin \sqrt{10t} = \frac{3}{7}$ $\Rightarrow\sqrt{10t} = 0.8481$ or $\sqrt{10t} = 2.29353$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$t_1 = 0.2682$, $t_2 = 0.72527$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\Rightarrow$ time $= 0.457$ (seconds)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(4)

(14 marks)

**Notes:**

**(a)**

**M1:** Use of $T = \frac{\lambda x}{a}$

**M1:** Dependent on the preceding M1. Equate their tensions

**A1:** $\lambda a$

**M1:** Condone sign error

**A1:** Correct unsimplified equation in $e$ and $x$

**A1:** Equation with one error

**A1:** Full working to justify conclusion that it is SHM about the equilibrium position

**(b)**

**B1ft:** Seen or implied. Follow their $e$

**M1:** Correct method for max. speed

**A1:** 1.81 or better. Follow their $a, \omega$

**(c)**

**B1:** Seen or implied

**M1:** Use of $x = a \cos \omega t$

**M1:** Correct strategy for the required interval

**A1:** 0.457 or better
You must have:
Decision Mathematics Answer Book (enclosed), calculator

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions
- Use black ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- Write your answers for this paper in the Decision Mathematics answer book provided.
- Fill in the boxes at the top of the answer book with your name, centre number and candidate number.
- Do not return the question paper with the answer book.
- Answer all questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – there may be more space than you need.
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information
- A booklet ‘Mathematical Formulae and Statistical Tables’ is provided.
- There are 7 questions in this question paper. The total mark for this paper is 75.
- The marks for each question are shown in brackets – use this as a guide as to how much time to spend on each question.

Advice
- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
Answer ALL questions. Write your answers in the answer book provided.

1. A list of \( n \) numbers needs to be sorted into descending order starting at the left-hand end of the list.

   (a) Describe how to carry out the first pass of a bubble sort on the numbers in the list.

   Bubble sort is a quadratic order algorithm.

   A computer takes approximately 0.021 seconds to apply a bubble sort to a list of 2000 numbers.

   (b) Estimate the time it would take the computer to apply a bubble sort to a list of 50 000 numbers. Make your method clear.

   (Total for Question 1 is 4 marks)
2.

(a) Define what is meant by a **planar** graph. 

(b) Starting at A, find a Hamiltonian cycle for the graph in Figure 1.

Arc AG is added to Figure 1 to create the graph shown in Figure 2.

(c) Use the planarity algorithm to determine whether the graph shown in Figure 2 is planar. You must make your working clear and justify your answer.

(Total for Question 2 is 7 marks)
3. (a) Explain clearly the difference between the classical travelling salesperson problem and the practical travelling salesperson problem.

(b) Use Floyd’s algorithm to find a table of least distances. You should show both the distance table and the route table after each iteration.

(c) Explain how the final route table can be used to find the shortest route from D to B. State this route.

(d) Using the final distance table and the Nearest Neighbour algorithm starting at D, find a minimum route and state its length.

(e) Describe how the results of these algorithms differ.

(2) (Total for Question 4 is 14 marks)

3. (a) Explain clearly the difference between the classical travelling salesperson problem and the practical travelling salesperson problem.

(b) Starting by deleting B and all of its arcs, find a lower bound for Preety’s route.

(c) find x, making your method clear.

(d) Write down the smallest interval that you can be confident contains the optimal length of Preety’s route. Give your answer as an inequality.

(Total for Question 3 is 11 marks)
Figure 3

The network in Figure 3 shows the roads linking a depot, D, and three collection points A, B and C. The number on each arc represents the length, in miles, of the corresponding road. The road from B to D is a one-way road, as indicated by the arrow.

(a) Explain clearly if Dijkstra’s algorithm can be used to find a route from D to A.

The initial distance and route tables for the network are given in the answer book.

(b) Use Floyd’s algorithm to find a table of least distances. You should show both the distance table and the route table after each iteration.

(c) Explain how the final route table can be used to find the shortest route from D to B. State this route.

There are items to collect at A, B and C. A van will leave D to make these collections in any order and then return to D. A minimum route is required.

Using the final distance table and the Nearest Neighbour algorithm starting at D,

(d) find a minimum route and state its length.

Floyd’s algorithm and Dijkstra’s algorithm are applied to a network. Each will find the shortest distance between vertices of the network.

(e) Describe how the results of these algorithms differ.

(Total for Question 4 is 14 marks)
5. A garden centre makes hanging baskets to sell to its customers. Three types of hanging basket are made, **Sunshine**, **Drama** and **Peaceful**. The plants used are categorised as **Impact**, **Flowering** or **Trailing**.

Each **Sunshine** basket contains 2 **Impact** plants, 4 **Flowering** plants and 3 **Trailing** plants. Each **Drama** basket contains 3 **Impact** plants, 2 **Flowering** plants and 4 **Trailing** plants. Each **Peaceful** basket contains 1 **Impact** plant, 3 **Flowering** plants and 2 **Trailing** plants.

The garden centre can use at most 80 **Impact** plants, at most 140 **Flowering** plants and at most 96 **Trailing** plants each day.

The profit on **Sunshine**, **Drama** and **Peaceful** baskets are £12, £20 and £16 respectively. The garden centre wishes to maximise its profit.

Let $x$, $y$ and $z$ be the number of **Sunshine**, **Drama** and **Peaceful** baskets respectively, produced each day.

(a) Formulate this situation as a linear programming problem, giving your constraints as inequalities.

(b) State the further restriction that applies to the values of $x$, $y$ and $z$ in this context.

The Simplex algorithm is used to solve this problem. After one iteration, the tableau is

<table>
<thead>
<tr>
<th>b.v.</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
<th>$r$</th>
<th>$s$</th>
<th>$t$</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>$-\frac{1}{4}$</td>
<td>0</td>
<td>$-\frac{1}{2}$</td>
<td>1</td>
<td>0</td>
<td>$-\frac{3}{4}$</td>
<td>8</td>
</tr>
<tr>
<td>$s$</td>
<td>$\frac{5}{2}$</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>$-\frac{1}{2}$</td>
<td>92</td>
</tr>
<tr>
<td>$y$</td>
<td>$\frac{3}{4}$</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{4}$</td>
<td>24</td>
</tr>
<tr>
<td>$P$</td>
<td>3</td>
<td>0</td>
<td>$-6$</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>480</td>
</tr>
</tbody>
</table>

(c) State the variable that was increased in the first iteration. Justify your answer.

(d) Determine how many plants in total are being used after only one iteration of the Simplex algorithm.

(e) Explain why for a second iteration of the Simplex algorithm the 2 in the $z$ column is the pivot value.
After a second iteration, the tableau is

<table>
<thead>
<tr>
<th>b.v.</th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>r</th>
<th>s</th>
<th>t</th>
<th>Value</th>
</tr>
</thead>
<tbody>
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<td>r</td>
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<td>0</td>
<td>1</td>
<td>(\frac{1}{4})</td>
<td>(-\frac{7}{8})</td>
<td>31</td>
</tr>
<tr>
<td>s</td>
<td>(\frac{5}{4})</td>
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<td>0</td>
<td>(\frac{1}{2})</td>
<td>(-\frac{1}{4})</td>
<td>46</td>
</tr>
<tr>
<td>y</td>
<td>(\frac{1}{8})</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>(-\frac{1}{4})</td>
<td>(\frac{3}{8})</td>
<td>1</td>
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<tr>
<td>P</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>(\frac{7}{2})</td>
<td>756</td>
</tr>
</tbody>
</table>

(f) Use algebra to explain why this tableau is optimal.  

(1)

(g) State the optimal number of each type of basket that should be made.  

(1)

The manager of the garden centre is able to increase the number of Impact plants available each day from 80 to 100. She wants to know if this would increase her profit.  

(h) Use your final tableau to determine the effect of this increase.  (You should not carry out any further calculations.)  

(2)

(Total for Question 5 is 15 marks)
6.

A project is modelled by the activity network shown in Figure 4. The activities are represented by the arcs. The number in brackets on each arc gives the time, in days, to complete that activity. Each activity requires one worker. The project is to be completed in the shortest possible time.

(a) Calculate the early time and the late time for each event, using Diagram 1 in the answer book.

(b) On Grid 1 in the answer book, complete the cascade (Gantt) chart for this project.

(c) On Grid 2 in the answer book, draw a resource histogram to show the number of workers required each day when each activity begins at its earliest time.

The supervisor of the project states that only three workers are required to complete the project in the minimum time.

(d) Use Grid 2 to determine if the project can be completed in the minimum time by only three workers. Give reasons for your answer.

(Total for Question 6 is 12 marks)
7. A linear programming problem in $x$, $y$ and $z$ is described as follows.

Maximise $P = 3x + 2y + 2z$

subject to $2x + 2y + z \leq 25$

$x + 4y \leq 15$

$x \geq 3$

(a) Explain why the Simplex algorithm cannot be used to solve this linear programming problem.

The big-M method is to be used to solve this linear programming problem.

(b) Define, in this context, what $M$ represents. You must use correct mathematical language in your answer.

The initial tableau for a big-M solution to the problem is shown below.

<table>
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<th>$x$</th>
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<th>$z$</th>
<th>$s_1$</th>
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<th>$s_3$</th>
<th>$t_1$</th>
<th>Value</th>
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<td>1</td>
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<td>0</td>
<td>25</td>
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<td>$s_2$</td>
<td>1</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
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<tr>
<td>$t_1$</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>$P$</td>
<td>$-(3 + M)$</td>
<td>$-2$</td>
<td>$-2$</td>
<td>0</td>
<td>0</td>
<td>$M$</td>
<td>0</td>
<td>$-3M$</td>
</tr>
</tbody>
</table>

(c) Explain clearly how the equation represented in the b.v. $t_1$ row was derived.

(d) Show how the equation represented in the b.v. $P$ row was derived.
The tableau obtained from the first iteration of the big-M method is shown below.

<table>
<thead>
<tr>
<th>b.v.</th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>s₁</th>
<th>s₂</th>
<th>s₃</th>
<th>t₁</th>
<th>Value</th>
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</thead>
<tbody>
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<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
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<td>19</td>
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<tr>
<td>s₂</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>–1</td>
<td>3</td>
</tr>
<tr>
<td>P</td>
<td>0</td>
<td>–2</td>
<td>–2</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>3 + M</td>
<td>9</td>
</tr>
</tbody>
</table>

(e) Solve the linear programming problem, starting from this second tableau. You must

- give a detailed explanation of your method by clearly stating the row operations you use and
- state the solution by deducing the final values of \( P, x, y \) and \( z \).

(7)

(Total for Question 7 is 12 marks)
The tableau obtained from the first iteration of the big-M method is shown below.

\[
\begin{array}{ccccccccc}
& & & & & & & & \\
& s_1 & & & & & & & \\
& 2 & & & & & & & \\
& 3 & & & & & & & t \\
& 1 & \text{Value} & & & & & & \\
\hline
s_1 & 0 & 2 & 1 & 1 & 0 & 2 & -2 & 19 \\
s_2 & 0 & 4 & 0 & 0 & 1 & 1 & -1 & 12 \\
x_1 & 0 & 0 & 0 & 0 & -1 & 1 & 3 & \ \\
P & -2 & -2 & 0 & 0 & -3 & 3 & M & 9 \\
\end{array}
\]

(e) Solve the linear programming problem, starting from this second tableau. You must
• give a detailed explanation of your method by clearly stating the row operations
  you use and
• state the solution by deducing the final values of
  \( P \), \( x \), \( y \) and \( z \).

(Total for Question 7 is 12 marks)
Question 1 continued

(Total for Question 1 is 4 marks)
2.

![Diagram of a geometric figure]

Figure 1

(Total for Question 2 is 7 marks)
Question 2 continued

Figure 2

(Total for Question 2 is 7 marks)
### Question 3 Continued

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(Total for Question 3 is 11 marks)
### Question 3 continued

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(Total for Question 3 is 11 marks)
4.

![Diagram](image)

**Figure 3**

(b)

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(Total for Question 4 is 14 marks)
Question 5 continued

(Total for Question 5 is 15 marks)
Diagram 1

Key:
- Early event time
- Late event time
Question 6 continued

Grid 1

(There is a spare grid on the next page)

Grid 2
Question 6 continued

Copy of Grid 1

(Total for Question 6 is 12 marks)
7.

<table>
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<td>P</td>
<td>0</td>
<td>−2</td>
<td>−2</td>
<td>0</td>
<td>0</td>
<td>−3</td>
<td>3 + M</td>
<td>9</td>
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<table>
<thead>
<tr>
<th>b.v.</th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>s₁</th>
<th>s₂</th>
<th>s₃</th>
<th>t₁</th>
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<th>s₁</th>
<th>s₂</th>
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Question 7 continued

<table>
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<th>x</th>
<th>y</th>
<th>z</th>
<th>s₁</th>
<th>s₂</th>
<th>s₃</th>
<th>t₁</th>
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<th>y</th>
<th>z</th>
<th>s₁</th>
<th>s₂</th>
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<th>t₁</th>
<th>Value</th>
<th>Row Ops</th>
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</table>

(Total for Question 7 is 12 marks)

TOTAL FOR PAPER IS 75 MARKS
## Paper 3D/4D: Decision Mathematics 1 Mark Scheme

<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1(a)</strong></td>
<td>In the first pass we compare the first value with the second value and we swap these values if the second is larger than the first</td>
<td>B1</td>
<td>2.4</td>
</tr>
<tr>
<td></td>
<td>We then compare the value which is now second with the third value and swap if the third is larger than the second. We continue in this way until we reach the end of this list</td>
<td>B1</td>
<td>2.4</td>
</tr>
<tr>
<td><strong>(b)</strong></td>
<td></td>
<td>(2)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( t = 0.021 \times \left( \frac{50000}{2000} \right)^2 )</td>
<td>M1</td>
<td>1.1a</td>
</tr>
<tr>
<td></td>
<td>( t = 13.125 ) (seconds)</td>
<td>A1</td>
<td>1.1b</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(2)</td>
<td></td>
</tr>
<tr>
<td><strong>Notes:</strong></td>
<td></td>
<td>(4 marks)</td>
<td></td>
</tr>
<tr>
<td><strong>(a)</strong></td>
<td>Comparing first value with second value, swap if second is larger (oe) – in their reasoning it must be clear that the first value in the list is being compared with the second value in the list and swapping if the second is larger than the first</td>
<td>B1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Compare second with third, (third with fourth), and so on until the end of the list – must be clear in their reasoning that after the first comparison the second value in the list is compared with the third value and so on until the end of the list</td>
<td>B1</td>
<td></td>
</tr>
<tr>
<td><strong>(b)</strong></td>
<td>Correct method seen – accept 25 for 50000/2000</td>
<td>M1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>cao</td>
<td>A1</td>
<td></td>
</tr>
</tbody>
</table>
Question | Scheme | Marks | AOs
--- | --- | --- | ---
2(a) | A planar graph is a graph that can be drawn so that… | B1 | 1.2
| … no arc meets another arc except at a vertex | B1 | 1.2

(b) | e.g. ABCDEGFA | B1 | 1.1b

(c) | Creates two lists of arcs | M1 | 2.1
| e.g. BG AD | M1 | 1.1b
| CG BD | M1 | 1.1b
| EG AE | A1 | 1.1b
| CE AF | A1 | 1.1b

Since no arc appears in both lists, the graph is planar (or draws a planar version) | A1 | 2.4

Notes:
(a) B1: A clear indication that a planar graph ‘can be drawn’ – allow this mark even if candidate implies that arcs can cross each other
| B1: cao – no arc meets another arc except at a vertex – technical language must be correct
(b) B1: Any correct Hamiltonian cycle (must start and finish at A) – must contain 8 vertices with every vertex appearing only once (except A)
(c) M1: Creates two list of arcs (with at least three arcs in each list) which contain no common arcs
| M1: Four arcs (in each list) and within each list there are no crossing arcs
| A1: cao
| A1: Correct reasoning that no arc appears in both lists + so the graph is therefore planar

(7 marks)
### 3(a)

E.g. in the practical problem each vertex must be visited at least once. In the classical problem each vertex must be visited just once.

<table>
<thead>
<tr>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>B2,1,0</td>
<td>2.4 2.4</td>
</tr>
</tbody>
</table>

### (b)

Prim’s algorithm on reduced network starting at A: AD, AF, AE, CE, CG

Lower bound = $107 + 17 + 25 = 149$ (km)

<table>
<thead>
<tr>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1</td>
<td>1.1b</td>
</tr>
<tr>
<td>A1</td>
<td>1.1b</td>
</tr>
</tbody>
</table>

### (c)

NNA from A: $A - D - F - E - C - G - B - A = 126 + x$

NNA from C: $C - E - A - D - F - B - G - C = 139 + x$

$(126 + x) + (139 + x) = 331 \Rightarrow x = 33$

<table>
<thead>
<tr>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1</td>
<td>1.1b</td>
</tr>
<tr>
<td>A1</td>
<td>1.1b</td>
</tr>
</tbody>
</table>

### (d)

$149 < \text{optimal} \leq 159$

<table>
<thead>
<tr>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1</td>
<td>2.2b</td>
</tr>
<tr>
<td>A1</td>
<td>1.1b</td>
</tr>
</tbody>
</table>

### Notes:

(a)

**B1**: Understands the difference is connected to the number of times each vertex may be visited (but maybe incorrectly attributed). Must be an attempt at a difference (so must refer to both the classical and practical problems explicitly). Technical language (vertex/node) must be correct. Need not imply each/every/all (oe) vertices for this first mark.

**B1**: Correctly reasons which is classical and which is practical and correctly states the difference. Must imply that each/every/all (oe) vertices are visited, so for example, ‘the practical problem visits a vertex at least once while the classical visits a vertex only once’ is B1B0 (note that B0B1 is not possible in (a)).

(b)

**M1**: Correctly applying Prim’s algorithm from node A for the first four arcs (or five nodes).

**M1**: Candidates weight of their RMST + 17 + 25 (the two smallest arcs incident to B)

**A1**: cao (condone lack of units)

(c)

**M1**: Either one route, must return to A

**A1**: One correct route, must return to A and corresponding length correct (do not is in part (c) if correct lengths seen but are then doubled)

**A1**: Both routes correct and their corresponding lengths correct

**A1**: cao for $x$

(d)

**M1**: Their numbers correctly used, accept any inequalities or any indication of interval from their 149 to their 159 (so 149 – 159 can score this mark). This mark is dependent on two routes seen in (c), however, neither of the two totals need to be correct. Please note that UB > LB for this mark.

**A1**: cao (no follow through on their values) including correct inequalities or equivalent set notation (but condone $149 \leq \text{optimal} \leq 159$)
<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>4(a)</strong></td>
<td>Yes Dijkstra's algorithm can be applied to either a directed or undirected network</td>
<td>B1</td>
<td>3.5b</td>
</tr>
<tr>
<td><strong>(b)</strong></td>
<td></td>
<td>(1)</td>
<td></td>
</tr>
<tr>
<td><strong>Initial tables</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>1st iteration</strong></td>
<td></td>
<td></td>
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<tr>
<td><strong>2nd iteration</strong></td>
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<tr>
<td><strong>3rd iteration</strong></td>
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<tr>
<td><strong>4th iteration</strong></td>
<td>no changes therefore optimal</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(c)</strong></td>
<td>Start at D (4th) row and read across to the B (2nd) column, there is a C there so the route starts DC. Look at the C row, B column and you see B</td>
<td></td>
<td>B1 2.4</td>
</tr>
<tr>
<td></td>
<td>The route is therefore DCB</td>
<td></td>
<td>B1 2.2a</td>
</tr>
</tbody>
</table>
### Question Scheme Marks AOs

<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>(d)</td>
<td>D – C – B – A – B – D</td>
<td>M1 2.2a</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Length 19 (miles)</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td>(e)</td>
<td>Dijkstra’s algorithm finds the shortest distances from <strong>one</strong> vertex to <strong>all</strong> the others. Floyd’s algorithm finds the shortest distance between <strong>every pair</strong> of vertices.</td>
<td>B1 2.5</td>
<td>B1 2.5</td>
</tr>
</tbody>
</table>

### Question 4 notes:

(a)
- **B1:** cao (must include mention of ‘directed’ network)
- **M1:** No change in the first row and first column of both tables with at least one value in the distance table reduced and one value in the route table changed

(b)
- **A1:** cao
- **M1:** No change in the second row and second column of both tables with at least two values in the distance table reduced and two values in the route table changed
- **A1ft:** Correct second iteration follow through from the candidate’s first iteration
- **M1:** No change in the third row and third column of both tables with at least one value in the distance table reduced and one value in the route table changed
- **A1ft:** Correct third iteration follow through from the candidate’s second iteration
- **A1:** cao (no change after the fourth iteration) – all previous marks must have been awarded in this part

(c)
- **B1:** Clear indication of how the final route table can be used to get from D to B (therefore must mention the correct rows and columns in their reasoning)
- **B1:** Completely correct argument + correct route (DCB)

(d)
- **M1:** Deduce correctly their minimum route from their final distance table (dependent on all M marks in (a)) must begin and end at D
- **A1:** cao (length of 19)

(e)
- **B1:** cao – must use correct language ‘**one** vertex to **all** other vertices’
- **B1:** cao – must use correct language ‘**every pair** of vertices’
<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>5(a)</td>
<td>Maximise $P = 12x + 20y + 16z$</td>
<td>B1 3.3</td>
<td></td>
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<td></td>
<td>Subject to $2x + 3y + z \leq 80$</td>
<td>M1 3.3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$4x + 2y + 3z \leq 140$</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$3x + 4y + 2z \leq 96$</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$x, y, z \geq 0$</td>
<td>B1 3.3</td>
<td></td>
</tr>
<tr>
<td>(b)</td>
<td>The values must all be integers</td>
<td>B1 3.3</td>
<td></td>
</tr>
<tr>
<td>(c)</td>
<td>Variable $y$ entered the basic variable column…</td>
<td>M1 2.4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>…so $y$ was increased first</td>
<td>A1 2.2a</td>
<td></td>
</tr>
<tr>
<td>(d)</td>
<td>$(80 + 140 + 96) - (8 + 92) = 216$ plants</td>
<td>B1 3.2a</td>
<td></td>
</tr>
<tr>
<td>(e)</td>
<td>The next pivot must come from a column which has a negative value in the objective row so therefore the pivot must come from column $z$</td>
<td>M1 2.4</td>
<td></td>
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<tr>
<td></td>
<td>The pivot must be positive and the least of $92/2 = 46$ and $24/0.5 = 48$ so the pivot must be the 2 (from column $z$)</td>
<td>A1 2.2a</td>
<td></td>
</tr>
<tr>
<td>(f)</td>
<td>$P + 10.5x + 3s + 3.5t = 756$ so increasing $x$, $s$ or $t$ will decrease profit</td>
<td>B1 2.4</td>
<td></td>
</tr>
<tr>
<td>(g)</td>
<td>Make 1 Drama basket and 46 Peaceful baskets</td>
<td>B1 2.2a</td>
<td></td>
</tr>
<tr>
<td>(h)</td>
<td>The slack variable, $r$, associated with this type of plant, is currently at 31. Increasing the number of Impact plants by a further 20 would have no effect</td>
<td>M1 3.1b</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>A1 3.2a</td>
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</table>

(15 marks)
### Question 5 notes:

(a)  
**B1:** Correct objective function/expression (accept in pence rather than pounds e.g. 1200x + 2000y + 1600z)  
**M1:** Correct coefficients and correct right-hand side for at least one inequality – accept any inequality or equals  
**A1:** Two correct (non-trivial) inequalities  
**A1:** All three non-trivial inequalities correct  
**B1:**  

(b)  
**B1:** cao  

(c)  
**M1:** Correct reasoning that y has become a basic variable  
**A1:** Correct deduction that y was therefore increased first  

(d)  
**B1:** cao  

(e)  
**M1:** Correct reasoning given that the pivot value must come from column z  
**A1:** Correctly deduce (from correctly stated calculations) that the pivot value is the 2 in column z  

(f)  
**B1:** States correct objective function and mention of increasing x, s or t will decrease profit  

(g)  
**B1:** cao – in context so not in terms of y and z  

(h)  
**M1:** Identifies the slack variable r and its current value of 31  
**A1:** Correct interpretation that increasing the number of Impact plants would have no effect
<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>6(a)</strong></td>
<td>See diagram on next page. Top and bottom boxes</td>
<td>M1</td>
<td>2.1</td>
</tr>
<tr>
<td>Top boxes correct</td>
<td>A1</td>
<td>1.1b</td>
<td></td>
</tr>
<tr>
<td>Bottom boxes correct</td>
<td>A1</td>
<td>1.1b</td>
<td></td>
</tr>
<tr>
<td>(3)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(b)</strong></td>
<td>See diagram below</td>
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<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>At least 8 activities + 4 floats with clear distinction between activity and their corresponding float</td>
<td>M1</td>
<td>2.5</td>
<td></td>
</tr>
<tr>
<td>Correct critical activities + 4 correct non-critical activities</td>
<td>A1</td>
<td>1.1b</td>
<td></td>
</tr>
<tr>
<td>All 13 correct</td>
<td>A1</td>
<td>1.1b</td>
<td></td>
</tr>
<tr>
<td>(3)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(c)</strong></td>
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<td></td>
</tr>
<tr>
<td>Bars correct to time = 13</td>
<td>M1</td>
<td>1.1b</td>
<td></td>
</tr>
<tr>
<td>Bars correct from 14 to 24</td>
<td>A1</td>
<td>1.1b</td>
<td></td>
</tr>
<tr>
<td>(3)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(d)</strong></td>
<td>Until time 4 only A and B can happen. After time 4, there are 6 worker-days to cover, but only 4 worker-days available. Hence the project cannot be completed by time 24 with three workers.</td>
<td>B1</td>
<td>3.1a</td>
</tr>
<tr>
<td></td>
<td>M1</td>
<td>2.4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>A1</td>
<td>2.2a</td>
<td></td>
</tr>
<tr>
<td>(3)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(12 marks)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Diagram for Question 6(a)

- (a) See diagram on next page.
  - Top and bottom boxes correct
  - M1
  - A1
  - 2.1
  - 1.1b

- (b) See diagram below
  - At least 8 activities + 4 floats with clear distinction between activity and their corresponding float
  - M1
  - 2.5
  - Correct critical activities + 4 correct non-critical activities
  - A1
  - 1.1b
  - All 13 correct
  - A1
  - 1.1b

- (c) M1
  - Bars correct to time = 13
  - A1
  - 1.1b
  - Bars correct from 14 to 24
  - A1
  - 1.1b

- (d) Until time 4 only A and B can happen.
  - After time 4, there are 6 worker-days to cover, but only 4 worker-days available.
  - Hence the project cannot be completed by time 24 with three workers.

- M1
  - 3.1a
  - 2.4
  - 2.2a

- (12 marks)
(a) Simplex can only work with \( \leq \) constraints.

(b) \( M \) is an arbitrary large real number.

(c) \( x + s - t = 3 \) where \( s \) is a surplus variable and \( t \) is an artificial variable.

(d) Let \( P(x, y, z, M) = x + y + z - M \) (where \( M \) is an arbitrary large number).

\[
3x + y + z - M = 3\quad (3) \\
2x + y + z - M = 3\quad (3) \\
M - x - y - z = M \\
9/2 \quad 11/2 \quad 1/2 \quad 1 \\
3/2 \quad 0 \quad 1/2 \quad 1/2 \quad 0 \quad 0 \quad 1 \\
25/2 \quad 31/2 \quad -1 \quad -1 \quad 1 \\
47/2 \quad 41/2 \quad -1 \quad 1/2 \quad 1 \\
75/2 \quad 41/2 \quad -1/2 \quad -3/2 \quad 0 \quad 0 \quad 1 \\
47/2 \quad 3 \quad 0 \quad 1 \\
\]

(b/v) \( x, y, z \) values:

<table>
<thead>
<tr>
<th>Value</th>
<th>Row Ops</th>
</tr>
</thead>
<tbody>
<tr>
<td>3s</td>
<td>0 \quad 1 \quad ½ \quad ½ \quad 0 \quad 1</td>
</tr>
<tr>
<td>2s</td>
<td>0 \quad 3 \quad -1/2 \quad -1/2 \quad 1 \quad 0 \quad 0 \quad 5/2 \quad 21</td>
</tr>
<tr>
<td></td>
<td>1 \quad 1/2 \quad 2 \quad 19/2 \quad 11 \</td>
</tr>
<tr>
<td></td>
<td>2 \quad 4 \quad 0 \quad 0 \quad 1 \quad 1</td>
</tr>
<tr>
<td></td>
<td>0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 25/2 \quad 31</td>
</tr>
</tbody>
</table>

Question 6 notes:

(a) **M1:** All top boxes and all bottom boxes completed. For the top boxes all values must be increasing in the direction of the arrows for both the activities and the dummies. For the bottom boxes all values must be decreasing in the opposite direction to the arrows for both the activities and the dummies. While the values need not be correct each value must be increasing or decreasing (as appropriate) in a logical and sequential manner.

**A1:** cao for top boxes

**A1:** cao for bottom boxes

**M1:** At least 8 activities including 4 floats. Scheduling diagram scores M0 – clear distinction must be shown between the notation used for an activity and its float.

(b) **A1:** Correct critical activities and 4 correct non-critical activities

**A1:** cao (all 13 correct activities)

(c) **M1:** Plausible histogram with no holes or overhangs (must go to at least 10 on the time axis).

**A1:** Histogram correct to time 13

**A1:** Histogram correct from time 14 to time 24

(d) **B1:** Considering an appropriate process to adjust Grid 2 so that no activity must be completed by a 4th worker, for example, a correct argument that until time 4 only activities A and B can happen (so no activity can use the spare worker before time 4)

**M1:** Uses their histogram to explain when the number of workers is greater or less than the minimum number found in (b)

**A1:** Correctly deduces that the project cannot be completed by time 24 – this mark is dependent on a correct histogram seen in (d)
**Question 7(a)**

Simplex can only work with $\leq$ constraints

B1 3.5b

**Question 7(b)**

M is an arbitrary large real number

B1 2.5

**Question 7(c)**

$x \geq 3 \Rightarrow x - s_3 + t_1 = 3$ where $s_3$ is a surplus variable and $t_1$ is an artificial variable

B1 2.4

**Question 7(d)**

Let $P = 3x + 2y + 2z - Mt_1$ (where $M$ is an arbitrary large number)

$\therefore P = 3x + 2y + 2z - M(3 - x + s_3)$

$= (3 + M)x + 2y + 2z - Ms_3 - 3M$

$\Rightarrow P - (3 + M)x - 2y - 2z + Ms_3 = -3M$

M1 A1 2.1 1.1b

**Question 7(e)**

<table>
<thead>
<tr>
<th>b.v.</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$t_1$</th>
<th>Value</th>
<th>Row Ops</th>
</tr>
</thead>
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<td>$s_3$</td>
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<td>1</td>
<td>½</td>
<td>½</td>
<td>0</td>
<td>1</td>
<td>−1</td>
<td>19/2</td>
<td>$r_i = (1/2)r_1$</td>
</tr>
<tr>
<td>$s_2$</td>
<td>0</td>
<td>3</td>
<td>−1/2</td>
<td>−1/2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>5/2</td>
<td>$R_2 - r_1$</td>
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<tr>
<td>$x$</td>
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<td>1</td>
<td>½</td>
<td>½</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>25/2</td>
<td>$R_3 + r_1$</td>
</tr>
<tr>
<td>$P$</td>
<td>0</td>
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<td>−1/2</td>
<td>3/2</td>
<td>0</td>
<td>0</td>
<td>$M$</td>
<td>75/2</td>
<td>$R_4 + 3r_1$</td>
</tr>
</tbody>
</table>

<table>
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<tr>
<th>b.v.</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$t_1$</th>
<th>Value</th>
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<td>1</td>
<td>0</td>
<td>2</td>
<td>−2</td>
<td>19</td>
<td>$r_i = 2r_1$</td>
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<td>4</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
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<td>12</td>
<td>$R_2 + (1/2)r_1$</td>
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<tr>
<td>$x$</td>
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<td>0</td>
<td>−1</td>
<td>1</td>
<td>3</td>
<td>$R_3 - (1/2)r_1$</td>
<td></td>
</tr>
<tr>
<td>$P$</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>$M$</td>
<td>−1</td>
<td>$R_4 + (1/2)r_1$</td>
</tr>
</tbody>
</table>

$P = 47, x = 3, y = 0, z = 19$

B1ft 1.1b

(12 marks)
### Question 7 notes:

**(a)**
**B1:** Correctly states the limitation of the Simplex model – Simplex involves iterations which allow movement from one vertex in the feasible region to another vertex (in the feasible region). If all constraints are of the form $y \leq$ this means that the origin is always a feasible solution and therefore can act as the initial starting point for the problem. However, the constraint $x \geq 3$ means that the origin is not feasible and so the algorithm is unable to begin.

**(b)**
**B1:** cao including the correct mathematical language (must include ‘arbitrary’, ‘large’ and ‘real’)

**(c)**
**(B1:** Correctly states both the inequality $x \geq 3$ and the equation $x - s_i + t_i = 3$ together with an explanation of the meaning behind the variables $s_i$ and $t_i$

**(d)**
**M1:** $P = 3x + 2y + 2z - Mt_i$ and substitutes their expression for $t_i$

**A1:** Correct mathematical argument including sufficient detail to allow the line of reasoning to be followed to the correct conclusion – dependent on previous B mark in (c)

**(e)**
**M1:** Correct pivot located, attempt to divide row. If negative value used then no marks

**A1:** Pivot row correct (including change of b.v.) and row operations used at least once, one of columns $y, z, s_i, t_i$ or Value correct

**A1:** cao for values (ignore b.v. column and Row Ops)

**M1:** Pivot row consistent (following their previous table) including change of b.v. and row operations used at least once, one of columns $y, s_i, s_i, t_i$ or Value correct

**A1:** cao on final table (ignore Row Ops)

**B1:** The correct Row Operations explained either in terms of the ‘old’ or ‘new’ pivot rows

**B1ft:** Correctly states the final values of $P, x, y$ and $z$ from their correct corresponding rows of the final table
You must have:
Decision Mathematics Answer Book (enclosed), calculator

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions
• Use black ink or ball-point pen.
• If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
• Write your answers for this paper in the Decision Mathematics answer book provided.
• Fill in the boxes at the top of the answer book with your name, centre number and candidate number.
• Do not return the question paper with the answer book.
• Answer all questions and ensure that your answers to parts of questions are clearly labelled.
• Answer the questions in the spaces provided – there may be more space than you need.
• You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
• Answers should be given to three significant figures unless otherwise stated.

Information
• There are 7 questions in this question paper. The total mark for this paper is 75.
• The marks for each question are shown in brackets – use this as a guide as to how much time to spend on each question.

Advice
• Read each question carefully before you start to answer it.
• Try to answer every question.
• Check your answers if you have time at the end.
1. (a) Find the general solution of the recurrence relation

\[ u_{n+2} = u_{n+1} + u_n, \quad n \geq 1 \]

Given that \( u_1 = 1 \) and \( u_2 = 1 \)

(b) find the particular solution of the recurrence relation.

(Total for Question 1 is 6 marks)
A company has three factories, A, B and C. It supplies mattresses to three shops, D, E and F. The table shows the transportation cost, in pounds, of moving one mattress from each factory to each shop. It also shows the number of mattresses available at each factory and the number of mattresses required at each shop. A minimum cost solution is required.

(a) Use the north-west corner method to obtain an initial solution.

(b) Show how the transportation algorithm is used to solve this problem.

You must state, at each appropriate step, the
• shadow costs,
• improvement indices,
• route,
• entering cell and exiting cell,

and explain clearly how you know that your final solution is optimal.

(Total for Question 2 is 12 marks)
3. Four workers, A, B, C and D, are to be assigned to four tasks, P, Q, R and S.

Each worker must be assigned to at most one task and each task must be done by just one worker.

The amount, in pounds, that each worker would earn while assigned to each task is shown in the table below.

<table>
<thead>
<tr>
<th></th>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>32</td>
<td>32</td>
<td>33</td>
<td>35</td>
</tr>
<tr>
<td>B</td>
<td>28</td>
<td>35</td>
<td>31</td>
<td>37</td>
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<tr>
<td>C</td>
<td>35</td>
<td>29</td>
<td>33</td>
<td>36</td>
</tr>
<tr>
<td>D</td>
<td>36</td>
<td>30</td>
<td>36</td>
<td>33</td>
</tr>
</tbody>
</table>

The Hungarian algorithm is to be used to find the maximum total amount which may be earned by the four workers.

(a) Explain how the table should be modified.

(b) Reducing rows first, use the Hungarian algorithm to obtain an allocation which maximises the total earnings, stating how each table was formed.

(c) Formulate the problem as a linear programming problem. You must define your decision variables and make your objective function and constraints clear.

(Total for Question 3 is 13 marks)
   A player gives 5 tokens to play and then picks a card. If they pick a 2, 3, 4, 5 or 6 then they gain 15 tokens. If any other card is picked they lose.
   
   If they lose, the card is replaced and they can choose to pick again for another 5 tokens. This time if they pick either an ace or a king they gain 40 tokens. If any other card is picked they lose.
   
   Daniel is deciding whether to play this game.
   
   (a) Draw a decision tree to model Daniel’s possible decisions and the possible outcomes.
   (b) Calculate Daniel’s optimal EMV and state the optimal strategy indicated by the decision tree.

(Total for Question 4 is 8 marks)
A two person zero-sum game is represented by the pay-off matrix for player A given above.

(a) Explain, with justification, how this matrix may be reduced to a 3 × 3 matrix.

(b) Find the play-safe strategy for each player and verify that there is no stable solution to this game.

The game is formulated as a linear programming problem for player A.

The objective is to maximise $P = V$, where $V$ is the value of the game to player A.

One of the constraints is that $p_1 + p_2 + p_3 \leq 1$, where $p_1$, $p_2$, $p_3$ are the probabilities that player A plays 1, 2, 3 respectively.

(c) Formulate the remaining constraints for this problem. Write these constraints as inequalities.

The Simplex algorithm is used to solve the linear programming problem.

The solution obtained is $p_1 = 0$, $p_2 = \frac{3}{7}$, $p_3 = \frac{4}{7}$

(d) Calculate the value of the game to player A.
6.

Figure 1 shows a capacitated, directed network. The number on each arc \((x, y)\) represents the lower \((x)\) capacity and upper \((y)\) capacity of that arc.

(a) Calculate the value of the cut \(C_1\) and cut \(C_2\)

(b) Explain why the flow through the network must be at least 12 and at most 16

(c) Explain why arcs DG, AG, EG and FG must all be at their lower capacities.

(d) Determine a maximum flow pattern for this network and draw it on Diagram 1 in the answer book. You do not need to use the labelling procedure.

(e) (i) State the value of the maximum flow through the network.

(ii) Explain why the value of the maximum flow is equal to the value of the minimum flow through the network.

Node E becomes blocked and no flow can pass through it. To maintain the maximum flow through the network the upper capacity of exactly one arc is increased.

(f) Explain how it is possible to maintain the maximum flow found in (d).

(Total for Question 6 is 12 marks)
7. A company assembles boats.

They can assemble up to five boats in any one month, but if they assemble more than three they will have to hire additional space at a cost of £800 per month.

The company can store up to two boats at a cost of £350 each per month.

The overhead costs are £1500 in any month in which work is done.

Boats are delivered at the end of each month. There are no boats in stock at the beginning of January and there must be none in stock at the end of May.

The order book for boats is

<table>
<thead>
<tr>
<th>Month</th>
<th>January</th>
<th>February</th>
<th>March</th>
<th>April</th>
<th>May</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number ordered</td>
<td>3</td>
<td>2</td>
<td>6</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

Use dynamic programming to determine the production schedule which minimises the costs to the company. Show your working in the table provided in the answer book and state the minimum production cost.

(Total for Question 7 is 12 marks)
A company assembles boats. They can assemble up to five boats in any one month, but if they assemble more than three they will have to hire additional space at a cost of £800 per month. The company can store up to two boats at a cost of £350 each per month. The overhead costs are £1500 in any month in which work is done. Boats are delivered at the end of each month. There are no boats in stock at the beginning of January and there must be none in stock at the end of May. The order book for boats is:

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</table>

Use dynamic programming to determine the production schedule which minimises the costs to the company. Show your working in the table provided in the answer book and state the minimum production cost.

(Total for Question 7 is 12 marks)
1.
Question 1 continued

(Total for Question 1 is 6 marks)
2.

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<td>19</td>
<td>9</td>
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<td>C</td>
<td>11</td>
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<td>Required</td>
<td>38</td>
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(Total for Question 2 is 12 marks)
Question 2 continued

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(Total for Question 2 is 12 marks)
Question 3 continued

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Question 3 continued

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(Total for Question 3 is 13 marks)
Question 4 continued

(Total for Question 4 is 8 marks)
5.  

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(Total for Question 5 is 12 marks)
Question 6 continued

Diagram 1

(Total for Question 6 is 12 marks)
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Turn over
Question 7 continued
**Paper 4G: Decision Mathematics 2 Mark Scheme**

<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1(a)</strong></td>
<td>Auxiliary equation: $\lambda^2 - \lambda - 1 = 0$ and attempt to solve</td>
<td>M1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\lambda = \frac{1 \pm \sqrt{5}}{2} \Rightarrow u_n = A \left(\frac{1 + \sqrt{5}}{2}\right)^n + B \left(\frac{1 - \sqrt{5}}{2}\right)^n$, where $A$ and $B$ are arbitrary constants</td>
<td>M1 1.1b</td>
<td>A1 2.2a</td>
</tr>
<tr>
<td></td>
<td>(3)</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(b)</strong></td>
<td>Use given conditions to obtain two equations in $A$ and $B$</td>
<td>M1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Attempt to solve to obtain $A$ and $B$</td>
<td>M1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$u_n = \frac{1}{\sqrt{5}} \left{ \left(\frac{1 + \sqrt{5}}{2}\right)^n - \left(\frac{1 - \sqrt{5}}{2}\right)^n \right}$</td>
<td>A1 1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(3)</td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>(6 marks)</td>
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</tbody>
</table>

**Notes:**

(a)  
M1: writes down correct auxiliary equation and attempts to solve using either the formula or completing the square  
M1: writes down the general solution in the form $u_n = A(\lambda_1)^n + B(\lambda_2)^n$ using their roots $\lambda_1, \lambda_2$ - dependent on the first M mark  
A1: CAO – both lhs and rhs correct including defining $A$ and $B$ as (arbitrary) constants

(b)  
M1: uses the correct initial conditions to write down two equations in $A$ and $B$ – for reference these equations are $A \left(1 + \sqrt{5}\right) + B \left(1 - \sqrt{5}\right) = 2$ and $A \left(1 + \sqrt{5}\right)^2 + B \left(1 - \sqrt{5}\right)^2 = 4$  
M1: Attempts to solve these two equations (using a correct method but condone sign slips) to achieve a value for $A$ and $B$  
A1: CAO
### Question 2 Notes:

**B1:**
- CAO

**M1:**
- Finding all 6 shadow costs and the 4 improvement indices for the correct 4 entries – candidates must clearly identify these two sets of results.

**A1:**
- Shadow costs and II CAO

**M1:**
- A valid route, their most negative II chosen, only one empty square used, \( \theta \)'s balance

**A1:**
- CAO – including the deduction of all entering and exiting cells

**M1:**
- Finding all 6 shadow costs and the 4 improvement indices for the correct 4 entries – this mark is dependent on all previous M marks which will therefore indicate a correct mathematical argument leading from the initial solution to the confirmation of the optimal solution.

**A1:**
- Shadow costs and II CAO

### Scheme

<table>
<thead>
<tr>
<th>Question</th>
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<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>2(a)</strong></td>
<td><img src="image" alt="Table" /></td>
<td>B1 1.1b</td>
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| (i) |

### (b)

#### Shadow Costs

<table>
<thead>
<tr>
<th>Shadow costs</th>
<th>15</th>
<th>22</th>
<th>14</th>
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<tr>
<td>0 A</td>
<td>X</td>
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<td>-5</td>
</tr>
<tr>
<td>-4 B</td>
<td>X</td>
<td>X</td>
<td>X</td>
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<tr>
<td>4 C</td>
<td>-8</td>
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<table>
<thead>
<tr>
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<tbody>
<tr>
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<tr>
<td>B</td>
<td>13</td>
<td>4</td>
</tr>
<tr>
<td>C</td>
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**Entering CE, exiting CF**

#### Shadow Costs

<table>
<thead>
<tr>
<th>Shadow costs</th>
<th>15</th>
<th>22</th>
<th>14</th>
</tr>
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</tr>
<tr>
<td>0 A</td>
<td>X</td>
<td>-3</td>
<td>-5</td>
</tr>
<tr>
<td>-4 B</td>
<td>X</td>
<td>X</td>
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</tr>
<tr>
<td>-10 C</td>
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<tbody>
<tr>
<td>A</td>
<td>25</td>
<td></td>
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<tr>
<td>B</td>
<td>13</td>
<td>4</td>
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<tr>
<td>C</td>
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**Entering AF, exiting AD**

#### Shadow Costs

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<tr>
<td>0 A</td>
<td>5</td>
<td>2</td>
<td>X</td>
</tr>
<tr>
<td>1 B</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>-5 C</td>
<td>6</td>
<td>X</td>
<td>14</td>
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<tr>
<td>B</td>
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<td>4</td>
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<tr>
<td>C</td>
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**No negative IIs so optimal solution of £1085**

(12 marks)
**Question 2 notes:**

<table>
<thead>
<tr>
<th>B1:</th>
<th>CAO</th>
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<tbody>
<tr>
<td>M1:</td>
<td>Finding all 6 shadow costs and the 4 improvement indices for the correct 4 entries – candidates must clearly identify these two sets of results</td>
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<td>CAO – including the deduction of all entering and exiting cells</td>
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<tr>
<td>A1:</td>
<td>Shadow costs and II CAO</td>
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<tr>
<td>A1:</td>
<td>CSO including the correct reasoning that the solution is optimal because there are no negative IIs</td>
</tr>
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</table>
Question 3 notes:

(a) B1: valid statement regarding converting a max. problem to a min. problem

(b) B1: CAO

B1: Correct statements regarding row and column reduction

M1: Simplifying the initial matrix by reducing rows and then columns

A1: CAO

B1: Correct statements regarding both max. number of lines to cover zeros and augmentation

M1: Develop an improved solution – need to see one double covered +e; one uncovered –e; and one single covered unchanged. 3 lines needed to 4 lines needed (so getting to the optimal table)

A1: CSO on final table (so must have scored all previous marks in this part ) + deduction of the correct allocation

(c) B1: possible values of \(ij\) defined

B1: define the set of values for \(i\) and \(j\)

B1: Correct objective function and either 'minimise' or 'maximise' (dependent on if problem is defined in terms of original values or modified values)

M1: at least four equations, unit coefficient and equal to 1

A1: CAO (all eight equations)

Question | Scheme | Marks | AOs |
--- | --- | --- | --- |
3(a) | Subtract each entry from a constant (eg 40) | B1 | 2.4 |

(b) | \[
\begin{pmatrix}
P & Q & R & S \\
A & 5 & 5 & 4 & 2 \\
B & 9 & 2 & 6 & 0 \\
C & 2 & 8 & 4 & 1 \\
D & 1 & 7 & 1 & 4 \\
\end{pmatrix}
\]
e.g.: | B1 | 1.1b |

Reducing row A by 2, no reduction for row B, reduce row C by 1 and row D by 1. No reduction of columns P, R and S, reduce column Q by 2.

| P & Q & R & S \\
| A & 3 & 3 & 2 & 0 \\
| B & 9 & 2 & 6 & 0 \\
| C & 1 & 7 & 3 & 0 \\
| D & 0 & 6 & 0 & 3 |
| then | P & Q & R & S \\
| A & 3 & 1 & 2 & 0 \\
| B & 9 & 0 & 6 & 0 \\
| C & 1 & 5 & 3 & 0 \\
| D & 0 & 4 & 0 & 3 |

M1 | 2.1 |

A1 | 1.1b |

Three lines required to cover the zeros hence solution is not optimal – augment by 1

| P & Q & R & S \\
| A & 2 & 1 & 1 & 0 \\
| B & 8 & 0 & 5 & 0 \\
| C & 0 & 5 & 2 & 0 \\
| D & 0 & 5 & 0 & 4 |

A1 | 2.2a |

(c) \(x_{ij} = \begin{cases} 1 & \text{if worker } i \text{ does task } j \\ 0 & \text{otherwise} \end{cases}\)

Where \(i \in \{A,B,C,D\}\) and \(j \in \{P,Q,R,S\}\)

| 1 if worker \(i\) does task \(j\) | B1 | 3.3 |
| 0 otherwise | |

| e.g. Minimise | B1 | 3.3 |
| \(5x_{AP} + 5x_{AQ} + 4x_{AR} + 2x_{AS} + 9x_{BP} + 2x_{BQ} + 6x_{BR} + 2x_{CP} + 8x_{CQ} + 4x_{CR} + x_{CS} + x_{DP} + 7x_{DQ} + x_{DR} + 4x_{DS}\) | |

Subject to:

\[\sum x_{ip} = 1, \sum x_{iq} = 1, \sum x_{ir} = 1, \sum x_{is} = 1\]

\[\sum x_{ij} = 1, \sum x_{ij} = 1, \sum x_{ij} = 1, \sum x_{ij} = 1\]

A1 | 3.3 |

(5) (13 marks)
**Question 3 notes:**

<p>| | |</p>
<table>
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<tbody>
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<tr>
<td><strong>(c)</strong></td>
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<tr>
<td><strong>B1:</strong></td>
<td>possible values of $x_{ij}$ defined</td>
</tr>
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<td><strong>B1:</strong></td>
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<td>Scheme</td>
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<tr>
<td>4(a)</td>
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<td><img src="image" alt="Tree Diagram" /></td>
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</tbody>
</table>

**Notes:**

(a)  
M1: Tree diagram with at least three end pay-offs, two decision nodes and two chance nodes  
A1: Correct structure of tree diagram with each arc labelled correctly (including probabilities)  
M1: At least three end pay-offs consistent with their stated probabilities; all five attempted  
A1: CAO for end pay-offs  
M1: End chance node follow through their end pay-offs and other chance/decision nodes completed  
A1: CAO for decision and chance nodes including double lines through inferior options

(b)  
B1: Correct EMV  
B1: Correct analysis
Question Scheme Marks AOs
5(a) Column 2 dominates column 4 B1 2.5
Because $2 > -2, 0 > -1$ and $3 > 2$ B1 2.4

(b) Row minima: -2, -1, -1 max is -1 M1 1.1b
Column maxima: 4, 2, 3 min is 2 A1 1.1b
Play safe is A plays 2 or 3 and B plays 2 A1 1.1b
Row maximin (-1) ≠ Column minimax (2) so not stable A1 2.4

(c) e.g. 
\[
\begin{pmatrix}
4 & -2 & 3 \\
3 & -1 & 2 \\
-1 & 2 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
6 & 0 & 5 \\
5 & 1 & 4 \\
1 & 4 & 2
\end{pmatrix}
\]
Subject to $V - 6p_1 - 5p_2 - p_3 \leq 0$ B1 1.1b
$V - p_2 - 4p_3 \leq 0$ B1 3.3
$V - 5p_1 - 4p_2 - 2p_3 \leq 0$ B1 3.3

(d) Substitute $p$ values to obtain $V = \frac{19}{7}, \frac{19}{7}, \frac{20}{7} \ : V = \frac{19}{7}$ M1 3.4
Value of the game to player A $= \frac{19}{7} - 2 = \frac{5}{7}$ M1 3.4

Notes:
(a) B1: Correct statement – must include the word ‘dominate’
B1: Correct inequalities – must be clear that all three inequalities must hold
(b) M1: Attempt at row minima and column maxima – condone one error
A1: Correct max(row min) and min(col max)
A1: Correct play safe for both players
A1: Correct reasoning that the game is not stable (accept $-1 \neq 2 +$ statement)
(c) B1: Correct augmentation to make all entries non-negative
B1: At least one (of the three) equations or inequalities correct in $V, p_1, p_2, p_3$ (with all $p_i$ terms in the constraint equations having correct signs)
B1: CAO - all three constraints correct involving $V$ and $p_i$ expressed as inequalities
(d) M1: Substitute $p$ values to obtain three values for $V$
M1: Their least value of $V$ minus their augmented value
A1: CAO for the value of the game to player A
### Question 6

<table>
<thead>
<tr>
<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>6(a)</strong></td>
<td>B1</td>
<td>1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td>B1</td>
<td>1.1b</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(2)</td>
<td></td>
</tr>
<tr>
<td><strong>(b)</strong></td>
<td>B1</td>
<td>2.4</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1)</td>
<td></td>
</tr>
<tr>
<td><strong>(c)</strong></td>
<td>B1</td>
<td>2.4</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1)</td>
<td></td>
</tr>
<tr>
<td><strong>(d)</strong></td>
<td>M1</td>
<td>3.1a</td>
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</tr>
<tr>
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<td>A1</td>
<td>1.1b</td>
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<tr>
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<td></td>
<td>(2)</td>
<td></td>
</tr>
<tr>
<td><strong>(e)</strong></td>
<td>B1</td>
<td>1.1b</td>
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<td>B1</td>
<td>2.4</td>
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</tr>
<tr>
<td></td>
<td>B1</td>
<td>2.2a</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(3)</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(f)</strong></td>
<td>B1</td>
<td>2.1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>B1</td>
<td>2.4</td>
<td></td>
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<tr>
<td></td>
<td>B1</td>
<td>2.2a</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(3)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(12 marks)
**Question 6 notes:**

(a)  
B1: correct capacity for $C_1$  
B1: correct capacity for $C_2$

(b)  
B1: correct statement regarding the min. flow out of the sink and max. flow into the sink

(c)  
B1: correct statement regarding the flow into node G

(d)  
M1: consistent flow pattern ($\geq 12$) throughout the network - so the flow into each node must equal the flow out of each node (and this flow must be greater than or equal to 12 but not necessarily the maximum flow of 15) - one number only on each arc  
A1: CAO

(e)  
B1: CAO (for max. flow)  
B1: Consideration of both the min. flow from the source and the flow through node C  
B1: Completely correct argument that the max. flow = min. flow

(f)  
B1: Correct argument regarding increasing the upper capacity of arc BF and hence the flow in that arc  
B1: Correct reasoning regarding increasing the flow in arcs FH and HT  
B1: Correct deduction that the flow in GT decreases to 5 and conclude that all other arcs are unchanged
### Question 7

All M marks – must bring optimal result from previous stage into calculations so for the second stage (April) if none of their 2200, 1850 or 2300 (the optimal results from May) are used then M0. Ignore extra rows. Condone and credit rows that have been crossed out if they can still be read. Must have right 'ingredients' (storage costs, additional space costs, overhead cost) at least once per stage. Must have values in two of the three columns (State, Action, Dest). If no working seen then the number stated in the Value column must be correct to imply the correct method has been used.

<table>
<thead>
<tr>
<th>Month</th>
<th>January</th>
<th>February</th>
<th>March</th>
<th>April</th>
<th>May</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number made</td>
<td>3</td>
<td>3</td>
<td>5</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

Minimum production cost: £9450

(12 marks)
**Question 7 notes:**

All M marks – must bring optimal result from previous stage into calculations so for the second stage (April) if none of their 2200, 1850 or 2300 (the optimal results from May) are used then M0. Ignore extra rows. Condone and credit rows that have been crossed out if they can still be read. Must have right ‘ingredients’ (storage costs, additional space costs, overhead cost) at least once per stage. Must have values in two of the three columns (State, Action, Dest). If no working seen then the number stated in the Value column must be correct to imply the correct method has been used.

<table>
<thead>
<tr>
<th>Month</th>
<th>Action</th>
<th>Dest</th>
<th>State</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>May</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>700</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>700</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>700</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>350</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>350</td>
</tr>
</tbody>
</table>

**1M1:** First stage (May) completed. At least 3 rows, ‘something’ in each cell (but see M mark guidance above) including the correct structure (e.g. no value greater than 5 in the action column) in each of the first four columns

**1A1:** CAO for first stage.

**2M1:** Second stage (April) completed. At least 9 rows, something in each cell (see M mark guidance above) including the correct structure for the fifth (Value) column (e.g. bringing forward values from the previous stage)

**2A1:** CAO for second stage. No extra rows

**3M1:** Third stage (March) completed. At least 3 rows, something in each cell (see M mark guidance above)

**3A1ft:** CAO on the ft for third stage. No extra rows

**4M1:** Fourth stage (February) completed. At least 6 rows, something in each cell (see M mark guidance above)

**4A1ft:** CAO on the ft for fourth stage. No extra rows

**5M1:** Fifth stage (January) completed. At least 3 rows, something in each cell (see M mark guidance above)

**5A1:** CAO for the fifth stage. No extra rows

**1B1:** CAO – **but must have scored all previous M marks**

**2B1:** CAO – condone lack of units - **but must have scored all previous M marks**