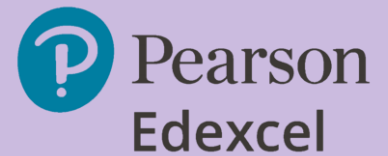


**Pearson Edexcel
Level 3 Advanced Subsidiary
GCE in Further Mathematics (8FM0)**



**Pearson Edexcel
Level 3 Advanced
GCE in Further Mathematics (9FM0)**



June 2019 – Further Pure Exemplar
Student answers with examiner comments

First teaching from September 2017

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About this booklet

This booklet has been produced to support mathematics teachers delivering the new Pearson Edexcel Level 3 Advanced Subsidiary and Advance Level GCE in Further Mathematics specification (8FM0 & 9FM0). The booklet looks at questions from the AS and A Level Further Mathematics – Further Pure June 2019 Examination Papers. It shows student responses to questions, and how the examining team follow the mark schemes to demonstrate how the students would be awarded marks on these questions.

How to use this booklet

Our examining team have selected student responses to all questions from the June 2018 Examination Papers. Following each question, you will find the mark scheme for that question and then a range of student responses with accompanying examiner comments on how the mark scheme has been applied and the marks awarded, and on common errors for this sort of question.

Student Response A

Student response

$$\frac{h}{3}(y_0 + 2y_{\text{even}} + 2y_{\text{odd}} + y_n)$$
$$\frac{0.4}{3}(0.4 + 4(0.8 + 1.6) + 2(1.2) + 2)$$
$$= \frac{12}{15} \left(\frac{72}{5} \right) = \frac{48}{25} = 1.92$$
$$h = \frac{2 - 0.4}{4} = \frac{2}{5}$$

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Examiner Comments

This candidate states the correct step length but there is no attempt to find any of the y values and the subsequent attempt at Simpson's rule uses x values. In this case only the first B mark was scored for the correct step length.

Examiner commentary on the student response

Marks awarded for the question or question parts

AS Further Mathematics – Further Pure 1 (8FM0 21)

Exemplar Question 1

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1. (a) Write down the t -formula for $\sin x$. (1)

(b) Using the answer to part (a),

- (i) find the exact value of $\sin x$ when

$$\tan\left(\frac{x}{2}\right) = \sqrt{2}$$

- (ii) show that

$$\cos x = \frac{1 - t^2}{1 + t^2} \quad (4)$$

- (c) Use the t -formulae to solve for $0 < \theta \leq 360^\circ$

$$7\sin \theta + 9\cos \theta + 3 = 0$$

giving your answers to one decimal place.

(4)

(Total for Question 1 is 9 marks)

Mean Score 6.8 out of 9

Examiner Comments

Q1 was accessible with some candidates struggling to gain access to Q1(b)(ii).

In Q1(a), nearly all candidates recalled a correct $\sin x = \frac{2t}{1+t^2}$.

In Q1(b)(i), most candidates used $\tan\left(\frac{x}{2}\right) = \sqrt{2}$ to deduce and substitute $t = \sqrt{2}$ into their answer from Q1(a). Many achieved $\sin x$ as an exact $\frac{2}{3}\sqrt{2}$.

In Q1(b)(ii), most candidates either applied $\sin x = \frac{2t}{1+t^2}$ to $\sin^2 x + \cos^2 x \equiv 1$ or applied Pythagoras' Theorem to a right-angled triangle with angle x , opposite edge $2t$ and hypotenuse $1+t^2$. Some of these attempts contained incorrect algebra or were incomplete because of errors in forming and factorising a quartic expression or an error in using a numerator of $t^2 - 1$. A few candidates applied $\tan x = \frac{\sin x}{\cos x}$ with $\tan x = \frac{2t}{1-t^2}$, $\sin x = \frac{2t}{1+t^2}$ and were generally more successful in showing that $\cos x = \frac{1-t^2}{1+t^2}$. Those candidates who applied $\cos x \equiv \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)$ to a right-angled triangle with angle $\frac{x}{2}$, edges 1 , t and $\sqrt{1+t^2}$, did not receive any credit as they had not used their answer to part (a). Some candidates received no credit because they attempted to verify that $\cos x = \frac{1-t^2}{1+t^2}$ was true, usually with $t = \sqrt{2}$.

In Q1(c), most candidates provided a fully correct solution to achieve both angles $x = 143.1^\circ, 292.6^\circ$. Many substituted $\cos \theta = \frac{1-t^2}{1+t^2}$ and their $\sin \theta = \frac{2t}{1+t^2}$ into the given $7 \sin \theta + 9 \cos \theta + 3 = 0$. Nearly all candidates formed and attempted to solve a quadratic equation in t . At this stage some candidates manipulated a correct $\tan\left(\frac{\theta}{2}\right) = -\frac{2}{3}$, 3 to give an incorrect $\theta = \arctan(-2)$ or $\theta = \arctan(6)$. Other candidates who achieved a correct $\frac{\theta}{2} = \{71.5650\dots, 146.3099\dots\}$ then halved (rather than doubled) their results to give an incorrect $\theta = \{35.8^\circ, 73.1^\circ, \dots\}$. Other common errors included failing to round their final answers to one decimal place; premature rounding earlier on in their working which led a loss of accuracy in their value(s) for θ ; additional solutions found in the range $0 < \theta \leq 360^\circ$; and leaving their final answer(s) in radians. Some candidates used the alternative method of substituting their values of t into the t -formulae for $\sin \theta$ or $\cos \theta$. Only a few of these candidates achieved both correct values for θ .

Mark Scheme

Question	Scheme	Marks	AOs
1 (a)	$\{\sin x = \frac{2t}{1+t^2}\}$	B1	1.2
		(1)	
(b)(i)	$\left\{ \tan\left(\frac{x}{2}\right) = \sqrt{2} \Rightarrow t = \sqrt{2} \Rightarrow \right\} \sin x = \frac{2(\sqrt{2})}{1+(\sqrt{2})^2}$ or $\frac{2(\sqrt{2})}{1+2}$	M1	1.1b
	$\sin x = \frac{2}{3}\sqrt{2}$ or $\frac{1}{3}\sqrt{8}$ or $\sqrt{\frac{8}{9}}$	A1	1.1b
		(2)	
(ii) Way 1	$\left\{ \cos x \equiv \frac{\sin x}{\tan x} \Rightarrow \right\} \cos x = \frac{\frac{2t}{1+t^2}}{\frac{2t}{1-t^2}}; = \frac{1-t^2}{1+t^2}$ * cso	M1;	1.1b
		A1*	2.1
		(2)	
(ii) Way 2	$\left\{ \tan x \equiv \frac{\sin x}{\cos x} \Rightarrow \right\} \frac{2t}{1-t^2} = \frac{\frac{2t}{1+t^2}}{\cos x}; \Rightarrow \cos x = \frac{1-t^2}{1+t^2}$ * cso	M1;	1.1b
		A1*	2.1
		(2)	
(ii) Way 3	$\{\sin^2 x + \cos^2 x \equiv 1 \Rightarrow \} \left(\frac{2t}{1+t^2}\right)^2 + \cos^2 x = 1$	M1	1.1b
	$\cos^2 x = 1 - \left(\frac{2t}{1+t^2}\right)^2 = \frac{(1+t^2)^2 - 4t^2}{(1+t^2)^2} = \frac{1-2t^2+t^4}{(1+t^2)^2} = \frac{(1-t^2)^2}{(1+t^2)^2}$ $\Rightarrow \cos x = \frac{1-t^2}{1+t^2}$ * cso	A1	2.1
		(2)	
(ii) Way 4	$\{o^2 + a^2 = h^2 \Rightarrow \} (2t)^2 + a^2 = (1+t^2)^2$	M1	1.1b
	$a^2 = (1+t^2)^2 - (2t)^2 = 1-2t^2+t^4 = (1-t^2)^2$ $a = 1-t^2 \Rightarrow \cos x = \frac{1-t^2}{1+t^2}$ * cso	A1	2.1
		(2)	
(c)	$\{7\sin\theta + 9\cos\theta + 3 = \} 7\left(\frac{2t}{1+t^2}\right) + 9\left(\frac{1-t^2}{1+t^2}\right) + 3$	M1	1.1b
	$7\left(\frac{2t}{1+t^2}\right) + 9\left(\frac{1-t^2}{1+t^2}\right) + 3 = 0 \Rightarrow 14t + 9 - 9t^2 + 3 + 3t^2 = 0$ $\Rightarrow 6t^2 - 14t - 12 = 0 \Rightarrow 3t^2 - 7t - 6 = 0 \Rightarrow (t-3)(3t+2) = 0 \Rightarrow t = \dots$	M1	1.1b
	Either $\left\{ t = 3 \Rightarrow \frac{\theta}{2} = \arctan(3) \Rightarrow \right\} \theta = 2\arctan(3)$ or $\left\{ t = -\frac{2}{3} \Rightarrow \frac{\theta}{2} = 180^\circ + \arctan\left(-\frac{2}{3}\right) \Rightarrow \right\} \theta = 2\left(180^\circ + \arctan\left(-\frac{2}{3}\right)\right)$	M1	1.1b
	$\frac{\theta}{2} = \{71.5650\dots, 146.3099\dots\} \Rightarrow \theta = \{143.1301\dots, 292.6198\dots\}$		
	$\theta = 143.1^\circ, 292.6^\circ$ (1dp)	A1	1.1b
		(4)	

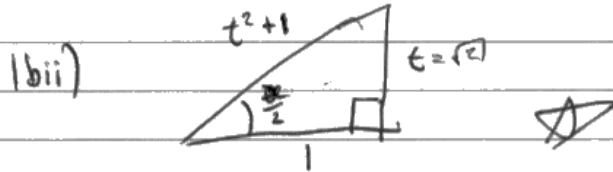
(9 marks)

Notes for Question 1	
(a)	
B1:	See scheme
(b)(i)	
M1:	Complete substitution of $t = \sqrt{2}$ into their expression from part (a)
A1:	Correct exact answer. See scheme.
Note:	Give M0 A0 for writing down the correct exact answer without any evidence of substituting $t = \sqrt{2}$ into $\sin x = \frac{2t}{1+t^2}$
Note:	For reference, $\sin x = \frac{2}{3}\sqrt{2} = 0.9428\dots$
(b)(ii)	Way 1, Way 2 and Way 3
M1:	Uses a correct trigonometric identity (or correct trigonometric identities) to find a correct expression which connects only $\cos x$ (or $\cos^2 x$) and t
A1*:	Correct proof
(b)(ii)	Way 4
M1:	Uses $\sin x = \frac{o}{h}$ and a correct Pythagoras method to express the adjacent edge of a triangle in terms of t .
A1*:	Correct proof
(c)	
M1:	Uses at least one of $\sin \theta = \frac{2t}{1+t^2}$ or $\cos \theta = \frac{1-t^2}{1+t^2}$ to express $7\sin \theta + 9\cos \theta + 3$ in terms of t only
M1:	Uses both correct formula $\sin \theta = \frac{2t}{1+t^2}$ and $\cos \theta = \frac{1-t^2}{1+t^2}$ in $7\sin \theta + 9\cos \theta + 3 = 0$, multiplies both sides by $1+t^2$, forms a 3TQ and uses a correct method (e.g. using the quadratic formula, completing the square or a calculator approach) for solving their 3TQ to give $t = \dots$
M1:	Uses both correct formula $\sin \theta = \frac{2t}{1+t^2}$ and $\cos \theta = \frac{1-t^2}{1+t^2}$ in $7\sin \theta + 9\cos \theta + 3 = 0$, adopts a correct <i>applied</i> strategy to find at least one value of θ within the range $0 < \theta \leq 360^\circ$ (or in radians $0 < \theta \leq 2\pi$) such that either <ul style="list-style-type: none"> • $\theta = 2\arctan(\text{their found } t)$, where their found $t > 0$ • $\theta = 2(180^\circ + \arctan(\text{their found } t))$, where their found $t < 0$ • $\theta = 2(180^\circ - \arctan \text{their found } t)$, where their found $t < 0$
A1:	Correct answer only of $\theta = 143.1^\circ, 292.6^\circ$
Note:	Give A0 for extra solutions given within the range $0 < \theta \leq 360^\circ$
Note:	Ignore extra solutions outside the range $0 < \theta \leq 360^\circ$ for the A mark
Note:	Give 3 rd M0 for $\frac{\theta}{2} = \{71.565\dots, 146.309\dots\}$ without attempting to find θ
Note:	Give 3 rd M0 for $\frac{\theta}{2} = \{71.565\dots, 146.309\dots\} \Rightarrow \theta = \{35.782\dots, 73.154\dots\}$
Note:	In degrees, $\frac{\theta}{2} = \{71.565\dots, 251.565\dots, -33.690\dots, 146.309\dots\}$
Note:	Working in radians gives $\frac{\theta}{2} = \{1.249\dots, 2.553\dots\} \Rightarrow \theta = \{2.498\dots, 5.107\dots\}$

Student Response A

$$(a) \sin x = \frac{2t}{1-t^2}$$

$$(b) \sin x = \frac{2(\sqrt{2})}{1-(\sqrt{2})^2} = \underline{\underline{-2\sqrt{2}}}$$



$$\cos x = \frac{1-t^2}{1+t^2}$$

$$\cos^2 x + \sin^2 x = 1$$

$$\frac{\sin x}{\cos x} = \tan x$$

$$(c) 7\left(\frac{2t}{1-t^2}\right) + 9\left(\frac{1-t^2}{1+t^2}\right) + 3 = 0$$

$$\frac{14t}{1-t^2} + \frac{9-9t^2}{1+t^2} + 3 = 0$$

$$14t(1+t^2) + (9-9t^2)(1-t^2) + 3(1-t^2)(1+t^2) = 0$$

$$14t + 14t^3 + (9-9t^2-9t^2+9t^4) + 3(1+t^2-t^2-t^4) = 0$$

$$8t^4 + 14t^3 - 18t^2 + 14t + 12 = 0$$

$$7\left(\frac{2t}{1-t^2}\right) + 9\left(\frac{1-t^2}{1+t^2}\right) + 3 = 0$$

$$\frac{14t}{1-t^2} + \frac{9-9t^2}{1+t^2} + 3 = 0$$

$$14t(t^2+1) + (9-9t^2)(1-t^2) + 3(t^2+1)(1-t^2) = 0$$

$$14t(t^2+1) + 9(1-t^2)^2 + 3(t^2+1)(1-t^2) = 0$$

$$(14t^3 + 14t) + 9(t^4 - 2t^2 + 1) + 3(-t^4 + 1) = 0$$

$$8t^4 + 14t^3 - 18t^2 + 14t + 12 = 0$$

$$3t^4 + 7t^3 - 9t^2 + 7t + 6 = 0$$

Examiner Comments**(a) B0****B0:** $\sin x = \frac{2t}{1-t^2}$ is incorrect.**(b) (i) M1 A0****M1:** Complete substitution of $t = \sqrt{2}$ into their part (a) expression for $\sin x$.**A0:** Incorrect answer.**(b)(ii) M0 A0****M0:** Does not use their correct trigonometry identity (or correct trigonometric identities) to find a correct equation which connects only $\cos x$ (or $\cos^2 x$) and t .**A0:** Follows M0.**(c) M1 M0 M0 A0****M1:** Uses at least one of $\sin \theta = \frac{2t}{1+t^2}$ or $\cos \theta = \frac{1-t^2}{1+t^2}$ to express $7 \sin \theta + 9 \cos \theta + 3 = 0$ in terms of t only.**M0 M0:** Does not apply both correct formula $\sin \theta = \frac{2t}{1+t^2}$ and $\cos \theta = \frac{1-t^2}{1+t^2}$ to $7 \sin \theta + 9 \cos \theta + 3 = 0$.**A0:** Follows M0 M0.

Student Response B

$$a) \sin x = \frac{2t}{1+t^2}$$

$$b) \frac{2t}{1+t^2} = \frac{1}{\sqrt{2}} \tan\left(\frac{x}{2}\right) = \sqrt{2}$$

$$\frac{x}{2} = \tan^{-1}(\sqrt{2})$$

$$\frac{x}{2} = 0.9553$$

$$x = 1.9106$$

S	A
T	C

$$\sin(1.9106) = \frac{2t}{1+t^2}$$

$$\therefore \sin(1.9106) = 0.9428$$

$$c) \sin^2 x + \cos^2 x = 1$$

$$\left(\frac{2t}{1+t^2}\right)^2 + \cos^2 x = 1$$

$$\cos^2 x = 1 - \left(\frac{2t}{1+t^2}\right)^2$$

$$\cos^2 x = 1 - \frac{4t^2}{(1+t^2)^2} = \frac{1+t^2 - 4t^2}{(1+t^2)^2}$$

$$\cos^2 x = \frac{1-t^2}{(1+t^2)^2}$$

$$= \frac{(1+t^2)^2 - 4t^2}{(1+t^2)^2}$$

$$= \frac{t^4 + 2t^2 + 1 - 4t^2}{t^4 + 2t^2 + 1}$$

$$= \frac{t^4 - 2t^2 + 1}{t^4 + 2t^2 + 1}$$

$$= \frac{(t-1)(t+1)}{(t-1)(t+1)}$$

$$= \frac{1-t^2}{1+t^2}$$

$$\begin{aligned} & (1+t^2)(1+t^2) \\ &= 1+t^2+t^2+t^4 \\ &= t^4+2t^2+1 \end{aligned}$$

$$c) 7 \sin \theta + 9 \cos \theta + 3 = 0$$

$$= 7 \left(\frac{2t}{1+t^2} \right) + 9 \left(\frac{1-t^2}{1+t^2} \right) + 3 = 0$$

$$= \frac{7(2t)}{1+t^2} + \frac{9(1-t^2)}{1+t^2} + \frac{3}{1} = 0$$

$$= \frac{7(2t) + 9(1-t^2) + 3(1+t^2)}{1+t^2} = 0$$

$$= 7(2t) + 9(1-t^2) + 3(1+t^2) = 0$$

$$14t + 9 - 9t^2 + 3 + 3t^2 = 0$$

$$-6t^2 + 14t + 12 = 0$$

$$6t^2 - 14t - 12 = 0 \quad \downarrow \div 2$$

$$3t^2 - 7t - 6 = 0 \quad \begin{matrix} p = -19 \\ q = -7 \end{matrix}$$

$$3t^2 + 2t - 9t - 6 = 0$$

$$t(3t+2) - 3(3t+2) = 0$$

$$(t-3)(3t+2) = 0$$

$$t = 3, \quad t = -\frac{2}{3}$$

$$x = 143.1^\circ, -67.4^\circ$$

$$\tan\left(\frac{x}{2}\right) = 3$$

$$\frac{x}{2} = \tan^{-1}(3)$$

$$x = 143.1^\circ$$

$$\tan\left(\frac{x}{2}\right) = -\frac{2}{3}$$

$$\frac{x}{2} = \tan^{-1}\left(-\frac{2}{3}\right)$$

$$x = -67.4^\circ$$

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Examiner Comments

(a) B1

B1: $\frac{2t}{1+t^2}$.

(b) (i) M0 A0

M0: Does not substitute $t = \sqrt{2}$ into $\frac{2t}{1+t^2}$.

A0: Follows M0.

(b)(ii) M1 A0 (Way 3)

M1: $\left(\frac{2t}{1+t^2}\right)^2 + \cos^2 x = 1$

A0: Candidate's working becomes incorrect at the stage $\cos^2 x = \frac{(t-1)(t+1)}{(t-1)(t+1)}$.

(c) M1 M1 M1 A0

M1 M1: Correct method leading to $t = 3, t = -\frac{2}{3}$.

M1: Correct applied method to find at least one value of θ within the range $0 < \theta \leq 360^\circ$.

Note: This candidate correctly applies $\theta = 2 \arctan(\text{their found } t)$, where their found $t > 0$.

Note: Condone $x = \dots$ rather than $\theta = \dots$

A0: Obtains 143.1° but does not obtain 292.6° .

Student Response C

$$1) a) \sin x = \frac{2t}{1+t^2}$$

$$b) (i) t = \sqrt{2}$$

$$\sin x = \frac{2\sqrt{2}}{1+(\sqrt{2})^2} = \frac{2\sqrt{2}}{3}$$

(ii)

$$\tan x = \frac{2t}{1-t^2}$$

$$\frac{\sin x}{\cos x} = \tan x$$

$$\cos x = \frac{\sin x}{\tan x} = \frac{\left(\frac{2t}{1+t^2}\right)}{\left(\frac{2t}{1-t^2}\right)} = \frac{2t}{1+t^2} \cdot \frac{1-t^2}{2t} = \frac{1-t^2}{1+t^2}$$

$$c) 7\left(\frac{2t}{1+t^2}\right) + 9\left(\frac{1-t^2}{1+t^2}\right) - 3 = 0$$

$$14t + 9 - 9t^2 + 3 + 3t^2 = 0$$

$$6t^2 - 14t - 9 = 0$$

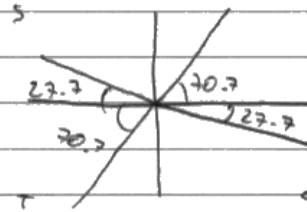
$$t = \frac{7 + \sqrt{103}}{6}$$

$$t = \frac{7 - \sqrt{103}}{6}$$

$$\tan\left(\frac{\theta}{2}\right) = t$$

$$0 < \theta < 360$$

$$\text{Let } y = \frac{\theta}{2} \quad 0 < \theta \leq 180$$



$$\tan(y) = \frac{7 + \sqrt{103}}{6}$$

$$y = 70.7$$

$$\theta = 2(70.7)$$

$$\tan(y) = \frac{7 - \sqrt{103}}{6}$$

$$= 141.4^\circ \text{ (1dp)}$$

$$y = -27.69 \quad \theta = 2(180 - 27.7)$$

$$= 304.6^\circ \text{ (1dp)}$$

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Examiner Comments
(a) B1

B1: $\frac{2t}{1+t^2}$

(b) (i) M1 A1
M1: Complete substitution of $t = \sqrt{2}$ into $\frac{2t}{1+t^2}$ to give $\frac{2\sqrt{2}}{1+(\sqrt{2})^2}$.

A1: Obtains the correct exact value $\frac{2\sqrt{2}}{3}$.

(b)(ii) M1 A1 (Way 1)
M1 A1: Uses $\cos x \equiv \frac{\sin x}{\tan x}$ with $\sin x = \frac{2t}{1+t^2}$ and $\tan x = \frac{2t}{1-t^2}$ to correctly show that

$$\cos x = \frac{1-t^2}{1+t^2}$$

(c) M1 M1 M1 A0
M1 M1: Applies both correct formula $\sin \theta = \frac{2t}{1+t^2}$ and $\cos \theta = \frac{1-t^2}{1+t^2}$ to

 $7 \sin \theta + 9 \cos \theta + 3 = 0$, multiplies both sides by $1+t^2$, forms a 3TQ (3 term-quadratic) and uses a correct quadratic formula method for solving their 3TQ equal to 0 to give $t = \dots$. Note:

 $t = \frac{7 \pm \sqrt{103}}{6}$ are the correct solutions of their quadratic equation $6t^2 - 14t - 9 = 0$.

M1: Correct applied method to find at least one value of θ within the range $0 < \theta \leq 360^\circ$. Note: This candidate correctly applies both $\theta = 2 \arctan(\text{their found } t)$, where their found $t > 0$ and $\theta = 2(180^\circ + \arctan(\text{their found } t))$, where their found $t < 0$
A0: Incorrect answer.

Exemplar Question 2

2. A student was set the following problem.

Use algebra to find the set of values of x for which

$$\frac{x}{x-24} > \frac{1}{x+11}$$

The student's attempt at a solution is written below.

$$x(x-24)(x+11)^2 > (x+11)(x-24)^2$$

$$x(x-24)(x+11)^2 - (x+11)(x-24)^2 > 0$$

$$(x-24)(x+11)[x(x+11) - x - 24] > 0$$

Line 3

$$(x-24)(x+11)[x^2 + 10x - 24] > 0$$

$$(x-24)(x+11)(x+12)(x-2) > 0$$

$$x = 24, x = -11, x = -12, x = 2$$

$$\{x \in \mathbb{R} : -12 < x < -11\} \cup \{x \in \mathbb{R} : 2 < x < 24\}$$

Line 7

There are errors in the student's solution.

- (a) Identify the error made

- (i) in line 3
(ii) in line 7

(2)

- (b) Find a correct solution to this problem.

(4)

(Total for Question 2 is 6 marks)

Mean Score 4.6 out of 6

Examiner Comments

Q2 was well-answered, although some candidates struggled to give clear explanations in Q2(a).

In Q2(a)(i), most candidates were aware of the sign error in line 3 and many explained that $x(x+11) - x - 24$ should have been written as $x(x+11) - (x-24)$ or $x(x+11) - x + 24$.

In Q2(a)(ii), most candidates realised that the student in the question had found the regions where the inequality is < 0 , but this was inadequately explained in some candidates' work, e.g. 'solved for $x < 0$ '. Some candidates said that the set notation was wrong, e.g. 'it should be \cap and not \cup '.

Many good solutions were seen in Q2(b) and only a few weak attempts were seen, such as invalid cross-multiplication or expansion to a quartic. Most candidates achieved the correct critical values and then went on to choose the appropriate regions. A small number failed to use set notation. A few candidates made the same sign error as the student in Q2(a) or thought that all they had to do in Q2(b) was to correct the student's line 7 error

Mark Scheme

Question	Scheme	Marks	AOs
2 (a)(i)	Line 3: Allow any of either <ul style="list-style-type: none"> • bracketing error • -24 should be 24 in the square brackets • $x(x+11) - x - 24$ should be $x(x+11) - (x - 24)$ • $x(x+11) - x - 24$ should be $x(x+11) - x + 24$ 	B1	2.3
(a)(ii)	Line 7: Allow any of either <ul style="list-style-type: none"> • should be $\{x \in \mathbb{R}: x < -12 \text{ or } -11 < x < 20 \text{ or } x > 24\}$ • they have found the regions where the inequality is < 0 • they have reversed the inequality 	B1	2.3
		(2)	
(b) Way 1	$(x - 24)(x + 11)[x(x + 11) - (x - 24)] > 0$	M1	1.1b
	$(x - 24)(x + 11)[x^2 + 10x + 24] > 0$		
	$(x - 24)(x + 11)(x + 6)(x + 4) > 0$	A1	1.1b
	Critical values $x = -11, -6, -4, 24$		
	$\{x \in \mathbb{R}: x < -11\} \cup \{x \in \mathbb{R}: -6 < x < -4\} \cup \{x \in \mathbb{R}: x > 24\}$	M1	2.2a
		A1	2.5
		(4)	
(b) Way 2	$\frac{x}{x - 24} > \frac{1}{x + 11} \Rightarrow \frac{x}{x - 24} - \frac{1}{x + 11} > 0 \Rightarrow \frac{x(x + 11) - (x - 24)}{(x - 24)(x + 11)} > 0$	M1	1.1b
	$\Rightarrow \frac{x^2 + 10x + 24}{(x - 24)(x + 11)} > 0 \Rightarrow \frac{(x + 6)(x + 4)}{(x - 24)(x + 11)} > 0$		
	Critical values $x = -11, -6, -4, 24$	A1	1.1b
	$\{x \in \mathbb{R}: x < -11\} \cup \{x \in \mathbb{R}: -6 < x < -4\} \cup \{x \in \mathbb{R}: x > 24\}$	M1	2.2a
		A1	2.5
		(4)	
(b) Way 3	Considering $x < -11$ $\frac{x}{x - 24} > \frac{1}{x + 11} \Rightarrow x^2 + 11x > x - 24 \Rightarrow x^2 + 10x + 24 > 0$ gives $x < -6$ or $x > -4$. Hence $x < -11$	M1	1.1b
	Considering $-11 < x < 24$ $\frac{x}{x - 24} > \frac{1}{x + 11} \Rightarrow x^2 + 11x < x - 24 \Rightarrow x^2 + 10x + 24 < 0$ gives $-6 < x < -4$. Hence $-6 < x < -4$		
	Considering $x > 24$ $\frac{x}{x - 24} > \frac{1}{x + 11} \Rightarrow x^2 + 11x > x - 24 \Rightarrow x^2 + 10x + 24 > 0$ gives $x < -6$ or $x > -4$. Hence $x > 24$	A1	1.1b
	Overall, $\{x \in \mathbb{R}: x < -11\} \cup \{x \in \mathbb{R}: -6 < x < -4\} \cup \{x \in \mathbb{R}: x > 24\}$	M1	2.2a
		A1	2.5
		(4)	

(6 marks)

Notes for Question 2	
(a)(i)	
B1:	See scheme
Note:	Give B0 for contradictory reasons
(a)(ii)	Way 1
B1:	See scheme
Note:	Give B0 for contradictory reasons
Note:	Allow “Should be $x < -12, -11 < x < 2, x > 24$ ”
Note:	Do not allow <ul style="list-style-type: none"> • “Should be $x < -12 \cap -11 < x < 2 \cap x > 24$” • They have found where $x < 0$ and not where $x > 0$ • “There should be 3 inequalities and not 2 inequalities” • “The sign is the wrong way around”
(b)	Way 1
M1:	Uses brackets {to correct the error made on line 3}, forms a 3TQ and uses a correct method of solving a 3TQ to give $x = \dots$
A1:	All four correct critical values for x
M1:	Deduces that the 2 “outsides” and the “middle interval” are required
A1:	Exactly 3 correct intervals. Their answer must be given in set notation. Accept equivalent set notation. E.g. Allow <ul style="list-style-type: none"> • $\{x \in \mathbb{R}: x < -11 \text{ or } -6 < x < -4 \text{ or } x > 24\}$ • $\{x < -11 \text{ or } -6 < x < -4 \text{ or } x > 24\}$ • $\{x < -11 \cup -6 < x < -4 \cup x > 24\}$ • $\mathbb{R} - ([-11, -6] \cup [-4, 24])$
Note:	Give final A0 for $\{x \in \mathbb{R}: x < -11\} \cap \{x \in \mathbb{R}: -6 < x < -4\} \cap \{x \in \mathbb{R}: x > 24\}$
Note:	Allow A1 for $\{x \in \mathbb{R}: x < -11, -6 < x < -4, x > 24\}$
(b)	Way 2
M1:	Gathers terms on one side and puts over a common denominator. Simplifies the numerator to $x(x+11) - (x-24)$ {and thereby corrects the error made in line 3}, forms a 3TQ and uses a correct method of solving a 3TQ to give $x = \dots$
A1:	See Way 1
M1:	See Way 1
A1:	See Way 1
(b)	Way 3
M1:	Considers each of the intervals $x < -11, -11 < x < 24, x > 24$ separately and evaluates which parts (if any) of these regions satisfy the original inequality
A1:	Obtains a correct inequality statement for each of the intervals $x < -11, -11 < x < 24, x > 24$
M1:	See Way 1
A1:	See Way 1

Student Response A

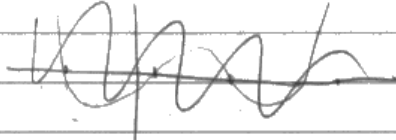
ai. There needs to be brackets around " $x - 24$ "

$$\Rightarrow (x - 24)(x + 11) [x(x + 11) - (x - 24)] > 0$$

OK write " $x + 24$ " instead

$$\Rightarrow (x - 24)(x + 11) [x(x + 11) - x + 24] > 0$$

ii ~~W~~. ~~looks like this~~



The student has found the set of values for which
 $(x - 24)(x + 11)(x + 12)(x - 2) < 0$
 instead of " $>$ "

b. $x(x - 24)(x + 11)^2 > (x + 11)(x - 24)^2$

$$x(x - 24)(x + 11)^2 - (x + 11)(x - 24)^2 > 0$$

$$(x - 24)(x + 11) [x(x + 11) - (x - 24)] > 0$$

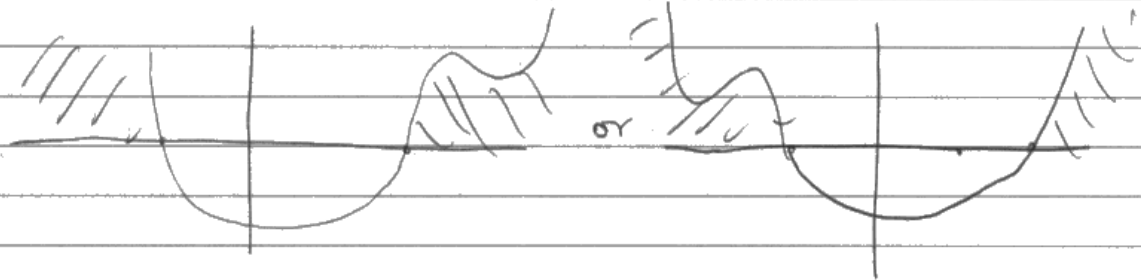
$$(x - 24)(x + 11) [x^2 + 11x - x + 24] > 0$$

$$(x - 24)(x + 11) [x^2 + 10x + 24] > 0$$

~~(x - 24)(x + 11)~~

$(x^2 + 10x + 24)$ has no real roots

\therefore ~~W~~ roots at $x = 24$, $x = -11$



$$\{x \in \mathbb{R} : x < -11\} \cup \{x \in \mathbb{R} : x > 24\}$$

2/6

Examiner Comments

(a) (i) B1

B1: Acceptable explanation.

(a)(ii) B1

B1: Acceptable explanation.

(b) M0 A0 M0 A0 (Way 1)

M0: Uses brackets to correct the error made on line 3, forms a 3TQ, but does not use a correct method of solving their $x^2 + 10x + 24 = 0$ to give $x = \dots$

A0: Follows M0.

M0: Does not obtain an inequality answer with 2 "outsides" and a "middle interval".

A0: Follows M0 M0.

Student Response B

a) i) they forgot to negate the $(x-24)$ bracket. it is supposed to be $-x+24$

ii) ~~$2 < x$ is supposed to be inclusive.~~

ii) it is meant to be intersection, not union $\rightarrow \cap$ not \cup in the set notation.

$$b) \frac{x}{x-24} > \frac{1}{x+11}$$

multiply both sides by $(x-24)^2(x+11)^2$

$$\frac{x(x-24)^2(x+11)^2}{x-24} > \frac{1(x-24)^2(x+11)^2}{x+11}$$

$$x(x-24)(x+11)^2 > (x-24)^2(x+11)$$

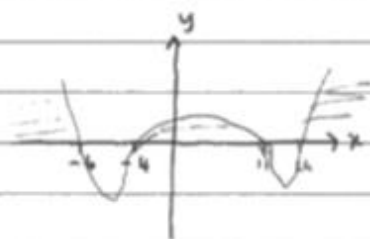
$$x(x-24)(x+11)^2 - (x-24)^2(x+11) > 0$$

$$(x-24)(x-11) [x(x+11) - (x-24)] > 0$$

$$(x-24)(x-11) [x^2+11x - x + 24] > 0$$

$$(x-24)(x-11) (x^2+10x+24) > 0$$

$$(x-24)(x-11)(x+4)(x+6) > 0$$



$$x = 24, x = 11, x = -4, x = -6$$

$$x < -6$$

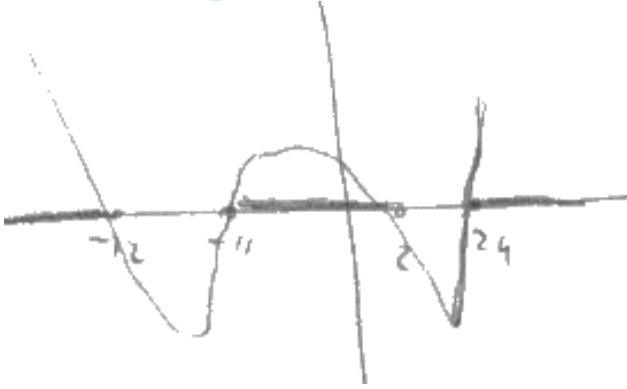
$$-4 \leq x < 11$$

$$x > 24$$

$$\{x \in \mathbb{R}; x < -6\} \cup \{x \in \mathbb{R}; -4 \leq x < 11\} \cup \{x \in \mathbb{R}; x > 24\}$$

Examiner Comments**(a) (i) B1****B1:** Acceptable explanation.**(a)(ii) B0****B0:** Incorrect explanation.**(b) M1 A0 M1 A0 (Way 1)****M1:** Uses brackets to correct the error made on line 3, forms a 3TQ and uses a correct method of solving a $3TQ = 0$ to give $x = \dots$ **A0:** Incorrect critical value of 11.**M1:** Deduces that the 2 “outsides” and the “middle interval” are required.**A0:** Incorrect answer.

Student Response C



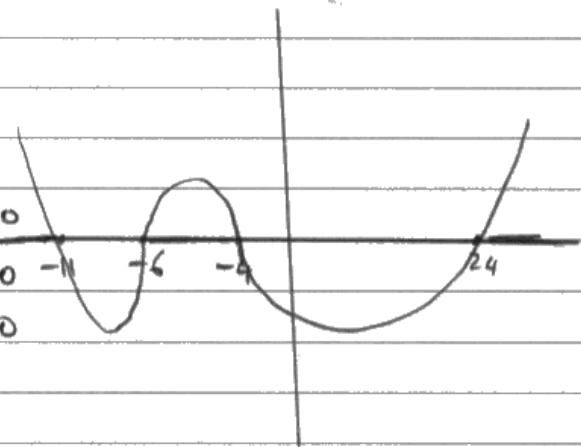
a) i) The student removed the brackets from $(x-24)$ so when she factorised ~~out~~ of the equation, the ~~fact~~ x was misused but the -24 wasn't. It should have been $-x+24$

ii) The student worked out the set of values for which

$$\frac{x}{x-24} < \frac{1}{x+11}$$

b) $\frac{x}{x-24} > \frac{1}{x+11}$

$$\begin{aligned} x(x-24)(x+11)^2 &> (x-24)^2(x+11) \\ x(x-24)(x+11)^2 - (x-24)^2(x+11) &> 0 \\ (x-24)(x+11)(x(x+11) - (x-24)) &> 0 \\ (x-24)(x+11)(x^2+11x - x + 24) &> 0 \\ (x-24)(x+11)(x^2+10x+24) &> 0 \\ (x-24)(x+11)(x+4)(x+6) &> 0 \\ x = 24 \quad x = -11 \quad x = -4 \quad x = -6 \end{aligned}$$



$$\{x \in \mathbb{R} : x < -11\} \cup \{x \in \mathbb{R} : -6 < x < -4\} \cup \{x \in \mathbb{R} : x > 24\}$$

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Examiner Comments**(a) (i) B1****B1:** Acceptable explanation.**(a)(ii) B1**

B1: Acceptable explanation. Note that candidate's explanation "The student worked out the set of values for which $\frac{x}{x-24} < \frac{1}{x+11}$ " is equivalent to "student has found regions where the inequality is < 0 ".

(b) M1 A1 M1 A1 (Way 1)**M1 A1 M1 A1:** A correct method leading to a correct answer which is expressed using set notation.

Exemplar Question 3

3. Julie decides to start a business breeding rabbits to sell as pets.

Initially she buys 20 rabbits. After t years the number of rabbits, R , is modelled by the differential equation

$$\frac{dR}{dt} = 2R + 4 \sin t \quad t > 0$$

Julie needs to have at least 40 rabbits before she can start to sell them.

Use two iterations of the approximation formula

$$\left(\frac{dy}{dx} \right)_n \approx \frac{y_{n+1} - y_n}{h}$$

to find out if, according to the model, Julie will be able to start selling rabbits after 4 months.

(7)

(Total for Question 3 is 7 marks)

Mean Score 5.4 out of 7

Examiner Comments

Q3 was well-answered with many candidates scoring full marks.

Most candidates applied two iterations of $R_{n+1} = R_n + h \left(\frac{dR}{dt} \right)_n$ for $n = 0, 1$ with $R_0 = 20$ and found an estimate for the number of rabbits at 4 months. Common errors included using an incorrect value for h such as 0.2, 2 or $\frac{1}{12}$; using $t_0 = \frac{1}{6}$, $t_1 = \frac{1}{3}$ (instead of $t_0 = 0$, $t_1 = \frac{1}{6}$) in their first and second iteration, respectively; or incorrectly finding $\left(\frac{dR}{dt} \right)_0$ as $2(20) + 4 \sin \left(\frac{1}{6} \right)$. A few candidates applied 4 iterations with $h = \frac{1}{12}$ and some candidates were unaware that they should be working in radians. Nearly all candidates gave a correct conclusion in context for their estimated value of R_2 .

Mark Scheme

Question	Scheme	Marks	AOs
3	{The population after 4 months is required over two iterations} $\Rightarrow h = \frac{1}{6}$	B1	3.3
	$\{t_0 = 0, R_0 = 20 \Rightarrow \left(\frac{dR}{dt}\right)_0 = 2(20) + 4\sin 0 \{= 40\}$	M1	3.4
	$\left\{\frac{R_1 - 20}{\left(\frac{1}{6}\right)} = "40" \Rightarrow R_1 = 20 + \left(\frac{1}{6}\right)(40)\right\}$	M1	1.1b
	$R_1 = \frac{80}{3}$ or awrt 26.7 or $20 + (\text{their } h)(40)$	A1ft	1.1b
	$\left(\frac{dR}{dt}\right)_1 = 2\left(\frac{80}{3}\right) + 4\sin\left(\frac{1}{6}\right) \{= 53.9969\dots\}$	M1	1.1b
	$R_2 = R_1 + h\left(\frac{dR}{dt}\right)_1 = \frac{80}{3} + \frac{1}{6}(53.9969\dots) = 35.666\dots = 35$ or 36 rabbits	A1	1.1b
	$R_2 = 35.666\dots \approx 35$ or $36 < 40$ Julie will not be able to start to sell her rabbits after 4 months.	B1ft	3.2a
		(7)	
(7 marks)			
Notes for Question 3			
B1:	Translates the situation given to state (or use) the correct value for the step length h		
M1:	Uses the model to find the initial value of $\frac{dR}{dt}$ using the initial condition $t_0 = 0, R_0 = 20$		
M1:	Applies the approximation formula with $R_0 = 20$, their stated h , their $\left(\frac{dR}{dt}\right)_0$ to find a numerical expression for R_1		
A1:	depends on both previous M marks At 2 months, finds the approximation for R as $\frac{80}{3}$ or awrt 26.7		
Note:	Only give the following follow through. i.e. Allow A1ft for $20 + (\text{their } h)(40)$ for their stated h		
M1:	Attempts to find a numerical expression for $\left(\frac{dR}{dt}\right)_1$ with their $\frac{80}{3}$ and $t_1 = \text{their } h$		
A1:	Applies the approximation formula for a second time to give R_2 as a truncated 35 or a value in the interval $[35.5, 36]$		
B1ft:	Attempts two iterations of their $R_{n+1} = R_n + h\left(\frac{dR}{dt}\right)_n$ to find a value for R_2 . Compares their value of R_2 with 40 (which can be implied) and draws a conclusion about whether Julie will be able to start to sell her rabbits after 4 months.		
Note:	Give final B0 for applying more than or fewer than two iterations before comparing		
Note:	Using $h = \frac{1}{12}$ yields $R_1 = 23.3333\dots, R_2 = 27.2499\dots, R_3 = 31.8469\dots, R_4 = 37.2372\dots$		
Note:	Give special case final A1 for giving R_4 as a truncated 37 or a value in the interval $[37, 37.4]$		

Note:	Therefore, using $h = \frac{1}{12}$ with four iterations can gain a maximum B0 M1 M1 A1 M1 A1 B0
Note:	Answers in the range [35.5, 36] can follow from an incorrect method. E.g. Give final M0 A0 for using: $h = \frac{1}{6}, \left(\frac{dR}{dt}\right)_1 = 2\left(\frac{80}{3}\right) + 4\sin(0.1) = 53.73266... \Rightarrow R_2 = \frac{80}{3} + \frac{1}{6}(53.73266...) = 35.622...$

Student Response A

$$R = 20$$

$t = \text{years}$

1 year = 12 months

$$\frac{R}{4} = 23 \quad \frac{4}{12} = \frac{1}{3}$$

$$h = 0.5$$

$$y_{n+1} \approx y_n + h \left(\frac{dy}{dx} \right)$$

$$\frac{dR}{dt} = 2(20) + 4 \sin \frac{1}{3} = 40.02327093$$

$$y_{n+1} = 20 + 0.5 (40.02327093)$$

$y_{n+1} = 40.011 \therefore \text{yes} \therefore \text{Julie can start selling her rabbits.}$

1/7

Examiner Comments

B0: Does not state or use the correct step length $h = \frac{1}{6}$.

M0: Uses an incorrect $t_0 = \frac{1}{6}$ to find $\left(\frac{dR}{dt} \right)_0$.

M1: Correct substitution of $R_0 = 20$, their $h = 0.5$ and their $\left(\frac{dR}{dt} \right)_0 = 40.02327093$ into the approximation formula $\left(\frac{dR}{dt} \right)_0 \approx \frac{R_1 - R_0}{h}$ in a complete method to find a numerical expression for R_1 (which is labelled as $y_{n+1} = \dots$).

A0: Follows from the first M0.

M0: Does not attempt to find a numerical value for $\left(\frac{dR}{dt} \right)_1$.

A0: Follows previous M0.

B0: Does not attempt two iterations of $R_{n+1} = R_n + h \left(\frac{dR}{dt} \right)_n$ to find a value for R_2 .

Student Response B

$$3) \quad \frac{4}{12} \frac{R}{12} = \frac{R}{12} =$$

$$\frac{dR}{dt} = 2(20) + 4 \sin 0$$

$$= \frac{dR}{dt} = 40$$

$$y_{n+1} = y_n + h \left(\frac{dy}{dx} \right)_n$$

$$y_{n+1} = 20 + \frac{1}{6} (40)$$

$$= \frac{80}{3}$$

$$\begin{array}{l} t, R \\ R, t \\ (t, R) \end{array}$$

$$\left(\frac{1}{6}, \frac{80}{3} \right)$$

$$\frac{dy}{dx} = 2 \left(\frac{80}{3} \right) + 4 \left(\sin \left(\frac{1}{6} \right) \right)$$

$$= 54$$

$$(t, R)$$

$$\left(\frac{1}{3}, 63 \right)$$

$$y_{n+1} = 54 + \frac{1}{6} (54) =$$

$$= 63$$

5/7

Examiner Comments**B1 M1 M1 A1 M1 A0 B0****B1:** Correct step length $h = \frac{1}{6}$.**M1 M1 A1:** Uses $t_0 = 0$, $R_0 = 20$ and a correct $\left(\frac{dR}{dt}\right)_0 = 40$ in a correct method to find a correct approximation (labelled as $y_{n+1} = \frac{80}{3}$) for the number of rabbits at 2 months.**M1:** Correct substitution of $R_1 = \frac{80}{3}$, $t_1 = h = \frac{1}{6}$ into $\left(\frac{dR}{dt}\right)_1 = 2R_1 + 4\sin t_1$ to give $2\left(\frac{80}{3}\right) + 4\sin\left(\frac{1}{6}\right)$.**A0:** 63 (which is not in the range $[35.5, 36]$) is an incorrect estimate for R_2 .**B0:** Attempts two iterations of $R_{n+1} = R_n + h\left(\frac{dR}{dt}\right)_n$ to find a value for R_2 . Candidate's conclusion based on their $R_2 = 63$ is incorrect.

Student Response C

$$\frac{dR}{dt} = 2R + 4\sin t$$

$$h = 2 \text{ months} = \frac{2}{12} = \frac{1}{6}$$

$$t_0 = 0$$

$$R_0 = 20$$

$$\left(\frac{dR}{dt}\right)_0 = 40$$

$$t_1 = \frac{1}{6}$$

$$R_1 = \frac{80}{3} = 26.667$$

$$\left(\frac{dR}{dt}\right)_1 = 53.997$$

$$t_2 = \frac{1}{3}$$

$$R_2 = 35.666$$

after 4 months Julie would have 35 rabbits so

$35 < 40 \therefore$ she wouldn't be able to start

selling the rabbits

7/7

Examiner Comments

B1 M1 M1 A1 M1 A1 B1

B1: Correct step length $h = \frac{1}{6}$.

M1 M1 A1: Uses $t_0 = 0$, $R_0 = 20$ and a correct $\left(\frac{dR}{dt}\right)_0 = 40$ in a correct method to find a correct approximation $R_1 = \frac{80}{3}$ for the number of rabbits at 2 months.

M1: Candidate's $\left(\frac{dR}{dt}\right)_1 = 53.997$ implies a correct substitution of $R_1 = \frac{80}{3}$, $t_1 = h = \frac{1}{6}$ into $\left(\frac{dR}{dt}\right)_1 = 2R_1 + 4\sin t_1$ to find a numerical expression for $\left(\frac{dR}{dt}\right)_1$.

A1: Obtains a correct $R_2 = 35.666$.

B1: Attempts two iterations of $R_{n+1} = R_n + h\left(\frac{dR}{dt}\right)_n$ to find a value for R_2 followed by a correct conclusion in context.

Exemplar Question 4

4.

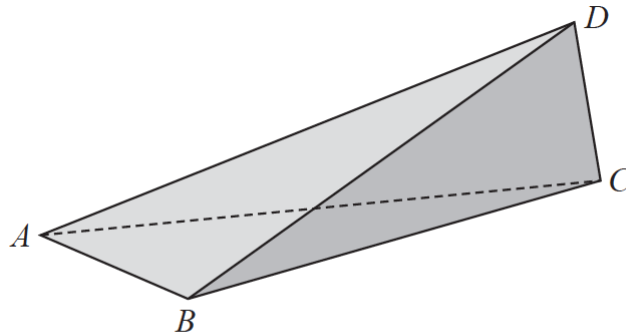


Figure 1

Figure 1 shows a sketch of a solid doorstop made of wood. The doorstop is modelled as a tetrahedron.

Relative to a fixed origin O , the vertices of the tetrahedron are $A(2, 1, 4)$, $B(6, 1, 2)$, $C(4, 10, 3)$ and $D(5, 8, d)$, where d is a positive constant and the units are in centimetres.

(a) Find the area of the triangle ABC .

(4)

Given that the volume of the doorstop is 21 cm^3

(b) find the value of the constant d .

(4)

(Total for Question 4 is 8 marks)

Mean Score 6.0 out of 8

Examiner Comments

Q4 was accessible to most candidates, but it was clear that there were a few candidates who had not revised the vector cross product formula.

In Q4(a), most candidates found the area of triangle ABC by applying the method $\frac{1}{2}|\overline{AB} \times \overline{AC}|$. Some candidates found the correct answer by applying the vector cross product between two other edges of triangle ABC . Some candidates applied the formula $\frac{1}{2}|\overline{AB}||\overline{AC}|\sin(\hat{BAC})$, where angle \hat{BAC} had been found by using the scalar product formula. A few candidates applied $\frac{1}{2}(\text{base})(\text{height})$ after correctly deducing that ABC was isosceles. Common errors included incorrect cross product calculations; finding $|\overline{AB} \times \overline{AC}|$ even after previously stating a correct formula $\frac{1}{2}|\overline{AB} \times \overline{AC}|$; or misconstruing the area formula as $\frac{1}{2}|\overline{OA} \times \overline{OB}|$.

Q4(b) proved more challenging since no solution was possible without the application of the scalar triple product. Some candidates used a complete method of applying $\frac{1}{6}|\overline{AD} \cdot (\overline{AB} \times \overline{AC})| = 21$ to find the value of d , although some used $\frac{1}{3}$, $\frac{1}{2}$ or 1 instead of $\frac{1}{6}$. There were several manipulation and calculation errors seen in the evaluation of the scalar triple product in terms of d . Those that used $\overline{AB} \times \overline{AC} = 18\mathbf{i} + 0\mathbf{j} + 36\mathbf{k}$ from Q4(a) usually found a correct $d = 6$ after solving $\frac{1}{6} \left| \begin{pmatrix} 3 \\ 7 \\ d-4 \end{pmatrix} \cdot \begin{pmatrix} 18 \\ 0 \\ 36 \end{pmatrix} \right| = 21$. A common error was the confusion between the modulus of a scalar and the magnitude of a vector, leading to some candidates writing $\sqrt{(54)^2 + (3d-144)^2} = 126$. Another common error was to simplify $|36d - 90|$ to give $36d + 90$ or $\sqrt{(36d)^2 - (90)^2}$.

Mark Scheme

Question	Scheme	Marks	AOs
4	$A(2, 1, 4), B(6, 1, 2), C(4, 10, 3), D(5, 8, d)$		
(a) Way 1	Uses appropriate vectors in a correct method to make a complete attempt to find the area of triangle ABC .	M1	3.1b
	$\overline{AB} = \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix}, \overline{AC} = \begin{pmatrix} 2 \\ 9 \\ -1 \end{pmatrix}, \left\{ \overline{BC} = \begin{pmatrix} -2 \\ 9 \\ 1 \end{pmatrix} \right\}$	M1	1.1b
	e.g. $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & -2 \\ 2 & 9 & -1 \end{vmatrix} = \dots$ or $\begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 9 \\ -1 \end{pmatrix} = \dots$	M1	1.1b
	$= 18\mathbf{i} + 0\mathbf{j} + 36\mathbf{k}$		
	Area $ABC = \frac{1}{2}\sqrt{(18)^2 + (0)^2 + (36)^2}$		
	$\{= 20.1246\dots\} = 9\sqrt{5} \text{ (cm}^2\text{)} \text{ or awrt } 20.1 \text{ (cm}^2\text{)}$	A1	2.2a
	(4)		
(a) Way 2	Uses appropriate vectors to find an angle or perpendicular height in triangle ABC and uses a correct method to make a complete attempt to find the area of triangle ABC .	M1	3.1b
	$\overline{AB} = \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix}, \overline{AC} = \begin{pmatrix} 2 \\ 9 \\ -1 \end{pmatrix}, \left\{ \overline{BC} = \begin{pmatrix} -2 \\ 9 \\ 1 \end{pmatrix} \right\}$	M1	1.1b
	Uses a correct method to find an angle or perpendicular height in triangle ABC	M1	1.1b
	Note: $BAC = 27.905\dots, ABC = 76.047\dots, BCA = 76.047\dots$ or perpendicular height = 9		
	Area $ABC = \frac{1}{2}\sqrt{86}\sqrt{20} \sin 76.047\dots$ or $\frac{1}{2}\sqrt{86}\sqrt{86} \sin 27.905\dots$ or $\frac{1}{2}\sqrt{20}(9)$		
	$\{= 20.1246\dots\} = 9\sqrt{5} \text{ (cm}^2\text{)} \text{ or awrt } 20.1 \text{ (cm}^2\text{)}$	A1	2.2a
	(4)		
(b)	Finds appropriate vectors to form the equation volume tetrahedron $ABCD = 21$ to give a linear equation in d	M1	3.1a
	Note: The volume must include $\frac{1}{6}$		
	e.g. $\left \begin{pmatrix} 3 \\ 7 \\ d-4 \end{pmatrix} \cdot \begin{pmatrix} 18 \\ 0 \\ 36 \end{pmatrix} \right = \dots$ or $\begin{vmatrix} 4 & 0 & -2 \\ 2 & 9 & -1 \\ 3 & 7 & d-4 \end{vmatrix} = \dots$	M1	1.1b
	$= 54 + 36d - 144 $ or $ 4(9d - 36 + 7) - 2(14 - 27) \{= 36d - 90 \}$	A1	1.1b
	$\left\{ \frac{1}{6} 36d - 90 = 21 \Rightarrow 36d - 90 = 126 \Rightarrow \right\} d = 6$	A1	1.1b
(4)			

(8 marks)

Question	Scheme	Marks	AOs
4	$A(2, 1, 4), B(6, 1, 2), C(4, 10, 3), D(5, 8, d)$		
(a) Way 3	Complete attempt to find the area of triangle ABC by applying $\frac{1}{2} \vec{OA} \times \vec{OB} + \vec{OB} \times \vec{OC} + \vec{OC} \times \vec{OA} $ or equivalent	M1	3.1b
	$\vec{OA} \times \vec{OB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 4 \\ 6 & 1 & 2 \end{vmatrix} = \dots \quad \text{and} \quad \vec{OB} \times \vec{OC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 1 & 2 \\ 4 & 10 & 3 \end{vmatrix} = \dots,$ $\text{and } \vec{OC} \times \vec{OA} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 10 & 3 \\ 2 & 1 & 4 \end{vmatrix} = \dots$	M1	1.1b
	$\{\vec{OA} \times \vec{OB} + \vec{OB} \times \vec{OC} + \vec{OC} \times \vec{OA}\} = \begin{pmatrix} -2 \\ 20 \\ -4 \end{pmatrix} + \begin{pmatrix} -17 \\ -10 \\ 56 \end{pmatrix} + \begin{pmatrix} 37 \\ -10 \\ -16 \end{pmatrix}$	M1	1.1b
	$\text{Area } ABC = \frac{1}{2}\sqrt{(18)^2 + (0)^2 + (36)^2}$		
	$\{= 20.1246\dots\} = 9\sqrt{5} \text{ (cm}^2\text{)} \text{ or awrt } 20.1 \text{ (cm}^2\text{)}$	A1	2.2a
			(4)
Notes for Question 4			
(a)	Way 1		
M1:	Complete correct process of taking the vector product between 2 edges of triangle ABC , applying Pythagoras and multiplying the result by 0.5		
M1:	Uses a correct method to find any 2 edges of triangle ABC		
M1:	Attempts to take the vector cross product between 2 edges of triangle ABC		
A1:	Deduces the correct area of either $9\sqrt{5} \text{ (cm}^2\text{)}$ or awrt $20.1 \text{ (cm}^2\text{)}$		
(a)	Way 2		
M1:	See scheme		
M1:	Uses a correct method to find any 2 edges of triangle ABC		
M1:	Either <ul style="list-style-type: none"> • finds an angle in ABC by using a correct scalar product method • finds an angle in ABC by using the cosine rule in the correct direction • realises triangle ABC is isosceles and applies Pythagoras in the correct direction to find the perpendicular height 		
A1:	Deduces the correct area as either $9\sqrt{5} \text{ (cm}^2\text{)}$ or awrt $20.1 \text{ (cm}^2\text{)}$		
Note:	For Way 1 and Way 2, using any of \vec{OA}, \vec{OB} or \vec{OC} in their vector product is M0 M0 A0 A0		
(a)	Way 3		
M1:	See scheme		
M1:	Attempts to apply $\vec{OA} \times \vec{OB}, \vec{OB} \times \vec{OC}$ and $\vec{OC} \times \vec{OA}$		
A1:	Attempts to add (as vectors) the results of applying $\vec{OA} \times \vec{OB}, \vec{OB} \times \vec{OC}$ and $\vec{OC} \times \vec{OA}$		
A1:	Deduces the correct area as either $9\sqrt{5} \text{ (cm}^2\text{)}$ or awrt $20.1 \text{ (cm}^2\text{)}$		

Notes for Question 4 Continued	
(b)	
M1:	See scheme
M1:	Uses appropriate vectors in an attempt at the scalar triple product
A1:	Correct applied expression for the scalar triple product (allow \pm and ignore modulus sign)
A1:	Correct solution leading to $d = 6$
Note:	Using any of \overrightarrow{OA} , \overrightarrow{OB} , \overrightarrow{OC} or \overrightarrow{OD} in their scalar triple product is M0 M0 A0 A0
Note:	Some vector product calculations for reference:
	$\left \overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) \right = \begin{vmatrix} 3 & 7 & d-4 \\ 4 & 0 & -2 \\ 2 & 9 & -1 \end{vmatrix} = \left \begin{pmatrix} 3 \\ 7 \\ d-4 \end{pmatrix} \cdot \begin{pmatrix} 18 \\ 0 \\ 36 \end{pmatrix} \right = 54 + 36d - 144 = 36d - 90 $
	$\left \overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) \right = \begin{vmatrix} 4 & 0 & -2 \\ 2 & 9 & -1 \\ 3 & 7 & d-4 \end{vmatrix} = \left \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 9d-29 \\ 5-2d \\ -13 \end{pmatrix} \right = 36d - 116 + 26 = 36d - 90 $
	$\left \overrightarrow{AC} \cdot (\overrightarrow{AB} \times \overrightarrow{AD}) \right = \begin{vmatrix} 2 & 9 & -1 \\ 4 & 0 & -2 \\ 3 & 7 & d-4 \end{vmatrix} = \left \begin{pmatrix} 2 \\ 9 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 14 \\ 10-4d \\ 28 \end{pmatrix} \right = 28 + 90 - 36d - 28 = 90 - 36d $

Student Response A

$$\begin{aligned} \text{1a) } \vec{AB} &= b-a \\ &= (4, 0, 2) \end{aligned}$$

$$\begin{aligned} \vec{AC} &= c-a \\ &= (2, 9, -1) \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} i & j & k \\ 4 & 0 & 2 \\ 2 & 9 & -1 \end{pmatrix} &= i(-18) - j(-8) + k(-8) \\ &= -18i + 8j - 8k \\ \sqrt{18^2 + 8^2 + 8^2} &= 2\sqrt{113} \\ &= 21.3 \end{aligned}$$

$$\begin{aligned} \text{b) } \vec{AD} &= d-a \\ &= (3, 7, d-4) \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} 3 & 7 & d-4 \\ 4 & 0 & 2 \\ 2 & 9 & -1 \end{pmatrix} &= 3(-18) - 7(-8) + (d-4)(-8) \end{aligned}$$

$$= -54 + 56 - 8d + 32$$

$$= 21$$

$$= 34 - 8d$$

$$= 21$$

$$8d = 13$$

$$d = \frac{13}{8}$$

3/8

Examiner Comments**(a) M0 M1 M1 A0 (Way 1)**

M0: Applies an incorrect method $|\overrightarrow{AB} \times \overrightarrow{AC}|$ (instead of a correct method $\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$) to find the area of triangle ABC .

M1: Uses a correct method (e.g. $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$) to find \overrightarrow{AB} and \overrightarrow{AC} . Note: The error in finding the z -component of \overrightarrow{AB} is condoned.

M1: Attempts to evaluate their $\overrightarrow{AB} \times \overrightarrow{AC}$.

A0: Follows from the first M0.

(b) M0 M1 A0 A0

M0: Does not apply $\frac{1}{6}|\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})| = 21$.

M1: Uses appropriate vectors \overrightarrow{AD} and $\overrightarrow{AB} \times \overrightarrow{AC}$ in an attempt to evaluate their
$$\begin{vmatrix} 3 & 7 & d-4 \\ 4 & 0 & 2 \\ 2 & 9 & -1 \end{vmatrix}.$$

A0: Candidate's un-simplified $3(-18) - 7(-8) - (d-4)(-8)$ is incorrect.

A0: Follows from the first M0.

Student Response B

$$a. A = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \quad B = \begin{pmatrix} 6 \\ 2 \\ 2 \end{pmatrix} \quad C = \begin{pmatrix} 4 \\ 10 \\ 3 \end{pmatrix}$$

$$\vec{AB} = B - A = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$$

$$\vec{AC} = C - A = \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix}$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 4 & 0 & -2 \\ 2 & 6 & 2 \end{vmatrix} = i \begin{vmatrix} 0 & -2 \\ 6 & 2 \end{vmatrix} - j \begin{vmatrix} 4 & -2 \\ 2 & -1 \end{vmatrix} + k \begin{vmatrix} 4 & 0 \\ 2 & 6 \end{vmatrix}$$

$$\vec{AB} \times \vec{AC} = i(0 - -12) - j(-4 - -4) + k(24 - 0)$$

$$\vec{AB} \times \vec{AC} = \begin{pmatrix} 12 \\ 0 \\ 24 \end{pmatrix}$$

$$|\vec{AB} \times \vec{AC}| = \left| \begin{pmatrix} 12 \\ 0 \\ 24 \end{pmatrix} \right| = \sqrt{12^2 + 0^2 + 24^2} = 12\sqrt{5}$$

Area of a triangle = $\frac{1}{2} |a \times b|$
 Area of triangle ABC = $\frac{1}{2} \times 12\sqrt{5} = 6\sqrt{5}$

$$\frac{a \cdot b}{|a||b|} = \cos \theta$$

area of a triangle = $\frac{1}{2} ab \sin C$

$$\vec{AB} = \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix} \quad \vec{AC} = \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix}$$

$$\vec{AB} \cdot \vec{AC} = \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} = 4(2) + 0(6) + -2(2) = 8 - 4 = 4$$

$$|\vec{AB}| = \sqrt{4^2 + 0^2 + 2^2} = 2\sqrt{5}$$

$$|\vec{AC}| = \sqrt{2^2 + 6^2 + 2^2} = \sqrt{40} = 2\sqrt{10}$$

$$\theta = \cos^{-1} \left(\frac{4}{2\sqrt{5} \times 2\sqrt{10}} \right)$$

$$\theta = 76.0472...^\circ = 76^\circ \text{ to } 3\text{ s.f.}$$

$$\text{Area of triangle ABC} = \frac{1}{2} \times |\vec{AB}| \times |\vec{AC}| \times \sin(76^\circ)$$

$$= 20.1246118$$

$$= 20.12 \text{ cm}^2 \text{ to } 4\text{ s.f.}$$

$$= 20.1 \text{ cm}^2 \text{ to } 3\text{ s.f.}$$

b. Volume of a tetrahedron = $\frac{1}{6} |a \cdot (b \times c)|$

$$\vec{AB} = \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix} \quad \vec{AC} = \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} \quad \vec{AD} = \begin{pmatrix} 3 \\ 7 \\ d-4 \end{pmatrix}$$

$$\vec{AB} \cdot (\vec{AC} \times \vec{AD}) = \begin{vmatrix} 4 & 0 & -2 \\ 2 & 6 & 2 \\ 3 & 7 & (d-4) \end{vmatrix}$$

$$= 4 \begin{vmatrix} 6 & 2 \\ 7 & (d-4) \end{vmatrix} - 0 \begin{vmatrix} 2 & 2 \\ 3 & (d-4) \end{vmatrix} + -2 \begin{vmatrix} 2 & 6 \\ 3 & 7 \end{vmatrix}$$

$$= 4(6d - 36 - 14) + 0 - 2(14 - 18)$$

$$= \begin{pmatrix} 36d-116 \\ 0 \\ 26 \end{pmatrix}$$

$$21 = \sqrt{26^2 + 0^2 + (36d-116)^2}$$

$$(1296d^2 - 8352d + 13456) + 676 = 441$$

$$1296d^2 - 8352d + 13691 = 0$$

$$d = 1.7$$

6/8

Examiner Comments**(a) M1 M1 M1 A1 (Way 2)**

M1 M1 M1: A correct scalar product method for finding angle BAC followed by a correct method for finding the area of triangle ABC by applying $\frac{1}{2} |\overrightarrow{AB}| |\overrightarrow{AC}| \sin BAC$.

A1: Obtains an answer which rounds to 20.1.

(b) M0 M1 A1 A0

M0: Does not apply $\frac{1}{6} |\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})| = 21$.

M1: Uses appropriate vectors \overrightarrow{AB} and $\overrightarrow{AC} \times \overrightarrow{AD}$ in an attempt to evaluate

$$\begin{vmatrix} 4 & 0 & -2 \\ 2 & 9 & -1 \\ 3 & 7 & d-4 \end{vmatrix}$$

A1: Candidate's un-simplified $4(9d - 36 - -7) + 0 - 2(14 - 27)$ is correct.

A0: Follows from the first M0.

Student Response C

$$a) \vec{AB} = \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix} \quad \vec{AC} = \begin{pmatrix} 2 \\ 9 \\ -1 \end{pmatrix}$$

$$\vec{AB} \times \vec{AC} = \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 9 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} (0 \times -1) - (-2 \times 9) \\ -((4 \times -1) - (-2 \times 2)) \\ (4 \times 9) - (0 \times 2) \end{pmatrix}$$

$$= \begin{pmatrix} 18 \\ 0 \\ 36 \end{pmatrix}$$

$$\sqrt{18^2 + 0^2 + 36^2} = 18\sqrt{5}$$

$$\frac{1}{2} \times 18\sqrt{5} = 9\sqrt{5}$$

$$b) \vec{AD} = \begin{pmatrix} 3 \\ 7 \\ d-4 \end{pmatrix}$$

$$\frac{1}{6} \left| \begin{pmatrix} 3 \\ 7 \\ d-4 \end{pmatrix} \cdot \begin{pmatrix} 18 \\ 0 \\ 36 \end{pmatrix} \right| = \frac{1}{6} (3 \times 18) + (7 \times 0) + (36 \times (d-4))$$

$$= \frac{1}{6} (54 + 36d - 144)$$

$$21 = \frac{1}{6} (1 - 90 + 36d)$$

$$126 = -90 + 36d$$

Examiner Comments**(a) M1 M1 M1 A1 (Way 1)****M1 M1 M1 A1:** A correct method of applying $\frac{1}{2}|\overline{AB} \times \overline{AC}|$ leading to a correct answer of $9\sqrt{5}$.**(b) M1 M1 A1 A0****M1:** Complete strategy of applying $\frac{1}{6}|\overline{AD} \cdot (\overline{AB} \times \overline{AC})| = 21$ which leads to a linear equation in d .**M1:** Uses appropriate vectors \overline{AD} and $\overline{AB} \times \overline{AC}$ in an attempt to evaluate $\begin{pmatrix} 3 \\ 7 \\ d-4 \end{pmatrix} \cdot \begin{pmatrix} 18 \\ 0 \\ 36 \end{pmatrix}$.**A1:** Candidate's un-simplified $54 + 36d - 144$ is correct.**A0:** Does not find $d = 6$.

Exemplar Question 5

5.

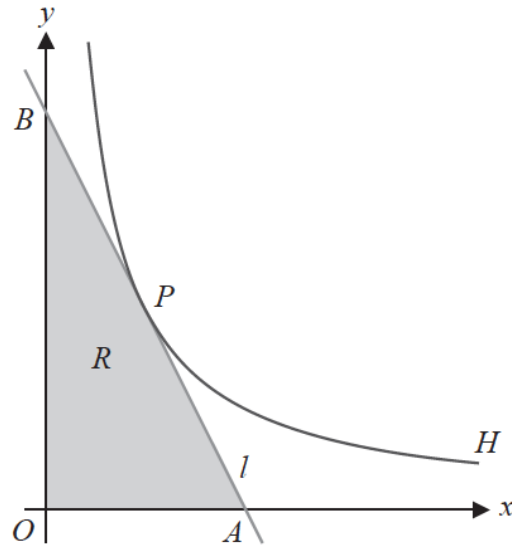


Figure 2

Figure 2 shows a sketch of part of the rectangular hyperbola H with equation

$$xy = c^2 \quad x > 0$$

where c is a positive constant.

The point $P\left(ct, \frac{c}{t}\right)$ lies on H .

The line l is the tangent to H at the point P .

The line l crosses the x -axis at the point A and crosses the y -axis at the point B .

The region R , shown shaded in Figure 2, is bounded by the x -axis, the y -axis and the line l .

Given that the length OB is twice the length of OA , where O is the origin, and that the area of R is 32, find the exact coordinates of the point P .

(10)

(Total for Question 5 is 10 marks)

Mean Score 6.8 out of 10

Examiner Comment

Q5 required an extended response and its unstructured nature proved too demanding for those who did not formulate a clear strategy. This meant that Q5 discriminated well across higher ability candidates and there was a significant minority who made no attempt or no creditable attempt at this question. A huge range of methods were viable, most of which were seen, although Way 1, as described in the mark scheme, was the most popular and successful route to a correct answer.

A significant number of candidates progressed as far as finding an equation for l (the tangent to H at P) and finding the points where l cut through the x -axis and the y -axis. Errors seen for those who made further progress included using $OA = 2OB$ rather than $OB = 2OA$; using $OA:OB = 1:3$; applying (base)(height) when finding the area of triangle OAB ; believing $t^2 = \frac{1}{2} \Rightarrow t = \frac{1}{4}$; or simplifying $\frac{1}{2}(x)(2x)$ to give either $\frac{1}{2}x^2$, $2x^2$ or $2x$. On the whole, it was pleasing to see a significant number of well-presented correct solutions.

Mark Scheme

Question	Scheme	Marks	AOs
5	$H: xy = c^2, c > 0; P\left(ct, \frac{c}{t}\right)$ lies on $H; OB = 2OA; \text{Area}(OAB) = 32$		
Way 1	<p>Either $y = \frac{c^2}{x} = c^2x^{-1} \Rightarrow \frac{dy}{dx} = -c^2x^{-2}$ or $-\frac{c^2}{x^2}$</p> <p>or $xy = c^2 \Rightarrow x\frac{dy}{dx} + y = 0$</p> <p>or $x = cp, y = \frac{c}{p} \Rightarrow \frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx} = -\left(\frac{c}{p^2}\right)\left(\frac{1}{c}\right)$; condone $t \equiv p$</p> <p>and so, at $P\left(ct, \frac{c}{t}\right), m_T = -\frac{1}{t^2}$</p>	M1	3.1a
	$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$	M1	1.1b
	or $\frac{c}{t} = -\frac{1}{t^2}(ct) + b \Rightarrow y = -\frac{1}{t^2}x + \text{their } b \Rightarrow y = -\frac{1}{t^2}x + \frac{2c}{t}$	A1	1.1b
	$y = 0 \Rightarrow x = 2ct \{\Rightarrow x_A = 2ct\}, x = 0 \Rightarrow y = \frac{2c}{t} \{\Rightarrow y_B = \frac{2c}{t}\}$	M1	1.1b
		A1	1.1b
	$\{OB = 2OA \Rightarrow\} \frac{2c}{t} = 2(2ct) \Rightarrow t = \dots$	M1	2.1
	$\left\{t^2 = \frac{1}{2} \Rightarrow\right\} t = \frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$ or awrt 0.707	A1	1.1b
	$\{\text{Area}(OAB) = 32 \Rightarrow\} \frac{1}{2}(2ct)\left(\frac{2c}{t}\right) = 32 \Rightarrow c = \dots \{\Rightarrow c = 4\}$	M1	2.1
	Deduces the numerical value x_p and y_p using their values of t and c	M1	2.2a
	$P(2\sqrt{2}, 4\sqrt{2})$ or $P(\text{awrt } 2.83, \text{awrt } 5.66)$ or $x = 2\sqrt{2}$ and $y = 4\sqrt{2}$	A1	1.1b
	(10)		
Way 2	Same requirement as the 1 st M mark in Way 1	M1	3.1a
	e.g. $\left\{t = \frac{1}{\sqrt{2}} \Rightarrow P\left(\frac{c}{\sqrt{2}}, \sqrt{2}c\right) \Rightarrow\right\} y - \sqrt{2}c = -2\left(x - \frac{c}{\sqrt{2}}\right)$	M1	1.1b
	using $m_T = -2$ and their P which has been found by a correct method	A1	1.1b
	$y = 0 \Rightarrow x = \sqrt{2}c \{\Rightarrow x_A = \sqrt{2}c\}, x = 0 \Rightarrow y = 2\sqrt{2}c \{\Rightarrow y_B = 2\sqrt{2}c\}$	M1	1.1b
		A1	1.1b
	$\{OB = 2OA \Rightarrow\} m_T = -2$ and their $m_T = -\frac{1}{t^2} = -2 \Rightarrow t = \dots$	M1	2.1
	$\left\{t^2 = \frac{1}{2} \Rightarrow\right\} t = \frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$ or awrt 0.707 $\left\{\Rightarrow P\left(\frac{c}{\sqrt{2}}, \sqrt{2}c\right)\right\}$	A1	1.1b
	$\{\text{Area}(OAB) = 32 \Rightarrow\} \frac{1}{2}\sqrt{2}c(2\sqrt{2}c) = 32 \Rightarrow c = \dots \{\Rightarrow c = 4\}$	M1	2.1
	Deduces the numerical value x_p and y_p using their values of t and c	M1	2.2a
	$P(2\sqrt{2}, 4\sqrt{2})$ or $P(\text{awrt } 2.83, \text{awrt } 5.66)$ or $x = 2\sqrt{2}$ and $y = 4\sqrt{2}$	A1	1.1b
	(10)		
(10 marks)			

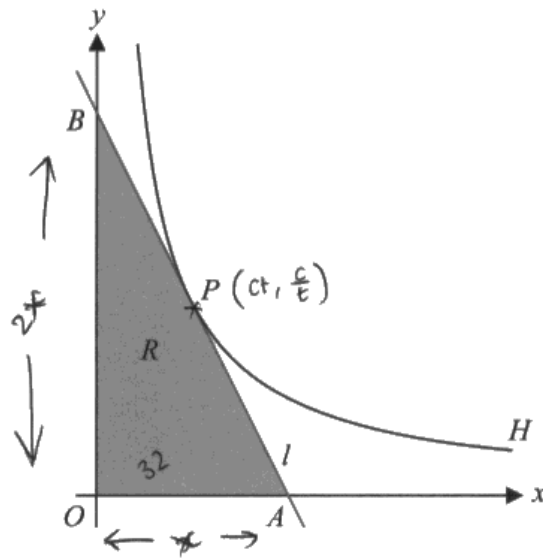
Question	Scheme	Marks	AOs
5	$H: xy = c^2, c > 0$; $P\left(ct, \frac{c}{t}\right)$ lies on H ; $OB = 2OA$; $\text{Area}(OAB) = 32$		
Way 3	Same requirement as the 1 st M mark in Way 1	M1	3.1a
	e.g. $y - 8\sqrt{2} = -2(x - 0)$ or $y - 0 = -2(x - 4\sqrt{2})$	M1	1.1b
	using $m_T = -2$ and either their $A(4\sqrt{2}, 0)$ or their $B(0, 8\sqrt{2})$ which have been found by a correct method	A1	1.1b
	$\{\text{Area}(OAB) = 32, OB = 2OA \Rightarrow\} \frac{1}{2}(x)(2x) = 32 \Rightarrow x = \dots$	M1	2.1
	$x = 4\sqrt{2} \{\Rightarrow x_A = 4\sqrt{2}\}$ or $y = 8\sqrt{2} \{\Rightarrow y_B = 8\sqrt{2}\}$	A1	1.1b
	$\{OB = 2OA \Rightarrow\} m_T = -2$ and their $m_T = -\frac{1}{t^2} = -2 \Rightarrow t = \dots$	M1	2.1
	$\left\{t^2 = \frac{1}{2} \Rightarrow\right\} t = \frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$ or awrt 0.707 $\left\{\Rightarrow P\left(\frac{c}{\sqrt{2}}, \sqrt{2}c\right)\right\}$	A1	1.1b
	$\sqrt{2}c - 8\sqrt{2} = -2\left(\frac{c}{\sqrt{2}} - 0\right) \Rightarrow c = \dots \{\Rightarrow c = 4\}$	M1	1.1b
	Deduces the numerical value x_p and y_p using their values of t and c	M1	2.2a
	$P(2\sqrt{2}, 4\sqrt{2})$ or $P(\text{awrt } 2.83, \text{awrt } 5.66)$ or $x = 2\sqrt{2}$ and $y = 4\sqrt{2}$	A1	1.1b
	(10)		
Way 4	Complete process substituting their $y - 8\sqrt{2} = -2(x - 0)$ or $y - 0 = -2(x - 4\sqrt{2})$ into $xy = c^2$ and applying $b^2 - 4ac = 0$ to their resulting $2x^2 - 8\sqrt{2}x + c^2 = 0$	M1	3.1a
	e.g. $y - 8\sqrt{2} = -2(x - 0)$ or $y - 0 = -2(x - 4\sqrt{2})$	M1	1.1b
	using $m_T = -2$ and either their $A(4\sqrt{2}, 0)$ or their $B(0, 8\sqrt{2})$ which have been found by a correct method	A1	1.1b
	$\{\text{Area}(OAB) = 32, OB = 2OA \Rightarrow\} \frac{1}{2}(x)(2x) = 32 \Rightarrow x = \dots$	M1	2.1
	$x = 4\sqrt{2} \{\Rightarrow x_A = 4\sqrt{2}\}$ or $y = 8\sqrt{2} \{\Rightarrow y_B = 8\sqrt{2}\}$	A1	1.1b
	dependent on 2nd M mark	dM1	2.1
	$\{xy = c^2 \Rightarrow\} x(-2x + 8\sqrt{2}) = c^2 \{\Rightarrow 2x^2 - 8\sqrt{2}x + c^2 = 0\}$		
	or $\{xy = c^2 \Rightarrow\} \frac{1}{2}(8\sqrt{2} - y)y = c^2 \{\Rightarrow y^2 - 8\sqrt{2}y + 2c^2 = 0\}$	A1	1.1b
	$\{b^2 - 4ac = 0 \Rightarrow\} (8\sqrt{2})^2 - 4(2)(c^2) = 0 \Rightarrow c = \dots \{\Rightarrow c = 4\}$	M1	1.1b
	Deduces the numerical value x_p and y_p using their value of c	M1	2.2a
$P(2\sqrt{2}, 4\sqrt{2})$ or $P(\text{awrt } 2.83, \text{awrt } 5.66)$ or $x = 2\sqrt{2}$ and $y = 4\sqrt{2}$	A1	1.1b	
	(10)		
Note:	For the final M1 mark in Way 1, Way 2, Way 3 and Way 4 Allow final M1 for a correct method which gives any of $x_p = 2\sqrt{2}$ or $y_p = 4\sqrt{2}$ or $x_p = \text{awrt } 2.83$ or $y_p = \text{awrt } 5.66$ o.e.		

Notes for Question 5	
Way 1	
M1:	Establishes the gradient of the tangent by differentiating $xy = c^2$ <ul style="list-style-type: none"> • to give $\frac{dy}{dx} = \pm kx^{-2}; k \neq 0$, or • by the product rule to give $\pm x \frac{dy}{dx} \pm y$, or • by parametric differentiation to give $\left(\text{their } \frac{dy}{dt}\right) \times \frac{1}{\left(\text{their } \frac{dx}{dt}\right)}$, condoning $p \equiv t$ <p>and attempt to use $P\left(ct, \frac{c}{t}\right)$ to write down the gradient of the tangent to the curve in terms of t</p>
M1:	Correct straight line method for an equation of a tangent where $m_T (\neq m_N)$ is found by using calculus. Note: m_T must be a function of t for this mark
A1:	Correct equation of the tangent which can be simplified or un-simplified
M1:	Attempts to find either the x -coordinate of A or the y -coordinate of B
A1:	Both { x -coordinate of A is} $2ct$ and the { y -coordinate of B is} $\frac{2c}{t}$
M1:	See scheme
A1:	See scheme
M1:	See scheme
M1:	See scheme
A1:	See scheme
Way 2	
M1:	Same description as the 1 st M mark in Way 1
M1:	See scheme
A1:	Correct equation of the tangent which can be simplified or un-simplified
M1:	Attempts to find either the x -coordinate of A or the y -coordinate of B
A1:	Both { x -coordinate of A is} $\sqrt{2}c$ and the { y -coordinate of B is} $2\sqrt{2}c$
M1:	Recognising that the gradient of the tangent is -2 and puts this equal to their $\frac{dy}{dx}$ and finds $t = \dots$
A1:	See scheme
M1:	See scheme
M1:	See scheme
A1:	See scheme

Notes for Question 5

Way 3	
M1:	Same description as the 1 st M mark in Way 1
M1:	See scheme
A1:	Correct equation of the tangent which can be simplified or un-simplified
M1:	Uses $y = 2x$ and Area (OAB) = 32 to find either x_A or y_B
A1:	Either { x -coordinate of A is} $4\sqrt{2}$ or the { y -coordinate of B is} $8\sqrt{2}$
M1:	Recognising that the gradient of the tangent is -2 and puts this equal to their $\frac{dy}{dx}$ and finds $t = \dots$
A1:	See scheme
M1:	Substitutes their P (which is in terms of c , and has come from a correct method) into the equation of the tangent and finds $c = \dots$
M1:	See scheme
A1:	See scheme
Way 4	
M1:	See scheme
M1:	See scheme
A1:	Correct equation of the tangent which can be simplified or un-simplified
M1:	Uses $y = 2x$ and Area (OAB) = 32 to find either x_A or y_B
A1:	Either { x -coordinate of A is} $4\sqrt{2}$ or the { y -coordinate of B is} $8\sqrt{2}$
M1:	See scheme
A1:	See scheme
M1:	See scheme
M1:	See scheme
A1:	See scheme

Student Response A



L: $y = c^2 x^{-1}$

$$\frac{dy}{dx} (c^2 x^{-1}) = -c^2 x^{-2}$$

$$y - y_1 = m(x - x_1)$$

$$y - ct^{-1} = -c^2 x^{-2} (x - ct)$$

$$y - ct^{-1} = -c^2 x^{-1} + c^3 t x^{-2}$$

$$y = -c^2 x^{-1} + c^3 t x^{-2} + ct^{-1}$$

B: $y = -c^2 (0)^{-1} + c^3 t (0)^{-2} + ct^{-1}$

$$y = ct^{-1}$$

$$B(0, \frac{c}{t})$$

A: $0 = \frac{-c^2}{x} + \frac{c^3 t}{x^2} + \frac{c}{t}$

$$\frac{1}{2} (2x)(x) = 32$$

$$x^2 = 32$$

$$x = 4\sqrt{2} \quad \text{- must be +ve bc. dist}$$

$$\frac{c}{t} = \cancel{4\sqrt{2}} \quad 8\sqrt{2}$$

$$A: \quad 0 = \frac{x}{-c^2} + \frac{x^2}{c^3 t} + \frac{t}{c}$$

$$0 = -c^2 x^{-1} + c^3 t x^2 + c t^{-1}$$

$$0 = c (-c^2 x^{-1} + c^2 t x^2 + t^{-1})$$

2/10

Examiner Comments**M0 M0 M0 M1 A1 M0 A0 M0 M0 A0 (Way 3)**

M0: Does not use $x = ct$, $y = \frac{c}{t}$ to write down the gradient of the tangent to the curve in terms of t .

M0: Does not apply $m_t = -2$ to their tangent equation.

A0: Follows previous M0.

M1 A1: Correct complete method of applying $OB = 2OA$ and $\frac{1}{2}(OA)(OB) = 32$ leading to a correct x -coordinate of A (denoted by x) as $4\sqrt{2}$ and a correct y -coordinate of B (denoted by $\frac{c}{t}$) as $8\sqrt{2}$.

M0 A0 M0 M0 A0: Makes no further creditable progress.

Student Response B

$$y = \frac{c^2}{x} \quad \vec{OB} = 2\vec{OA} \quad (10)$$

$$\frac{1}{2}\vec{OA} \times \vec{OB} = 32$$

$$\frac{1}{2}\vec{OA} \times 2\vec{OA} = 32$$

$$\vec{OA}^2 = \cancel{64} 32$$

$$\vec{OA} = \sqrt{32}$$

$$\text{so } \underline{\underline{A = 4\sqrt{2}}}$$

$$\text{so } \cancel{A} A = (4\sqrt{2}, 0)$$

$$\text{so } B = (0, 8\sqrt{2})$$

$$L: m = \frac{8\sqrt{2}}{-4\sqrt{2}} \quad \text{so tangent @ P:}$$

$$= -2 \quad \begin{cases} y = -2(x - 4\sqrt{2}) \\ y = -2x + 8\sqrt{2} \end{cases}$$

but gradient @ P = -2

gradient of the curve: *

$$\frac{dy}{dx} = \frac{c^2}{x^2} \quad \frac{c^2}{x^2} = -2 \quad c^2 = -2x^2$$

$$\text{length } AB = \sqrt{(8\sqrt{2})^2 + (4\sqrt{2})^2} \\ = 4\sqrt{10}$$

$$y = -2x + 8\sqrt{2}$$

$$x(-2x + 8\sqrt{2}) = c^2 \quad \text{bf} = -2x^2$$

$$-2x^2 + 8\sqrt{2}x = c^2$$

$$\cancel{-2x^2 + 8\sqrt{2}x = c^2}$$

6/10

Examiner Comments**M0 M1 A1 M1 A1 M1 A1 M0 M0 A0 (Way 4)****M0:** Does not apply $b^2 - 4ac = 0$ to their equation $-2x^2 + 8\sqrt{2}x = c^2$.**M1 A1:** Uses $m_T = -2$ and their $A(4\sqrt{2}, 0)$ to give a correct tangent equation $y = -2(x - 4\sqrt{2})$.**M1 A1:** Complete correct method of applying $OB = 2OA$ and $\frac{1}{2}(OA)(OB) = 32$ leading to a correct x -coordinate of A as $4\sqrt{2}$ and a correct y -coordinate of B as $8\sqrt{2}$.**M1 A1:** Substitutes their tangent equation $y = -2x + 4\sqrt{2}$ into $xy = c^2$ to give a correct equation $x(-2x + 4\sqrt{2}) = c^2$.**M0:** Follows from the first M0.**M0 A0:** Does not proceed as far as obtaining a numerical value for x_p and y_p .

Student Response C

$$xy = c^2$$

$$y = c^2 x^{-1}$$

$$\frac{dy}{dx} = -c^2 x^{-2} = -\frac{c^2}{x^2}$$

when $x = ct$ $\frac{dy}{dx} = \frac{-c^2}{(ct)^2} = \frac{-c^2}{c^2 t^2} = -\frac{1}{t^2}$

l: $y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$

$$t^2 y - ct = -x + ct$$

x intercept of l when $y = 0$

$$t^2(0) - ct = -x + ct$$

$$-ct = -x + ct$$

$$-2ct = -x$$

$$2ct = x$$

$\therefore A: (2ct, 0)$

y intercept of l when $x = 0$

$$t^2 y - ct = -(0) + ct$$

$$t^2 y = 2ct$$

$$y = \frac{2ct}{t^2} = \frac{2c}{t}$$

$\therefore B: (0, \frac{2c}{t})$

area of R = $\frac{2ct \times \frac{2c}{t}}{2} = ct \times \frac{c}{t} = \frac{c^2 t}{t} = c^2$

$\therefore 32 = c^2 \rightarrow \pm 4\sqrt{2} = c$

4 marks

A: $(4\sqrt{2}t, \frac{4\sqrt{2}}{t})$

$$xy = c^2$$

$$(4\sqrt{2}t) \left(\frac{4\sqrt{2}}{t}\right) = 1$$

~~$t^2 y - ct = -x + ct$~~

~~$t^2 \left(\frac{4\sqrt{2}}{t}\right) - 4\sqrt{2}t = -(4\sqrt{2}t) - ct$~~

~~$4\sqrt{2}t = -ct$~~

$$\begin{array}{l}
 \frac{2c}{t} = 2(2ct) \\
 \frac{2c}{t} = 4ct \\
 2c = 4ct^2 \\
 \frac{2c}{4c} = t^2
 \end{array}
 \qquad
 \begin{array}{l}
 \frac{1}{2} = t^2 \\
 \pm \frac{\sqrt{2}}{2} = t
 \end{array}
 \qquad
 \begin{array}{l}
 \text{either } c = 4\sqrt{2} \\
 \text{and } t = \frac{\sqrt{2}}{2} \\
 \text{or } c = -4\sqrt{2} \\
 \text{and } t = -\frac{\sqrt{2}}{2}
 \end{array}$$

$$\therefore P = \left((4\sqrt{2})\left(\frac{\sqrt{2}}{2}\right), \frac{(4\sqrt{2})}{\left(\frac{\sqrt{2}}{2}\right)} \right) = (4, 8)$$

9/10

Examiner Comments

M1 M1 A1 M1 A1 M1 A1 M1 M1 A0 (Way 1)

M1 M1 A1: Correct method leading to a correct tangent equation $y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$.

M1 A1: Correct method leading to a correct x-coordinate of A as $2ct$ and a correct y-coordinate of B as $\frac{2c}{t}$.

M1 A1: Correct method of applying $OB = 2OA$ to give a correct $t = \frac{\sqrt{2}}{2}$.

M1: Correct complete method of applying $\frac{1}{2}(OA)(OB) = 32$ to find a value for c .

Note: It is noted that $\frac{2ct \times \frac{2c}{t}}{2}$ is incorrectly simplified to give c^2 , but candidate's complete method is correct.

M1: Applies their $c = 4\sqrt{2}$ and their $t = \frac{\sqrt{2}}{2}$ to deduce a numerical value x_p (from ct) and a numerical value y_p (from $\frac{c}{t}$).

A0: Incorrect answer.

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Exemplar Question 1

1. Given that

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$$

(a) find the characteristic equation for the matrix \mathbf{A} , simplifying your answer. (2)

(b) Hence find an expression for the matrix \mathbf{A}^{-1} in the form $\lambda\mathbf{A} + \mu\mathbf{I}$, where λ and μ are constants to be found. (3)

(Total for Question 1 is 5 marks)

Mean Score 3.6 out of 6

Examiner Comments

This question is assessing the use of the Cayley-Hamilton theorem (spec ref 3.3).

This question was generally done well by candidates who were well prepared for this exam

Part (a) was done well by the majority of candidates, with a few candidates losing a mark due to not forming an equation, missing $= 0$.

Part (b) the question stated ‘Hence’ requiring candidates to use their answer to part (a) and the Cayley-Hamilton Theorem. A few candidates found \mathbf{A}^{-1} and then tried to express in terms of \mathbf{A} and \mathbf{I} , gaining no marks. Some candidates did not fully apply the Cayley Hamilton Theorem writing $\mathbf{A}^2 - 5\mathbf{A} + 2 = 0$ instead of $\mathbf{A}^2 - 5\mathbf{A} + 2\mathbf{I} = 0$. Candidates who use the theorem correctly went on to achieve the correct result.

Mark Scheme

Question	Scheme	Marks	AOs
1(a)	$\det \begin{pmatrix} 3-\lambda & 2 \\ 2 & 2-\lambda \end{pmatrix} = 3-\lambda \quad 2-\lambda \quad -4 = 0$	M1	1.1b
	$\lambda^2 - 5\lambda + 2 = 0$	A1	1.1b
		(2)	
(b)	$\mathbf{A}^2 - 5\mathbf{A} + 2\mathbf{I} = 0$	B1ft	1.1b
	Multiplies through by \mathbf{A}^{-1} $\mathbf{A} - 5\mathbf{I} + 2\mathbf{A}^{-1} = 0$ and rearranges to get $\mathbf{A}^{-1} = \dots$ OR Rearranges to make \mathbf{I} the subject, takes out a factor of \mathbf{A} and rearranges to get $\mathbf{A}^{-1} = \dots$ $\mathbf{I} = \frac{5\mathbf{A} - \mathbf{A}^2}{2} = \mathbf{A} \frac{5\mathbf{I} - \mathbf{A}}{2} \Rightarrow \mathbf{A}^{-1} = \dots$ OR Rearranges to make \mathbf{I} the subject and multiplies through by \mathbf{A}^{-1} $\mathbf{I} = \frac{5}{2}\mathbf{A} - \frac{1}{2}\mathbf{A}^2 \Rightarrow \mathbf{A}^{-1} = \frac{5}{2}\mathbf{A}\mathbf{A}^{-1} - \frac{1}{2}\mathbf{A}^2\mathbf{A}^{-1}$	M1	3.1a
	Identifies $\mathbf{A}^{-1} = -\frac{1}{2}\mathbf{A} + \frac{5}{2}\mathbf{I}$	A1	1.1b
		(3)	
(5 marks)			
Notes			
(a)	M1: Complete method to find the characteristic equation, condone missing = 0 A1: Obtains a correct three term quadratic equation – may use any variable.		
(b)	B1ft: Uses Cayley Hamilton Theorem to produce equation replacing λ with \mathbf{A} and constant term with constant multiple of the identity matrix \mathbf{I} M1: A complete method using part (a) to find \mathbf{A}^{-1} Multiplies through by \mathbf{A}^{-1} and rearranges to get $\mathbf{A}^{-1} = \dots$ Or rearranges to make \mathbf{I} the subject, takes out a factor of \mathbf{A} , and rearranges to get $\mathbf{A}^{-1} = \dots$ Or rearranges to make \mathbf{I} the subject and multiplies through by \mathbf{A}^{-1} to get $\mathbf{A}^{-1} = \dots$ A1: Correct expression for \mathbf{A}^{-1} , must be using their answer to part (a).		

Student Response A

$$\det(A - \lambda I) \quad A\bar{x} = \lambda \bar{x}$$

$$= \begin{vmatrix} 3-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} \quad A - \lambda I = 0$$

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= (3-\lambda)(2-\lambda) - 4 = 0$$

$$6 - 3\lambda - 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 5\lambda + 2 = 0$$

$$(\lambda - 3)^2 - 9 + 2 = 0$$

$$(\lambda - 3)^2 = 7$$

$$\lambda = 3 \pm \sqrt{7}$$

1/5

Examiner Comments

In part (a)

M1: Complete method to find the characteristic equation, finding $\det(A - \lambda I)$

A0: Incorrect simplified characteristic equation

In part (b)

B0 ft: Does not use the Cayley Hamilton Theorem to produce equation replacing λ with A and constant term with constant multiple of the identity matrix I M0 A0: Does not use their equation to find an expression for A^{-1}

Student Response B

$$\begin{aligned} \text{a) } (3-\lambda)(2-\lambda)-4 &= 0 \\ 6-3\lambda-2\lambda+\lambda^2-4 &= 0 \\ \lambda^2-5\lambda+2 &= 0 \\ A^2-5A+2I &= 0 \end{aligned}$$

$$\begin{aligned} \text{b) } A^2 &= 5A - 2I \\ A^{-1} &= 5AA^{-1} - 2A^{-1} \\ A^{-1} &= 5A^{-2} - 2A^{-3} \end{aligned}$$

$$-5A = -A^2 - 2I$$

$$5A = A^2 + 2I$$

$$A = \frac{1}{5}(A^2 + 2I)$$

$$A^{-1} = \frac{1}{5}A^{-1} + 2A^{-1}$$

$$A^{-1} - 2A^{-1} = \frac{1}{5}A^{-1}$$

$$A^{-1}(1-2) = \frac{1}{5}A^{-1}$$

$$A^{-1}(-1) = -\frac{1}{5}A^{-1}$$

$$A^{-1} = \frac{1}{5}A + 0I$$

3/5

Examiner Comments

In part (a)

M1: Complete method to find the characteristic equation, finding $\det(\mathbf{A} - \lambda\mathbf{I})$

A1: Correct simplified characteristic equation

In part (b)

B1ft: Uses the Cayley Hamilton Theorem to produce equation replacing λ with \mathbf{A} and constant term with constant multiple of the identity matrix \mathbf{I} , seen in part (a) and used in part (b).M0 A0: Does not use a correct method to find an expression for \mathbf{A}^{-1} . There is an error on their 7th line of working for part (b).

Student Response C

$$a. \det(A - \lambda I) = 0$$

$$\det \begin{pmatrix} 3-\lambda & 2 \\ 2 & 2-\lambda \end{pmatrix} = 0$$

$$(3-\lambda)(2-\lambda) - 4 = 0$$

$$\lambda^2 - 5\lambda + 2 = 0$$

b.

$$A^2 - 5A + 2I = 0 \quad \text{Cayleigh-Hamilton}$$

$$\therefore A - 5I + 2A^{-1} = 0$$

$$\therefore A^{-1} = -\frac{1}{2}A + \frac{5}{2}I$$

$$\lambda = -\frac{1}{2} \quad \mu = \frac{5}{2}$$

5/5

Examiner Comments

In part (a)

M1: Complete method to find the characteristic equation, finding $\det(\mathbf{A} - \lambda\mathbf{I})$

A1: Correct simplified characteristic equation

In part (b)

B1ft: Uses the Cayley Hamilton Theorem to produce equation replacing λ with \mathbf{A} and constant term with constant multiple of the identity matrix \mathbf{I} , seen in part (a) and used in part (b).M0: Uses a correct method to find an expression for \mathbf{A}^{-1} .A1: Correct expression for \mathbf{A}^{-1}

Exemplar Question 2

2. (i) Determine all the possible integers a , where $a > 3$, such that

$$15 \equiv 3 \pmod{a} \quad (2)$$

- (ii) Show that if p is prime, x is an integer and $x^2 \equiv 1 \pmod{p}$ then either

$$x \equiv 1 \pmod{p} \quad \text{or} \quad x \equiv -1 \pmod{p} \quad (3)$$

- (iii) A company has £13 940 220 to share between 11 charities.

Without performing any division and showing all your working, decide if it is possible to share this money equally between the 11 charities.

(2)

(Total for Question 2 is 7 marks)

Mean Score 5.3 out of 7

Examiner Comments

This question is assessing number theory, modular arithmetic and divisibility tests. Most candidates were able to attempt parts (a) and (b) however no candidates scored full marks for part (c)

Part (a) The majority of candidates were able to find at least one correct value for a and with a few gave all three correct values.

Part (b) The majority of candidates scored no marks in this part; many did not know how to start the question of square rooted both sides. A handful of candidates scored the first mark for realising that $x^2 - 1$ is divisible by p . The crux of the proof is that **p is prime** so that either $(x - 1)$ is divisible by p or $(x + 1)$ is divisible by p . This is the rigours element of the proof which was missed by all candidates.

Part (c) the majority of candidates where able to apply a divisibility test for 11. For the final mark candidates where expected to refer back to the context of the question when giving their conclusion, £13 940 220 is not divisible by 11 therefore, it is **not is it possible** to share this **money equally** between the 11 **charities**

Mark Scheme

Question	Scheme	Marks	AOs	
2(i)	For any correct value for $a = 4, 6$ or 12	M1	1.1b	
	For all three correct values for a and no extras $a = 4, 6$ & 12	A1	1.1b	
		(2)		
(ii)	$x^2 - 1$ is divisible by p OR $x^2 - 1 \equiv 0 \pmod{p}$ OR $p / x^2 - 1$	B1	1.1b	
	$\therefore x-1$ $x+1$ is divisible by p and since p is prime either $x-1$ is divisible by p or $x+1$ is divisible by p OR $\therefore (x-1)(x+1) \equiv 0 \pmod{p}$ and since p is prime either $x-1 \equiv 0 \pmod{p}$ or $x+1 \equiv 0 \pmod{p}$ OR $\therefore p / x-1$ $x+1$ and since p is prime either $p / x-1$ or $p / x+1$	M1	2.1	
	$\therefore x \equiv 1 \pmod{p}$ or $x \equiv -1 \pmod{p}$ *	A1*	1.1b	
		(3)		
(iii)	For selecting and performing a divisibility test for dividing by 11			
	$1 - 3 + 9 - 4 + 0 - 2 + 2 - 0 = 3$ or $1 - 3 + 9 - 4 + 0 - 2 + 2 - 0 + 0 - 0 = 3$ 3 is not divisible by 11 or $11 \nmid 3$	Sum odd = 12 Sum even = 9 Difference = 3 which is not 0 or divisible by 11	M1	1.1b
	Fully correct method with reason (must have correct sum ± 3) and conclusion £13 940 220 is not divisible by 11 Therefore, it is not possible to share this money equally between the 11 charities	A1	3.2a	
		(2)		
(7 marks)				
Notes				
(i)	M1: For an understanding of mod notation and finding a correct value for $a = 4, 6$ or 12 A1: For all three correct values for a and no extras $a = 4, 6$ & 12			
(ii)	see scheme			
(iii)	M1: For applying a divisibility test for dividing by 11 to £13 940 220 or 139 402 2000p A1: Fully correct method and concludes not divisible by 11 and interprets conclusion in context			

Student Response A

$$(i) \quad 15 \equiv 3 \pmod{a}$$

$$15 \equiv 3 \pmod{12}$$

$$15 \equiv 3 \pmod{6}$$

$$(ii) \quad x^2 \equiv 1 \pmod{p}$$

$$\Rightarrow x^2 = 1 \pmod{p}$$

$$x = \pm 1 \pmod{p}$$

$$(iii) \quad \text{even placed values: } 3, 4, 2, 0$$

$$\text{odd placed values: } 1, 9, 0, 2$$

$$(1+9+0+2) - (3+4+2+0) = 12 - 9 \\ = 3$$

$11 \nmid 3 \therefore$ it is not possible to have out the money equally.

2/7

Examiner Comments

In part (a)

M1: At least correct value of a , which is 6 or 12

A0: Does not find all three correct values for a .

In part (b)

No valid attempt

In part (c)

M1: Carries out a divisibility test for 11

A0: Conclusion does not have all the three elements in bold, (the last one is missing) it is **not possible** to share this **money equally** between the 11 charities.

Advice is to repeat the demand of the question.

Student Response B

$$i) 15 \equiv 3 \pmod{a}$$

Ans $15 - 3 = 12$
12 | 9

$$\boxed{4, 6, 12}$$

$$ii) x^2 \equiv 1 \pmod{p} \quad x-1 = jp$$

$$x^2 - 1 = kp$$

$$\frac{x^2 - 1}{k} = \frac{x-1}{j}$$

$$j(x^2 - 1) = k(x-1)$$

$$\frac{1}{k} \cdot j(x^2 - 1) = (x-1)$$

$$\frac{j}{k} = \frac{x-1}{(x-1)(x+1)}$$

$$\frac{j}{k} = \frac{1}{x+1}$$

$$\frac{k}{j} = x+1$$

$$\frac{k}{j} - 1 = x$$

Hence $x^2 \equiv 1 \pmod{p}$ is equal to $x \equiv 1 \pmod{p}$
as there exists a rational number

$$iii) \begin{array}{cccccc} + & - & + & - & + & - \\ 1 & 3 & 9 & 4 & 2 & 2 & 0 \end{array}$$

$$1 - 3 + 9 - 4 - 2 + 2 - 0$$

$$7 - 6 + 2 = 3$$

3 does not divide 11

thus can't be split equally.

$$ii) x^2 - 1 = kp$$

$$x-1 = jp$$

$$x^2 - 1 = jp(x+1)$$

$$x^2 - 1 = jp(x+1)$$

$$x^2 - 1 = jp(x+1)$$

Hence $x^2 \equiv 1 \pmod{p}$ equal to $x \equiv 1 \pmod{p}$

As jp and kp both integers.

Examiner Comments

In part (a)

M1 A1: Finds all three correct values for a and no extra values.

In part (b)

B1: For implying that $x^2 - 1$ is divisible by p by $x^2 - 1 = kp$

The key part of this proof is that p is prime

M0 A0: Does not state that **since p is prime** then as $x^2 - 1 = (x - 1)(x + 1)$ either $x + 1$ or $x - 1$ must be divisible by p

In part (c)

M1: Correctly carries out a divisibility test for 11

A0: Conclusion does not have all the three elements in bold, it is **not possible** to share this **money equally** between the 11 **charities**.

Advice is the repeat the demand of the question

Student Response C

$$i) \quad 15 \equiv 3 \pmod{a}$$

~~$$15 \equiv 3 \pmod{a}$$~~

$$15 = na + 3$$

$$na = 15 - 3$$

$$na = 12$$

$$a = \frac{12}{n} \quad \text{where } n \in \mathbb{Z}^+$$

$$\therefore a = 12, 6, 4 \quad \text{if } n = 1, 2, 3 \quad \therefore a > 3$$

$$ii) \quad x^2 \equiv 1 \pmod{p}$$

~~$$x^2 \equiv 1 \pmod{p}$$~~

~~$$x^2 \equiv 1 \pmod{p}$$~~

~~$$x^2 - 1 \equiv 1 - 1 \pmod{p}$$~~

$$x^2 - 1 \equiv 1 - 1 \pmod{p}$$

$$x^2 - 1 \equiv 0 \pmod{p}$$

$$(x-1)(x+1) \equiv 0 \pmod{p}$$

$$\therefore x-1 \equiv 0 \pmod{p} \quad \text{or} \quad x+1 \equiv 0 \pmod{p}$$

$$x-1+1 \equiv 0+1 \pmod{p} \quad \text{or} \quad x+1-1 \equiv 0-1 \pmod{p}$$

$$\therefore x \equiv 1 \pmod{p} \quad \text{or} \quad x \equiv -1 \pmod{p}$$

iii) If it is possible to share this money equally

~~with 11 people~~ between 11 charities

$$\text{then } 11 \mid 13940220$$

$$1-3+9-4+0-2+2-0 = 3$$

$$\therefore 11 \nmid 3$$

$$\therefore 11 \nmid 13940220$$

Therefore you cannot share £13940220 equally between 11 charities.

5/7

Examiner Comments

In part (a)

M1 A1: Finds all three correct values for a and no extra values.

In part (b)

B1: For implying that $x^2 - 1$ is divisible by p by $x^2 - 1 \equiv \text{mod } p$

The key part of this proof is that p is prime

M0 A0: Does not state that **since p is prime** then as $x^2 - 1 = (x - 1)(x + 1)$ either $x + 1 \equiv \text{mod } p$ or $x - 1 \equiv \text{mod } p$.

In part (c)

M1: Correctly carries out a divisibility test for 11

A1: Full conclusion which has all the three elements in bold, it is **not possible** to share this **money equally** between the 11 **charities**.

Exemplar Question 3

3. A curve C in the complex plane is described by the equation

$$|z - 1 - 8i| = 3|z - 1|$$

- (a) Show that C is a circle, and find its centre and radius.

(4)

- (b) Using the answer to part (a), determine whether $z = 3 - 3i$ satisfies the inequality

$$|z - 1 - 8i| \geq 3|z - 1|$$

(2)

- (c) Shade, on an Argand diagram, the set of points that satisfies both

$$|z - 1 - 8i| \geq 3|z - 1| \quad \text{and} \quad 0 \leq \arg(z + i) \leq \frac{\rho}{4}$$

(4)

(Total for Question 3 is 10 marks)

Mean Score 4.8 out of 10

Examiner Comments

This question assesses complex numbers, loci and regions in the Argand diagram (spec ref 4.1) Part (a) the majority of candidates knew that was required however a few candidates made errors in the algebra. The demand of the questions was that candidates needed to show that C is a circle, this required the candidate to draw the conclusion that the equation that they produce is that of a circle for the final mark.

Part (b) demanded that the candidates used the answer to part (a), finding the distance of z to the centre of the circle and comparing it with the radius. Candidates who substituted z into the inequality gained no marks. The question tested whether candidates can make connections between the parts of the question. A few candidates who made the connection incorrectly thought that the inequality was not satisfied.

Part (c) Candidates are advised to indicate key coordinates on any diagrams, such as the centre of the circle and the starting coordinate of the half-line. The majority of candidates drew a circle, some put the centre of their circle in the first quadrant instead of the fourth. Again, many candidates knew to draw a half-line but lost marks as they did not indicate the coordinate where it started. Those candidates who in part (b) who thought the inequality was not satisfied shade the incorrect area outside the circle instead of inside.

Mark Scheme

Question	Scheme	Marks	AOs	
3(a)	$x-1^2 + y-8^2 = 9 \left[x-1^2 + y^2 \right]$ <p style="text-align: center;">Or</p> $\sqrt{x-1^2 + y-8^2} = 3\sqrt{x-1^2 + y^2}$	M1	2.1	
	$8x^2 - 16x + 8y^2 + 16y - 56 = 0$	A1	1.1b	
	$x^2 - 2x + y^2 + 2y - 7 = 0$ so $x-1^2 + y+1^2 = 9$ and finds the centre and radius	M1	1.1b	
	Therefore, a circle with centre (1, -1) and radius = 3	A1	2.2a	
	(4)			
(b)	Distance = $\sqrt{3-1^2 + -3--1^2} = \dots$ or finds $d^2 = 3-1^2 + -3--1^2 = \dots$	M1	1.1b	
	Distance = $\sqrt{8} = 2.828 < 3 \therefore z = 3 - 3i$ satisfies the inequality Or $8 < 9 \therefore z = 3 - 3i$ satisfies the inequality	A1	2.2a	
	(2)			
(c)		Circle with their centre and radius	M1	1.1b
		Circle with centre in the fourth quadrant	A1	1.1b
		Half line drawn from (0, -1) and passes through the x-axis within the circle	M1	1.1b
		Correct region shaded	A1	2.2a
(4)				
(10 marks)				

Notes

(a)**M1:** Obtains an equation in terms of x and y using the given information. Condone

$$x-1^2 + y-8^2 = 3 \left[x-1^2 + y^2 \right] \text{ for this mark.}$$

A1: Expands and simplifies the algebra, collecting terms and obtains a correct equation.**M1:** Completes the square for their equation to find the centre and radius.**A1:** Deduces that it is a circle (may be seen anywhere in their solution) with centre $(1, -1)$ and radius = 3**(b)****M1:** Finds the distance between $(3, -3)$ and their centre or d^2 (note: correct centre is $(1, -1)$)**A1:** Compares distance with 3 or compares d^2 with 9 and deduces that the inequality is satisfied – must be using correct centre and radius.**(c)****M1:** Circle for their centre and radius.**A1:** Correct circle with centre in the fourth quadrant and passing through all four quadrants. Condone dotted circle.**M1:** Half line drawn from $(0, -1)$ and passing the x -axis within the circle. Condone dotted line.**A1:** Correct region shaded with both half-line and circle correct and not dotted.**Special case:** M1A1M1A0 if no coordinates stated throughout and it is clear that the half-line intersects the coordinate axes level with the correct centre of the circle.

Student Response A

$$a) \quad |z-1-8i| = 3 |z-1|$$

~~$$|x+iy-1-8i| = 3|x+iy-1|$$~~

$$|x+iy-1-8i| = 3|x+iy-1|$$

$$\sqrt{(x-1)^2 + (y-8)^2} = 3\sqrt{(x-1)^2 + (y)^2}$$

$$(x-1)^2 + (y-8)^2 = 9((x-1)^2 + y^2)$$

~~$$x^2 - 2x + 1 + y^2 - 16y + 64 = 9x^2 - 18x + 9y^2$$~~

~~$$8x^2 - 16x - 56 + 8y^2$$~~

← fit the circle formula

it is shown to be a circle.

~~$$= x^2 - 2x - 7 + y^2$$~~

~~$$= (x-1)^2 - 1 - 7 + y^2$$~~

$$8 = (x-1)^2 + y^2$$

$$\text{centre: } (1, 0)$$

$$\text{radius: } 2\sqrt{2} = (\sqrt{8})$$

~~$$b) \quad 3-1-3i-8i = 2-11i$$~~

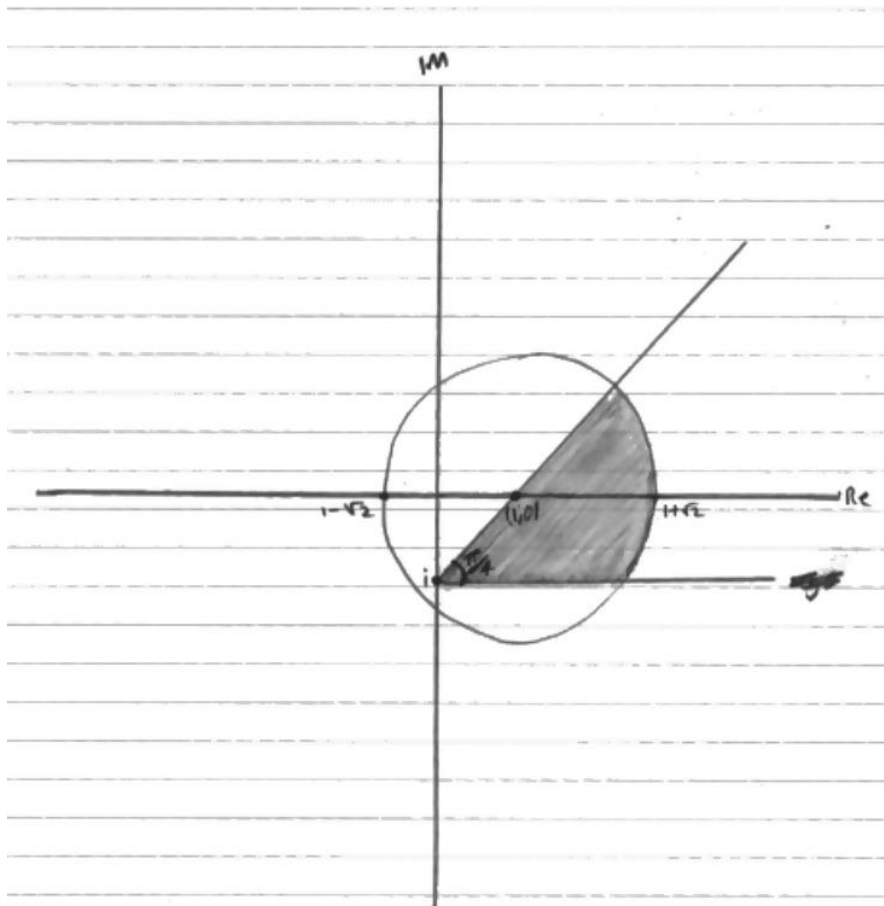
~~$$3-1-3i-8i = 2-11i$$~~

~~$$3|z-1| = 3|3-1-3i|$$~~

~~$$= 3(2-3i)$$~~

~~$$2-11i \neq 6-6i$$~~

~~∴~~ ∴ the point lies outside the curve.



4/10

Examiner Comments

In part (a)

M1: Correctly forms an equation commenting x and y using lengths.A0: Incorrect simplified equation, missing y term.

M1: Completed the square and finds the centre and radius of the circle

A0: They do state that they have an equation of a circle, however have an incorrect centre and radius.

In part (b)

M0 A0: The demand of the question is to use the answer to part (a), hence the centre coordinate, which does not happen.

In part (c)

M1: Draw a circle for their centre and radius.

A0: Incorrect circle for the question,

M1: The half-life drawn from $(0, -1)$

A0: Follows earlier A0

⑥

~~Handwritten scribble~~

$$z = 3 - 3i$$

$$x = 3$$

$$y = -3$$

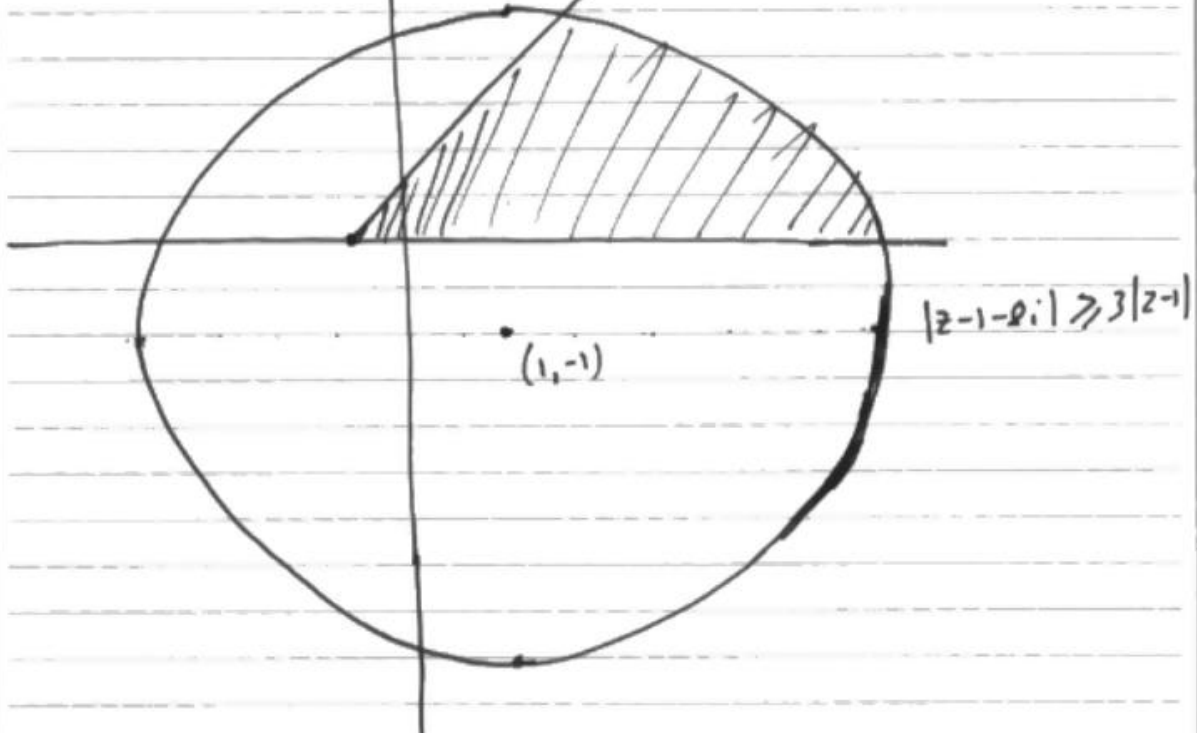
$$(x-1)^2 + (y+1)^2 = 4$$

$$(3-1)^2 + (-3+1)^2 =$$

$$2^2 + (-2)^2 = 8 \neq 4 \text{ so does not satisfy equation.}$$

⑦

$$0 \leq \arg(z+i) \leq \frac{\pi}{4}$$



6/10

Examiner Comments

In part (a)

M1: Correctly forms an equation commenting x and y using lengths.

A1: Correct simplified equation

M1: Completed the square and finds the centre and radius of the circle

A0: State the correct coordinate for the centre and radius of the circle, however part of the demand of the question is to show that it is a circle. The candidates does not draw the required conclusion that therefore it is a circle.

In part (b)

M1: Finds the distance between the centre of the circle and the coordinate $(3, -3)$.

A0: They need to compare their distance with 9 as an inequality, here they say $8 \neq 9$, should have $8 < 9$ therefore the inequality is satisfied.

In part (c)

M1 A1: Correct circle drawn.

M0: The half-life should be drawn from $(0, -1)$ not $(-1, 0)$

A0: Incorrect region shaded.

Student Response C

$$(a) \text{ let } z = x + iy$$

$$|(x-1) + i(y-8)| = 3 \quad |(x-1) + i(y-8)|$$

$$(x-1)^2 + (y-8)^2 = 9((x-1)^2 + y^2)$$

$$(x-1)^2 + (y^2 - 16y + 64) = 9(x-1)^2 + 9y^2$$

$$0 = 8(x-1)^2 + 8y^2 + 16y - 64$$

$$= 8x^2 - 16x + 8 + 8y^2 + 16y - 64$$

$$= 8x^2 - 16x + 8y^2 + 16y - 56$$

$$= x^2 - 2x + y^2 + 2y - 7$$

$$7 = x^2 - 2x + y^2 + 2y = (x-1)^2 + (y+1)^2 - 2$$

$$\therefore 9 = (x-1)^2 + (y+1)^2$$

$$r = 3 \quad C(1, -1)$$

$$(b) \quad z = 3 - 3i \quad \Rightarrow \quad x = 3 \quad y = -3$$

~~$$|3 - 3i - 1 - 8i| = |2 - 11i| = \sqrt{2^2 + 11^2} = 5\sqrt{5} \approx 11.18$$~~

~~$$3|3 - 3i - 1| = 3|2 - 3i| = 3\sqrt{13} \approx 3.60555$$~~

~~41218~~ $5\sqrt{5} > 3\sqrt{13} \quad \therefore$ Satisfys inequality

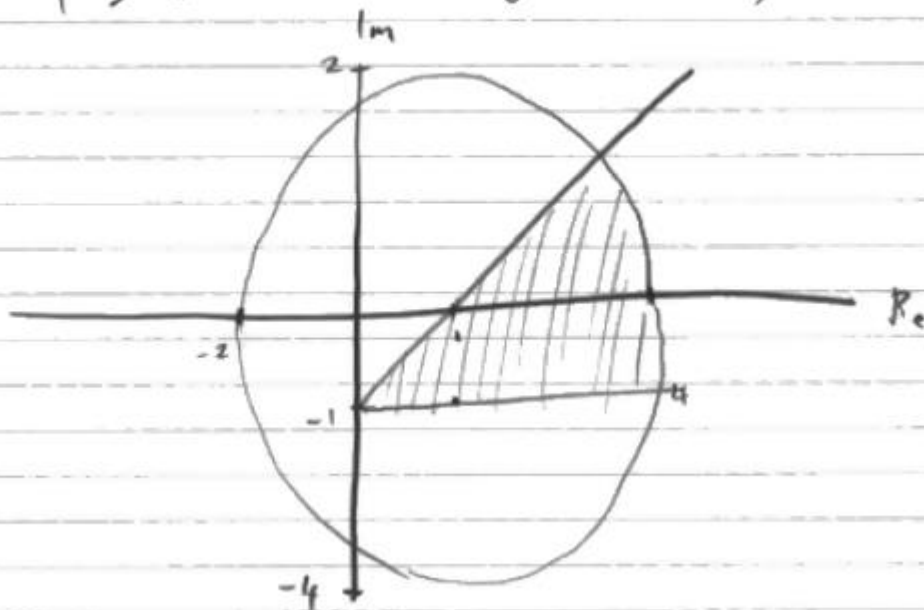
$$9 \geq (x-1)^2 + (y+1)^2$$

$$\geq (3-1)^2 + (-3+1)^2$$

$$\geq 4 + 4$$

$9 \geq 8 \quad \therefore$ Satisfys inequality

(c)



9/10

Examiner Comments

In part (a)

M1: Correctly forms an equation commenting x and y using lengths.

A1: Correct simplified equation

M1: Completed the square and finds the centre and radius of the circle

A0: State the correct coordinate for the centre and radius of the circle, however part of the demand of the question is to show that it is a circle. The candidates does not draw the required conclusion that therefore it is a circle.

In part (b)

M1: Finds the distance between the centre of the circle and the coordinate $(3, -3)$.

A1: Compares $8 \leq 9$ therefore satisfies the inequality.

In part (c)

M1 A1: Correct circle drawn.

M1: The half-life drawn from $(0, -1)$

A1: Correct region shaded.

Exemplar Question 4

4 The set $\{e, p, q, r, s\}$ forms a group, A , under the operation $*$

Given that e is the identity element and that

$$p * p = s \quad s * s = r \quad p * p * p = q$$

(a) show that

(i) $p * q = r$

(ii) $s * p = q$

(2)

(b) Hence copy and complete the Cayley table below.

$*$	e	p	q	r	s
e					
p					
q					
r					
s					

(2)

(c) Use your table to find $p * q * r * s$

(1)

A student states that there is a subgroup of A of order 3

(d) Comment on the validity of this statement, giving a reason for your answer.

(2)

(Total for Question 4 is 7 marks)

Mean Score 4.0 out of 7

Examiner Comments

This question is assessing groups, operations, Cayley tables, Lagrange's theorem and the order of a group (spec ref 1.2, 1.3, 1.4)

The majority of candidates were able to have a good attempt at this question on groups.

Part (a) Most of the candidates were able to prove at least of the two statements. A few did not start from the left-hand side so did not achieve the printed statement e.g. $p * p * p * p = s * s = r$ instead of $p * q = p * p * p * p = s * s = r$

Part (b) All the candidates were able to fill in some elements in the Cayley table, with most completing the first row and column using the identity element and the given results. Many then stopped at this point and did not complete the table.

Part (c) This required candidates to make reference to Lagrange's theorem, the order of a subgroup must be a factor of the order of the group. They needed to say that as 3 is not a factor of 5 the student's statement is incorrect. Again, candidates need to refer to the demand of the question, comment on the validity of the statement. Some candidates did not refer to the statement in their conclusion.

Mark Scheme

Question	Scheme	Marks	AOs																																				
4(a)	$p^*q = p^*p^*p^*p = s^*s = r$ OR $s^*s = r \Rightarrow p^*p^*p^*p = r \Rightarrow p^*q = r$	B1	2.1																																				
	$s^*p = p^*p^*p = q$ OR as $p^*p^*p = q$ and $p^*p = s \Rightarrow s^*p = q$	B1	2.1																																				
		(2)																																					
(b)	<table border="1"> <thead> <tr> <th>*</th> <th><i>e</i></th> <th><i>p</i></th> <th><i>q</i></th> <th><i>r</i></th> <th><i>s</i></th> </tr> </thead> <tbody> <tr> <td><i>e</i></td> <td><i>e</i></td> <td><i>p</i></td> <td><i>q</i></td> <td><i>r</i></td> <td><i>s</i></td> </tr> <tr> <td><i>p</i></td> <td><i>p</i></td> <td><i>s</i></td> <td><i>r</i></td> <td><i>e</i></td> <td><i>q</i></td> </tr> <tr> <td><i>q</i></td> <td><i>q</i></td> <td><i>r</i></td> <td><i>p</i></td> <td><i>s</i></td> <td><i>e</i></td> </tr> <tr> <td><i>r</i></td> <td><i>r</i></td> <td><i>e</i></td> <td><i>s</i></td> <td><i>q</i></td> <td><i>p</i></td> </tr> <tr> <td><i>s</i></td> <td><i>s</i></td> <td><i>q</i></td> <td><i>e</i></td> <td><i>p</i></td> <td><i>r</i></td> </tr> </tbody> </table>	*	<i>e</i>	<i>p</i>	<i>q</i>	<i>r</i>	<i>s</i>	<i>e</i>	<i>e</i>	<i>p</i>	<i>q</i>	<i>r</i>	<i>s</i>	<i>p</i>	<i>p</i>	<i>s</i>	<i>r</i>	<i>e</i>	<i>q</i>	<i>q</i>	<i>q</i>	<i>r</i>	<i>p</i>	<i>s</i>	<i>e</i>	<i>r</i>	<i>r</i>	<i>e</i>	<i>s</i>	<i>q</i>	<i>p</i>	<i>s</i>	<i>s</i>	<i>q</i>	<i>e</i>	<i>p</i>	<i>r</i>	M1 A1	1.1b 1.1b
	*	<i>e</i>	<i>p</i>	<i>q</i>	<i>r</i>	<i>s</i>																																	
	<i>e</i>	<i>e</i>	<i>p</i>	<i>q</i>	<i>r</i>	<i>s</i>																																	
	<i>p</i>	<i>p</i>	<i>s</i>	<i>r</i>	<i>e</i>	<i>q</i>																																	
	<i>q</i>	<i>q</i>	<i>r</i>	<i>p</i>	<i>s</i>	<i>e</i>																																	
	<i>r</i>	<i>r</i>	<i>e</i>	<i>s</i>	<i>q</i>	<i>p</i>																																	
<i>s</i>	<i>s</i>	<i>q</i>	<i>e</i>	<i>p</i>	<i>r</i>																																		
		(2)																																					
(c)	$p^*q^*r^*s = e$	B1	1.1b																																				
		(1)																																					
(d)	The order of a subgroup is a factor of the order of the group (Lagrange's Theorem)	M1	1.2																																				
	As 3 is not a factor of 5, the student's statement is wrong	A1	2.3																																				
		(2)																																					
(7 marks)																																							
Notes																																							
(a)	B1: Correct proof to achieve the printed statement B1: Correct proof to achieve the printed statement																																						
(b)	Marked B1 B1 on ePen M1: Finds at least 13 correct entries – usually the highlighted ones A1: Completely correct table																																						
(c)	B1: See scheme																																						
(d)	M1: Some indication that the order of a subgroup must be a factor of the order of the group. May say that 3 is not a factor of 5 or equivalent A1: Fully correct unambiguous statement that refers Lagrange's theorem and either <ul style="list-style-type: none"> • 3 is not a factor of 5 • 3 does not divide 5 • 5 is not divisible by 3 and comments that the student's statement is incorrect. No contradictory statements																																						

Student Response A

a)

i) $p * q = r$

$$p * q = s * s$$

$$= (p * p) * (p * p)$$

$$= (p * p * p) * p$$

$$= q * p$$

$$= r$$

if $p * q = s * s$
then $p * q = r$

ii) $s * p = p * p * p$

$$= (p * p) * p$$

$$= s * p$$

$$= q$$

if $s * p = p * p * p$ then $s * p = q$

(b) Hence complete the Cayley table below.

*	e	p	q	r	s
e	e	p	q	r	s
p	p	s	r	q	e
q	q	r	e		
r	r	e			
s	s	q			r

3/7

Examiner Comments

In part (a)

B1: Uses $q = p * p * p$ and $p * p = s$ then $s * s = r$ to show that $p * q = r$ B1: Use $p * p = s$ and $p * p * p = q$ to show that $s * p = q$

In part (b)

M1: At least 13 correct entries in the Cayley table.

A0: Does not complete the whole of the Cayley table

In part (c) and (d)

Not attempted

Student Response B

$$i) q = p \star p \star p$$

$$p \star p \star p \star p = p \star p \star p$$

$$p \star p = s$$

$$\therefore p \star p \star p \star p = s \star s = r$$

$$ii) s = p \star p$$

$$\therefore s \star p = p \star p \star p = q$$

$$\therefore s \star p = q$$

(b) Hence complete the Cayley table below.

*	e	p	q	r	s
e	e	p	q	r	s
p	p	s	r	e	q
q	q	r	s p	s	e
r	r	e	s	q	p
s	s	q	e	p	r

4/7

Examiner Comments

In part (a)

B0: Uses $q = p^*p^*p$ and $p^*p = s$ then $s*s = r$ but never achieves the printed answer $p^*q = r$

B1: Use $p^*p = s$ and $p^*p^*p = q$ to show that $s^*p = q$

In part (b)

M1 A1: Fully correct Cayley table

In part (c)

B1: Correct answer for $p^*q^*r^*s = e$

In part (d)

M0 A0: Does not consider the order of the group and the subgroup.

Student Response C

$$a. i. p * p = s$$

$$s * s = r$$

$$p * p * p = q$$

$$p * q = (p * p * p) * p = (p * p) * (p * p) \\ = s * s \\ = r$$

$$ii. s * p = q$$

$$p * p = s$$

$$p * p * p = q$$

(b) Hence complete the Cayley table below.

*	e	p	q	r	s
e	e	p	q	r	s
p	p	s	r	e	q
q	q	r	p	s	e
r	r	e	s	q	p
s	s	q	e	p	r

b.

$$\begin{array}{c}
 p \star r \\
 p \star s \star s \\
 p \star p \star p \star s \\
 q \star s \\
 q \star p \star p \\
 r \star p
 \end{array}$$

$$\begin{array}{c}
 q \star q = s \star s \star p \star p \\
 q \star q \\
 s \star s \\
 r \star r = s \star s \star s \\
 p \star p \star p \star p \\
 q \star q \star s \quad q \star p \star p \star s \\
 r \star p \star q \\
 r \star r \\
 q \star r = \dots \\
 q \star r \\
 q \star q \star s \\
 q \star q \star p \star s \\
 q \star q \star q \\
 q \star r \quad s \star p \star q \star r
 \end{array}$$

c. $p \star q \star r \star s$
 $r \star p$
 $e, p \star q \star r \star s = e$

d. Invalid, as A has order 5, and 3 doesn't divide 5, so that ~~the~~ subgroup can't exist

6/7

Examiner Comments

In part (a)

B1: Uses $q = p \star p \star p$ and $p \star p = s$ then $s \star s = r$ to show that $p \star q = r$

B1: Use $p \star p = s$ and $p \star p \star p = q$ to show that $s \star p = q$

In part (b)

M1 A1: Fully correct Cayley table

In part (c)

B1: Correct answer for $p \star q \star r \star s = e$

In part (d)

M1: Order of the group is 5 and 3 is not divisible by 5

A0: For this mark they need to comment on the validity of the statement, which this candidate does not do.

Exemplar Question 5

5. On Jim's 11th birthday his parents invest £1000 for him in a savings account.
The account earns 2% interest each year.

On each subsequent birthday, Jim's parents add another £500 to this savings account.

Let U_n be the amount of money that Jim has in his savings account n years after his 11th birthday, once the interest for the previous year has been paid and the £500 has been added.

- (a) Explain, in the context of the problem, why the amount of money that Jim has in his savings account can be modelled by the recurrence relation of the form

$$U_n = 1.02U_{n-1} + 500 \qquad U_0 = 1000 \qquad n \in \mathbb{Z}^+ \qquad (3)$$

- (b) State an assumption that must be made for this model to be valid. (1)

- (c) Solve the recurrence relation

$$U_n = 1.02U_{n-1} + 500 \qquad U_0 = 1000 \qquad n \in \mathbb{Z}^+ \qquad (5)$$

Jim hopes to be able to buy a car on his 18th birthday.

- (d) Use the answer to part (c) to find out whether Jim will have enough money in his savings account to buy a car that costs £4 500 (2)

(Total for Question 5 is 11 marks)

Mean Score 5.6 out of 11

Examiner Comments

This question is assessing the first order recurrence relations and solving them to obtain a closed form (spec ref 6.1, 6.2)

Part (a) when answering this part candidates needed to refer to the context of the question, e.g. not just saying 500 is added, but £500 is added each year.

Part (b) Only a few candidates were able to give a correct assumption for the model. Just saying that the interest does not change is insufficient, it is the interest **rate** that needs to remain the same. Candidates who said no money was removed from the account scored the mark.

Part (c) A minority of candidates did not know how to solve the recurrence relation and scored no further marks for this question. Those candidates who did know how to solve the recurrence relation did so very successfully by in splitting into CF + PS.

Part (d) needed candidates to use the answer to part (c) find the 7th term, which many did successfully who had an answer to part(c). With the conclusion candidates needed to compare the 7th term with £4 500 and comment on whether Jim will have enough money. Some candidates did not show the comparison and hence did not justify their conclusion.

Mark Scheme

Question	Scheme	Marks	AOs
5(a)	U_{n-1} is the amount in the saving account $n - 1$ years after Jim's 11 th birthday. This is increased by 2% each year, so is multiplied by 1.02 to give $1.02U_{n-1}$	B1	3.3
	Jim's parents invest £500 for each subsequent birthday so 500 is added	B1	3.4
	$U_0 = 1000$ as this is the amount invested on Jim's 11 th birthday	B1	1.1b
		(3)	
(b)	To use this model, one of, for example The interest rate stays the same each year Jim does not withdraw any money from the savings account Jim only saves the birthday money +£500 in this saving account, he does not invest any other money.	B1	3.5b
		(1)	
(c)	A complete method to solve the recurrence relation using $U_n = CF + PS = c \cdot 1.02^n + \lambda$	M1	3.1a
	$PS = \lambda \Rightarrow \lambda = 1.02\lambda + 500$ leading to $\lambda = \dots$	M1	1.1b
	$\lambda = -25\,000$	A1	1.1b
	Uses $U_0 = 1000$ and their value for λ to find the value of $1000 = c \cdot 1.02^0 - 25\,000$ $c = \dots 26\,000$	M1	1.1b
	$U_n = 26\,000 \cdot 1.02^n - 25\,000 \quad n \geq 0$	A1	1.1b
		(5)	
	Alternative 1		
	Realises that $U_n =$ term of a GP + sum of a GP both with $r = 1.02$	M1	3.1a
	Sum of a GP = $\frac{500(1-1.02^n)}{1-1.02}$ or $\frac{500(1.02^n-1)}{1.02-1}$	M1 A1	1.1b 1.1b
	Term of a GP = $1000 \cdot 1.02^n$ or $1000 \cdot 1.02^{n-1}$	M1	1.1b
	$U_n = 1000 \cdot 1.02^n - 25\,000(1-1.02^n)$ or $U_n = 1000 \cdot 1.02^n + 25\,000(1.02^n - 1)$	A1	1.1b
	(5)		
(d)	Uses $U_n = 26\,000 \cdot 1.02^n - 25\,000$, with either $n = 7$ or 8	M1	3.4
	$U_7 = 4865.83 > 4500$ therefore, Jim will have enough money in his savings account to buy a car costing £ 4500.	A1ft	2.2a
		(2)	

(11 marks)

Notes

(a)

B1: Need to explain that 2% interest rate linked to multiplication by scale factor 1.02**B1:** Need to explain that 500 is added due to receiving £500 each year**B1:** Needs to explain that $U_0 = 1000$ is the initial amount invested

(b)

B1: See main scheme

(c)

M1: A complete method to solve the recurrence relation using $U_n = CF + PS = c \cdot 1.02^n + \lambda$ **M1:** Uses $PS = \lambda \Rightarrow \lambda = 1.02\lambda + 500$ to find a value for λ **A1:** $\lambda = -25\,000$ **M1:** Uses U_0 and their value for λ to find a value of c **A1:** Fully correctly defined sequence $U_n = 26000 \cdot 1.02^n - 25\,000, \quad n \geq 0$ **Alternative 1****M1:** A correct form for U_n term of a GP + Sum of a GP both with $r = 1.02$ **M1:** For the sum of a GP with $a = 500, r = 1.02$ and uses n or $n - 1$ **A1:** Correct the sum of a GP with $a = 500, r = 1.02$ and n **M1:** For the term of a GP with $a = 1000, r = 1.02$ and uses n or $n - 1$ **A1:** Fully correctly defined sequence U_n

(d)

M1: Uses their U_n with either $n = 7$ or 8 **A1ft:** Finds U_7 compares with 4 500 and comes to an appropriate conclusion. Follow through on their value of U_7

Student Response A

d. Every year U_n , he will have the same (2)
 balance as last year plus 2% interest $U = 1.02 U_{n-1}$ plus
 £500 extra from his parents = $U = 1.02 U_{n-1} + 500$. U_0
 is the initial investment and the years must be natural
 numbers.

b. Jim doesn't spend any of the money

c. $U = 1.02^n U_0$

$$U_n = \lambda n + \mu$$

$$U_n = 1.02 U_{n-1} + 500$$

$$\lambda n + \mu = 1.02(\lambda(n-1) + \mu) + 500$$

$$0 = 1.02\lambda(n-1) + 0.02\mu - \lambda n$$

$$0 = 0.02\lambda n + 0.02\mu - 1.02\lambda$$

$$0.02\lambda = 0$$

$$0.02\mu + 1.02\lambda = 0 \quad \therefore \mu = 0$$

3/11

Examiner Comments

In part (a)

B1: Comments on last year's plus 2% interest therefore $1.02U_n$.

B1: Plus £500 extra from his parents

B0: Comments that U_0 is the initial investment but the value is not stated

In part (b)

B1: Jim does not spend any of the money

In part (c)

M0: Uses an incorrect general term of $U_n = (1.02)^n U_0 + \lambda n + \mu$

M0 A0: Incorrect PS used $\lambda n + \mu$

M0 A0: Does not use U_0 to find missing constant.

In part (d)

Not attempted

Student Response B

a) ~~The interest~~ The interest is based on the previous balance and £500 is added each time

b) Assumed that the interest is the same for each year and does not change.

$$c) \quad u_n = 1.02u_{n-1} + 500 \quad u_0 = 1000 \quad n \in \mathbb{Z}^+$$

$$u_n = C + CF + PS$$

$$= C(1.02)^n + PS$$

$$u_n = PS$$

$$u_n = \lambda$$

$$\lambda = 1.02\lambda + 500$$

$$-\frac{1}{50}\lambda = 500$$

$$\lambda = -25000$$

$$u_n = C(1.02)^n - 25000$$

$$1000 = C - 25000$$

$$26000 = C$$

$$u_n = 26000(1.02)^n - 25000$$

d

$$u_{18} = 26000(1.02)^{18} - 25000$$

$$= 12134.40244$$

5/11

Examiner Comments

In part (a)

B0: Does not comment on where the 1.02 has come from.

B0: Does not comment on where the £500 has come from

B0: No comment on U_0

In part (b)

B0: They have comments that the interest is the same each year. The amount of interest will not be the same but it assumes that the interest **rate** will be the same each year.

In part (c)

M1: Uses the general term is the Complementary function + Particular solution

$$U_n = c(1.02)^n + \lambda$$

M1: Correctly uses $PS = \lambda$ leading to $\lambda = 1.02\lambda + 500$ and finds a value for λ

A1: Correct value for λ

M1: Uses U_0 and their value for λ to find a value of c

A1: Correct sequence

In part (d)

M0 A0: Finds the 18th term and not either the 7th or 8th term. (note the first term is on the 11th birthday)

Student Response C

a, The previous year's balance is denoted by U_{n-1} .
It earns a 2% interest rate. It is therefore modelled as $1.02U_{n-1}$. On each subsequent birthday, £500 is added to the savings account so the balance is modelled by $1.02U_{n-1} + 500$.
 $U_0 = 1000$ because £1000 was put into the account when it first opened.

b,

$$G \quad U_n = 1.02U_{n-1} + 500 \quad U_0 = 1,000 \text{ } \odot$$

$$CF: \quad \cancel{C(1.02^n)} \quad C(1.02^n)$$

PS: form $\Rightarrow \lambda$

$$\lambda = 1.02\lambda + 500$$

$$-500 = 0.02\lambda$$

$$\lambda = -25,000 \text{ } \odot$$

$$U_n = C(1.02^n) - 25,000$$

$$n=0$$

$$U_0 = C(1.02^0) - 25,000 \text{ } \odot$$

$$1,000 = C - 25,000$$

$$C = 26,000 \text{ } \odot$$

$$\text{solution: } U_n = 26,000(1.02^n) - 25,000$$

at 18th birthday $\Rightarrow n=7$

$$U_7 = 26,000(1.02^7) - 25,000 \text{ } \odot$$

$$U_7 = 4865.827354$$

He has £4865.83 by his 18th birthday so he has enough money to buy the car.

Examiner Comments

In part (a)

B1: Comments on 2% interest therefore $1.02U_n$

B1: Comments that on each subsequent birthday £500 is added

B1: Comment that $U_0 = 1000$ as £1000 is put into the account when first opened.

In part (b)

B0: Not attempted

In part (c)

M1: Uses the general term is the Complementary function + Particular solution

$$U_n = c(1.02)^n + \lambda$$

M1: Correctly uses PS = λ leading to $\lambda = 1.02\lambda + 500$ and finds a value for λ

A1: Correct value for λ

M1: Uses U_0 and their value for λ to find a value of c

A1: Correct sequence

In part (d)

M1: Finds the value of the U_7

A0: They need to compare their value for U_7 with £4 500 the price of the car and then give a conclusion. Here there is no comparison.

A Level Further Mathematics – Further Pure 1 (9FM0 3A)

Exemplar Question 1

[back to Contents Page](#)

1. Use Simpson's rule with 4 intervals to estimate

$$\int_{0.4}^2 e^{x^2} dx$$

(5)

(Total for Question 1 is 5 marks)

Mean Score 4.4 out of 5

Examiner Comments

Candidates found this an accessible question at the beginning of the paper with the majority achieving full marks. Many candidates made a successful start and the first two marks, for finding h and evaluating the y -values, proved to be attainable by almost all candidates. Subsequent mistakes seen included the use of an incorrect number of intervals and mislabelling of their ordinates so that their odds and evens were confused in the formula. Several attempts were seen to use the trapezium rule.

A Level Further Mathematics (Further Pure 1) – 9FM0 3A Exemplar Question 1
Mark Scheme

Question	Scheme	Marks	AOs																														
1	Step length = 0.4	B1	1.1b																														
	<table border="1" style="width: 100%; border-collapse: collapse; margin: 0 auto;"> <thead> <tr> <th></th> <th>y_0</th> <th>y_1</th> <th>y_2</th> <th>y_3</th> <th>y_4</th> </tr> </thead> <tbody> <tr> <td>x</td> <td style="text-align: center;">0.4</td> <td style="text-align: center;">0.8</td> <td style="text-align: center;">1.2</td> <td style="text-align: center;">1.6</td> <td style="text-align: center;">2</td> </tr> <tr> <td>y</td> <td style="text-align: center;">$e^{0.16}$</td> <td style="text-align: center;">$e^{0.64}$</td> <td style="text-align: center;">$e^{1.44}$</td> <td style="text-align: center;">$e^{2.56}$</td> <td style="text-align: center;">e^4</td> </tr> <tr> <td></td> <td style="text-align: center;">1.173...</td> <td style="text-align: center;">1.896...</td> <td style="text-align: center;">4.220...</td> <td style="text-align: center;">12.935</td> <td style="text-align: center;">54.598</td> </tr> <tr> <td></td> <td></td> <td></td> <td style="text-align: center;">...</td> <td style="text-align: center;">...</td> <td></td> </tr> </tbody> </table>		y_0	y_1	y_2	y_3	y_4	x	0.4	0.8	1.2	1.6	2	y	$e^{0.16}$	$e^{0.64}$	$e^{1.44}$	$e^{2.56}$	e^4		1.173...	1.896...	4.220...	12.935	54.598					M1	1.1b
		y_0	y_1	y_2	y_3	y_4																											
	x	0.4	0.8	1.2	1.6	2																											
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		1.173...	1.896...	4.220...	12.935	54.598																											
																															
$y_0 + 4y_1 + 2y_2 + 4y_3 + y_4 = 123.54\dots$	M1	1.1b																															
$\int_{0.4}^2 e^{x^2} dx \approx \frac{0.4}{3} \times \{1.173\dots + 54.598\dots + 4(1.896\dots + 12.935\dots) + 2(4.220\dots)\}$ $\approx \frac{0.4}{3} \times "123.54\dots"$	dM1	1.1b																															
$= 16.5$	A1	1.1b																															
	(5)																																
(5 marks)																																	
Notes																																	
<p>B1: Correct step length of 0.4 which may be implied e.g. by their 0.4, 0.8, etc.</p> <p>M1: Attempts to find y values for their x values – may be in terms of e or numerical values. Must see an attempt to find at least 3 values.</p> <p>M1: Correct structure for y values of Simpson’s rule (ends + 2evens + 4odds) (must have an odd number of ordinates). Must be y values not x values.</p> <p>dM1: $\frac{"0.4"}{3} \times \text{their } 123.54\dots$ or for $\frac{h}{3} \times \text{their } 123.54\dots$ leading to a value and where h has clearly been defined earlier.</p> <p>Dependent on both previous method marks</p> <p>A1: Awrt 16.5</p> <p style="text-align: center;">Note that a minimum we would expect to see for full marks is:</p> <p style="text-align: center;">$h = 0.4$</p> <table border="1" style="width: 100%; border-collapse: collapse; margin: 0 auto;"> <thead> <tr> <th></th> <th>y_0</th> <th>y_1</th> <th>y_2</th> <th>y_3</th> <th>y_4</th> </tr> </thead> <tbody> <tr> <td>x</td> <td style="text-align: center;">0.4</td> <td style="text-align: center;">0.8</td> <td style="text-align: center;">1.2</td> <td style="text-align: center;">1.6</td> <td style="text-align: center;">2</td> </tr> <tr> <td>y</td> <td style="text-align: center;">$e^{0.16}$</td> <td style="text-align: center;">$e^{0.64}$</td> <td style="text-align: center;">$e^{1.44}$</td> <td style="text-align: center;">$e^{2.56}$</td> <td style="text-align: center;">e^4</td> </tr> <tr> <td></td> <td style="text-align: center;">1.173...</td> <td style="text-align: center;">1.896...</td> <td style="text-align: center;">4.220...</td> <td style="text-align: center;">12.935</td> <td style="text-align: center;">54.598</td> </tr> <tr> <td></td> <td></td> <td></td> <td style="text-align: center;">...</td> <td style="text-align: center;">...</td> <td></td> </tr> </tbody> </table> $A \approx \frac{h}{3} [y_0 + 4y_1 + 2y_2 + 4y_3 + y_4] = 16.5$ <p>(Note that a calculator gives 16.030... for the area)</p>					y_0	y_1	y_2	y_3	y_4	x	0.4	0.8	1.2	1.6	2	y	$e^{0.16}$	$e^{0.64}$	$e^{1.44}$	$e^{2.56}$	e^4		1.173...	1.896...	4.220...	12.935	54.598				
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Student Response A

$$\frac{h}{3}(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4)$$

$$\frac{0.4}{3}(0.4 + 4(0.8 + 1.6) + 2(1.2) + 2)$$

$$= \frac{2}{15} \left(\frac{72}{5} \right) = \frac{48}{25} = 1.92$$

$$h = \frac{2 - 0.4}{4} = \frac{2}{5}$$

1/5

Examiner Comments

This candidate states the correct step length but there is no attempt to find any of the y values and the subsequent attempt at Simpson's rule uses x values. In this case only the first B mark was scored for the correct step length.

Student Response B

$$\int_{0.4}^{2} e^{x^2} dx \quad y = e^{x^2}$$

x	0.4	0.8	1.2	1.6	<u>2</u>
y	1.173510871	1.896480879	4.22069587	12.93581732	54.59815

$$\frac{1}{3} \times \frac{3}{5} \left[1.173510871 + 54.59815 + 4(1.896480879 + 12.93581732) + 2 \times 4.22069587 \right]$$

$$\frac{1}{5} \left[55.77166087 + 59.3291928 + 8.441391634 \right]$$

$$\frac{1}{5} (123.54 \dots) = 24.70844906$$

$$= 24.71 \quad (2 \text{ d.p.})$$

3/5

Examiner Comments

This candidate scores the B mark for the correct step length (this was implied from their table of values). The appropriate y values are calculated which scores the first method mark and the structure for the y values for Simpson's rule is correct and so the second method mark is scored. When attempting the final value, this candidate uses $\frac{3}{5}$ for the step length rather than $\frac{2}{5}$ and so the final 2 marks are not scored.

Student Response C

x	0.4	0.8	1.2	1.6	2.0
e^{x^2}	1.17...	1.89...	4.22...	12.93...	54.59...

$$\int_{0.6}^2 e^{x^2} dx \approx \frac{1}{3} \times 0.4 \left[1.17... + 4(1.89... + 12.93...) + 2(4.22...) + 54.59... \right]$$

$$\approx \frac{2}{15} \left[123.54... \right]$$

$$\approx 16.4723 \quad (4 \text{ dp})$$

5/5

Examiner Comments

This candidate has a fully correct solution. Note that anything rounding to 16.5 was accepted for the final answer.

Exemplar Question 2

- 2 Given that k is a real non-zero constant and that

$$y = x^3 \sin kx$$

use Leibnitz's theorem to show that

$$\frac{d^5 y}{dx^5} = (k^2 x^2 + A)k^3 x \cos kx + B(k^2 x^2 + C)k^2 \sin kx$$

where A , B and C are integers to be determined.

(4)

(Total for Question 2 is 4 marks)**Mean Score 3.5 out of 4****Examiner Comments**

This question was the first of its kind on the new specification and it was obvious that most candidates had a good knowledge of how Leibnitz's theorem was applied, with only a few attempts at repeated differentiation seen. There were very few candidates who were unable to make good progress in this question.

Very few errors were seen in obtaining the derivatives that were required and there were similarly few errors seen in the structure of Leibnitz's theorem. The most common errors seen in this short question were errors when collecting terms to obtain the final answer, or transcription errors when students copied their expressions from one line to the next. For the candidates not familiar with Leibnitz Theorem, it was common to see no binomial coefficients present, or for them to be incorrectly attached to their terms.

Mark Scheme

Question	Scheme	Marks	AOs
2	$u = x^3 \Rightarrow \frac{du}{dx} = 3x^2, \frac{d^2u}{dx^2} = 6x, \frac{d^3u}{dx^3} = 6$	M1	1.1b
	$v = \sin kx \Rightarrow \frac{dv}{dx} = k \cos kx, \frac{d^2v}{dx^2} = -k^2 \sin kx, \frac{d^3v}{dx^3} = -k^3 \cos kx,$ $\frac{d^4v}{dx^4} = k^4 \sin kx, \frac{d^5v}{dx^5} = k^5 \cos kx$	M1	2.1
	$\frac{d^5y}{dx^5} = x^3 k^5 \cos kx + 5 \times 3x^2 \times k^4 \sin kx + \frac{5 \times 4}{2} \times 6x \times (-k^3 \cos kx) +$ $\frac{5 \times 4 \times 3}{3!} \times 6 \times (-k^2 \sin kx)$	M1	2.1
	$= (k^2 x^2 - 60)k^3 x \cos kx + 15(k^2 x^2 - 4)k^2 \sin kx$	A1	1.1b
		(4)	
(4 marks)			
Notes			
<p>M1: Differentiates $u = x^3$ three times. Need to see $x^3 \rightarrow \dots x^2 \rightarrow \dots x \rightarrow k$</p> <p>M1: Uses $v = \sin kx$ to establish the form of the derivatives. Need to see at least alternating $k \dots \sin kx$ and $k \dots \cos kx$ with increasing powers of k for at least 3 derivatives.</p> <p>M1: Uses a correct formula with 2 and 3! (or 6) with terms shown to disappear after the fourth term. This needs to be a correct application of the theorem so that the correct binomial coefficients need to go with the correct pairings of their derivatives. If there is any doubt, at least 3 terms should have the correct structure. Allow equivalent notation for the binomial coefficients e.g. $\binom{5}{0}, \binom{5}{1}$ etc. or ${}^5C_0, {}^5C_1$ etc.</p> <p>A1: Correct expression in the required form with correct values of A, B and C. Apply isw if necessary e.g. if a correct expression is followed by $A = 60, B = 15, C = -4$ (NB $A = -60, B = 15, C = -4$)</p> <p>If there is no use Leibnitz's theorem e.g. repeated differentiation of products, this scores no marks.</p>			

Student Response A

$$y = x^3 \sin kx.$$

~~$$\frac{dy}{dx} = 3x^2 \sin kx + x^3 \cos kx.$$~~

~~$$\frac{d^2y}{dx^2} = 6x \sin kx + 3x^2 \cos kx + (3x^2 \cos kx - x^3 \sin kx)$$~~
~~$$= 6x \sin kx + 3x^2 \cos kx + 3x^2 \cos kx - x^3 \sin kx.$$~~

~~$$\frac{d^3y}{dx^3} = 6 \sin kx + (6x \cos kx) +$$~~

$$\frac{dy}{dx} = 3x^2 \sin kx + kx^3 \cos kx.$$

~~$$\frac{d^2y}{dx^2} = 6x \sin kx + kx^3 \cdot (k \sin kx) +$$~~
~~$$= 6x \sin kx + k^2 x^3 \sin kx.$$~~

~~$$\frac{d^3y}{dx^3} = 6 \sin kx.$$~~

$$\frac{d^2y}{dx^2} = 6x \sin kx + 3x^2 \cdot k \cdot (-\cos kx) + 3kx^2 \cos kx + kx^3 \cdot k \cdot (-\sin kx)$$

$$= 6x \sin kx + 3x^2 k (-\cos kx) + 3kx^2 \cos kx - k^2 x^3 \sin kx$$

$$= 6x \sin kx - 3x^2 k \cos kx + 3kx^2 \cos kx - k^2 x^3 \sin kx$$

$$= 6x \sin kx - k^2 x^3 \sin kx.$$

$$\frac{d^3y}{dx^3} = kbx \cos kx + b \sin kx - (2k^2 x \sin kx + k^2 x^2 \cdot k \cdot \cos kx).$$

$$= 6bx \cos kx + 6 \sin kx - 2k^2 x \sin kx - k^3 x^2 \cos kx.$$

$$\frac{d^4 y}{dx^4} = bk \cos kx + \underbrace{(-bk^2 x \sin kx)} + \frac{bk \cos kx}{bk \cos kx} - \underbrace{2k^2 \sin kx} - 2k^3 x \cos kx$$

$$- \cancel{2k^2 x \cos kx} + k^4 x^2 \sin kx$$

$$= bk \cos kx - bk^2 x \sin kx - \cancel{2k^2 \sin kx} - 2k^2 \sin kx - 4k^3 x \cos kx + k^4 x^2 \sin kx$$

$$\frac{d^5 y}{dx^5} = \frac{bk^2}{bk \cos kx}$$

$$-bk^2 \sin kx - bk^2 \sin kx - bk^3 x \cos kx - 2k^3 \cos kx - 4k^3 \cos kx + 4k^4 x \sin kx$$

$$+ 2k^4 x \sin kx + k^5 x^2 \cos kx$$

$$= -12k^3 \sin kx - bk^3 x \cos kx + bk^4 x \sin kx + k^5 x^2 \cos kx - bk^3 \cos kx$$

$$= -12k^3 \sin kx + bk^4 x \sin kx + k^5 x^2 \cos kx - bk^3 x \cos kx - bk^3 \cos kx$$

$$= \sin kx (-12k^3 + bk^4 x) + \cos kx (k^5 x^2 - bk^3 x - bk^3)$$

or

0/4

Examiner Comments

This candidate does not use Leibnitz's Theorem and so no marks were scored.

Student Response B

 ~~$\frac{d^3y}{dx^3}$~~

$$u = x^3$$

$$v = \sin kx$$

$$\dot{u} = 3x^2$$

$$\dot{v} = k \cos kx$$

$$\ddot{u} = 6x$$

$$\ddot{v} = -k^2 \sin kx$$

$$\ddot{\ddot{u}} = 6$$

$$\ddot{\ddot{v}} = -k^3 \cos kx$$

$$\ddot{\ddot{\ddot{u}}} = 0$$

$$\ddot{\ddot{\ddot{v}}} = k^4 \sin kx$$

$$\ddot{\ddot{\ddot{\ddot{u}}}} = 0$$

$$\ddot{\ddot{\ddot{\ddot{v}}}} = k^5 \cos kx$$

$$\frac{d^5y}{dx^5} = \binom{5}{1} k^5 \cos kx \times \sin kx \times x^3$$

$$+ \binom{5}{2} k \cos kx \times 3x^2$$

$$+ \binom{5}{3} \times -k^2 \sin kx \times 6x$$

$$\times \binom{5}{4} \times -k^3 \cos kx + 6$$

$$+ 0 + 0 \dots$$

$$=$$

~~$$(k^2 x^2 +$$~~

$$5x^3 \sin kx + 30k(\cos kx)x^2$$

$$+ -60k^2 \sin(kx)x$$

$$-30k^3(\cos kx)$$

$$= (k^2 x^2 - 30)k^3 x \cos kx$$

$$+ (k^2 x^2 + \frac{-60}{0})k^2 \sin kx$$

2/4

Examiner Comments

This candidate scores the first 2 method marks for differentiating x^3 and $\sin kx$ appropriately. When applying Leibnitz's Theorem, the derivatives have been paired incorrectly and so the final 2 marks are not available.

Student Response C

$$y = x^3 \sin hx$$

$$\frac{d^3 y}{dx^3} = U \frac{d^3 V}{dx^3} + 3 \frac{dU}{dx} \times \frac{d^2 V}{dx^2} + 3 \frac{d^2 U}{dx^2} \times \frac{dV}{dx} + 10 \frac{d^3 U}{dx^3} \times \frac{d^0 V}{dx^0} + 3 \frac{d^2 U}{dx^2} \times \frac{d^2 V}{dx^2} + 3 \frac{dU}{dx} \times \frac{d^3 V}{dx^3} + \frac{d^3 U}{dx^3} \times V$$

$$U = x^3 \quad V = \sin(hx)$$

$$U' = 3x^2 \quad V' = h \cos(hx)$$

$$U'' = 6x \quad V'' = -h^2 \sin(hx)$$

$$U''' = 6 \quad V''' = -h^3 \cos(hx)$$

$$U^{(4)} = 0 \quad V^{(4)} = h^4 \sin(hx)$$

$$U^{(5)} = 0 \quad V^{(5)} = h^5 \cos(hx)$$

$$= h^5 x^3 \cos hx + 15x^2 h^4 \sin hx + 60x h^3 \cos(hx) + 60 h^2 (\sin hx)$$

$$= (h^2 x^3 + 60) h^3 x \cos(hx) + 15(h^2 x^2 + 4) h^2 \sin hx$$

4/4

Examiner Comments

This candidate has a fully correct solution.

Exemplar Question 3

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3.

$$\frac{dy}{dx} = x - y^2 \quad (\text{I})$$

(a) Show that

$$\frac{d^5y}{dx^5} = ay \frac{d^4y}{dx^4} + b \frac{dy}{dx} \frac{d^3y}{dx^3} + c \left(\frac{d^2y}{dx^2} \right)^2$$

where a , b and c are integers to be determined.

(4)

(b) Hence find a series solution, in ascending powers of x as far as the term in x^5 , of the differential equation (I), given that $y = 1$ at $x = 0$

(5)

(Total for Question 3 is 9 marks)**Mean Score 7.1 out of 9****Examiner Comments**

This was another question in which most candidates could make very good progress. Candidates appreciated the correct strategy required and found the first five derivatives, evaluated them at $x = 0$ and then substituted into the Maclaurin formula. A common error was to obtain $\frac{d^2y}{dx^2} = -2y \frac{dy}{dx}$ (losing the 1) but they were allowed to recover. Another error seen fairly often was the omission of the constant term in the second derivative but this did not prevent candidates from continuing to find the other derivatives required. Other errors noted by examiners included the miscopying of previous work and collecting terms incorrectly.

It was also common to see the product rule applied incorrectly in the 4th derivative which then led to the middle term on the 5th derivative giving a coefficient of -4 or -6 instead of -8 . In the subsequent work, there were errors in calculation of the values of the derivatives seen and also, less frequently, an incorrect structure for the Maclaurin series.

One calculation error seen on several candidates' work was in finding $\frac{d^2y}{dx^2} = 2$ (not 3). This gave the remaining derivatives as -6 , 24 and -120 leading to the series $1 - x + x^2 - x^3 + x^4 - x^5$. A few candidates quoted a series expansion without $1 + \dots$ despite obtaining all the other coefficients correctly and this was a costly error.

Mark Scheme

Question	Scheme	Marks	AOs
3(a)	$\frac{d^2 y}{dx^2} = 1 - 2y \frac{dy}{dx} \Rightarrow \frac{d^3 y}{dx^3} = -2y \frac{d^2 y}{dx^2} - 2 \left(\frac{dy}{dx} \right)^2$	M1 A1	1.1b 1.1b
	$\frac{d^4 y}{dx^4} = -2 \frac{dy}{dx} \frac{d^2 y}{dx^2} - 2y \frac{d^3 y}{dx^3} - 4 \frac{dy}{dx} \frac{d^2 y}{dx^2} = -6 \frac{dy}{dx} \frac{d^2 y}{dx^2} - 2y \frac{d^3 y}{dx^3}$	dM1	2.1
	$\begin{aligned} \frac{d^5 y}{dx^5} &= -6 \frac{dy}{dx} \frac{d^3 y}{dx^3} - 6 \left(\frac{d^2 y}{dx^2} \right)^2 - 2y \frac{d^4 y}{dx^4} - 2 \frac{dy}{dx} \frac{d^3 y}{dx^3} \\ &= -2y \frac{d^4 y}{dx^4} - 8 \frac{dy}{dx} \frac{d^3 y}{dx^3} - 6 \left(\frac{d^2 y}{dx^2} \right)^2 \end{aligned}$	A1	2.1
		(4)	
(b)	$x = 0, y = 1 \Rightarrow \left(\frac{dy}{dx} \right)_0 = 0 - 1^2 = -1$	B1	2.2a
	$\begin{aligned} \left(\frac{d^2 y}{dx^2} \right)_0 &= 1 - 2(1)(-1) = 3, \left(\frac{d^3 y}{dx^3} \right)_0 = -2(1)(3) - 2(-1)^2 = -8 \\ \left(\frac{d^4 y}{dx^4} \right)_0 &= -6(-1)(3) - 2(1)(-8) = 34, \\ \left(\frac{d^5 y}{dx^5} \right)_0 &= -2(1)(34) - 8(-1)(-8) - 6(3)^2 = -186 \end{aligned}$	M1 A1	1.1b 1.1b
	$y = y(0) + x \left(\frac{dy}{dx} \right)_0 + \frac{x^2}{2!} \left(\frac{d^2 y}{dx^2} \right)_0 + \frac{x^3}{3!} \left(\frac{d^3 y}{dx^3} \right)_0 + \frac{x^4}{4!} \left(\frac{d^4 y}{dx^4} \right)_0 + \frac{x^5}{5!} \left(\frac{d^5 y}{dx^5} \right)_0 + \dots$	M1	2.5
	With their values $(y =) 1 - x + \frac{3}{2}x^2 - \frac{8}{6}x^3 + \frac{34}{24}x^4 - \frac{186}{120}x^5 + \dots$ $(y =) 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 + \frac{17}{12}x^4 - \frac{31}{20}x^5 + \dots$	A1ft	1.1b
		(5)	

(9 marks)

Notes

(a)

M1: Attempts to find the second and third derivatives:

 This requires $\frac{d^2 y}{dx^2} = 1 \pm 2y \frac{dy}{dx}$ or $\frac{d^2 y}{dx^2} = \pm 2y \frac{dy}{dx}$ followed by $\frac{d^3 y}{dx^3} = \pm 2y \frac{d^2 y}{dx^2} \pm \dots$ or

$$\frac{d^3 y}{dx^3} = \pm \dots \pm 2 \left(\frac{dy}{dx} \right)^2$$

 A1: Correct second **and** third derivatives.

 dM1: Continues to differentiate to reach the 5th derivative. **This is dependent on the first method mark but there is no need to check the detail and the mark can be awarded as long as the 5th derivative is reached.**

 A1: Completes the process, collecting terms if necessary, to obtain the correct expression (NB $a = -2, b = -8, c = -6$)

Allow dash/dot notation for the derivatives but the final answer must be in the correct form.

Note that if $\frac{d^2y}{dx^2} = \pm 2y \frac{dy}{dx}$ is obtained initially, allow a full recovery in (a).

Note that (a) can be found using Leibnitz's theorem and the following scheme should be applied:

M1: $\frac{d^2y}{dx^2} = 1 \pm 2y \frac{dy}{dx}$ or $\frac{d^2y}{dx^2} = \pm 2y \frac{dy}{dx}$ followed by an attempt to differentiate y 3 times and $\frac{dy}{dx}$ 3 times.

A1: All correct

dM1: $\frac{d^5y}{dx^5} = -2 \frac{dy}{dx} \frac{d^4y}{dx^4} - 3 \times 2 \frac{dy}{dx} \frac{d^3y}{dx^3} - 3 \times 2 \left(\frac{d^2y}{dx^2} \right)^2 - 2y \frac{d^3y}{dx^3} \frac{dy}{dx}$ (correct application of Leibnitz)

A1: $= -2y \frac{d^4y}{dx^4} - 8 \frac{dy}{dx} \frac{d^3y}{dx^3} - 6 \left(\frac{d^2y}{dx^2} \right)^2$

As in the main scheme, if $\frac{d^2y}{dx^2} = \pm 2y \frac{dy}{dx}$ is obtained initially, allow a full recovery in (a).

Alternative for (a):

M1: $\frac{d^2y}{dx^2} = 1 - 2y \frac{dy}{dx} = 1 - 2y(x - y^2) = 1 - 2xy + 2y^3 \Rightarrow \frac{d^3y}{dx^3} = -2y - 2x \frac{dy}{dx} + 6y^2 \frac{dy}{dx}$

Score for the second derivative form as in the main scheme and then an attempt at the third derivative with at least 2 terms correct.

A1: Fully correct

Then as main scheme.

(b)

B1: Deduces the correct value for $y'(0)$

M1: Finds the values of all the other derivatives at $x = 0$ up to 5^{th} . There is no need to check their values as long as there is no obvious incorrect work, but values for all the derivatives up to the 5^{th} must be found

A1: All values correct (as single values – e.g. do not allow unsimplified)

M1: Applies the correct Maclaurin series for their values including the factorials up to the term in x^5

A1ft: Correct expansion, follow through their values for the derivatives. This does not have to be simplified but the factorials need to be evaluated. **Once a correct, or correct follow through expression is seen apply isw.**

Student Response A

$$a) \quad \frac{dy}{dx} = x - y^2$$

$$\frac{d^2y}{dx^2} = 1 - 2y \frac{dy}{dx}$$

$$\text{let } u = -2y, \quad v = \frac{dy}{dx}$$

$$\frac{du}{dx} = -2 \frac{dy}{dx} \quad \frac{dv}{dx} = \frac{d^2y}{dx^2} \cdot \frac{dy}{dx}$$

$$\frac{d^3y}{dx^3} = -2y \frac{d^2y}{dx^2} \cdot \frac{dy}{dx} - 2 \left(\frac{dy}{dx} \right)^2$$

$$= \frac{dv}{dx} \left[-2y \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} \right]$$

$$\text{let } u = \frac{dy}{dx} \quad v = -2y \frac{d^2y}{dx^2} - 2 \frac{dy}{dx}$$

$$\frac{du}{dx} = \frac{d^2y}{dx^2} \cdot \frac{dy}{dx}$$

$$b) \quad \mathbb{R} \quad y = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots$$

$$f(0) = 1$$

$$f'(0) = -1$$

$$f''(0) = -1$$

$$f'''(0) = -4$$

$$y = 1 - x$$

$$y = 1 - x - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \dots$$

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Examiner Comments

In part (a), this candidate scores the first method mark as they attempt to find the first and second derivatives and the first derivative is of the required form (correct in this case) and the second derivative has 2 terms, one of which is correct. The candidate then continues to differentiate again but does not reach the 5th derivative and so only the first mark is scored in part (a).

In part (b), the candidate obtains the correct value for the first derivative at $x = 0$ but does not find the values of all the other required derivatives and does not apply the Maclaurin expansion up to the term in x^5 . As a consequence, only the first mark is scored in part (b).

Student Response B

(a) $\frac{dy}{dx} = x - y^2$

~~$\frac{d^2y}{dx^2} = 1 - 2y \frac{dy}{dx}$~~

~~$\frac{d^3y}{dx^3} = -2 \frac{dy}{dx} - 2y \frac{d^2y}{dx^2}$~~

~~$\frac{d^4y}{dx^4} = -2 \frac{d^2y}{dx^2} - 2 \frac{d^2y}{dx^2} - 2y \frac{d^3y}{dx^3}$~~

~~$= -4 \frac{d^2y}{dx^2} - 2y \frac{d^3y}{dx^3}$~~

~~$\frac{d^5y}{dx^5} = -4 \frac{d^3y}{dx^3} - 2 \frac{d^4y}{dx^4} - 2 \frac{d^3y}{dx^3} - 2y \frac{d^4y}{dx^4}$~~

~~$= -2y \frac{d^4y}{dx^4} - 4 \frac{d^3y}{dx^3} \left(\frac{dy}{dx} \right) -$~~

$\frac{d^3y}{dx^3} = -2 \left(\frac{dy}{dx} \right) \left(\frac{dy}{dx} \right) - 2y \frac{d^2y}{dx^2} = -2 \left(\frac{dy}{dx} \right)^2 - 2y \frac{d^2y}{dx^2}$

$\frac{d^4y}{dx^4} = -2 \left(\frac{dy}{dx} \right)^2 - 2y \frac{d^3y}{dx^3}$

\downarrow

$= -4 \left(\frac{dy}{dx} \right) \left(\frac{d^2y}{dx^2} \right) - 2 \frac{d^2y}{dx^2} - 2y \frac{d^3y}{dx^3} - 2 \left(\frac{dy}{dx} \right) \left(\frac{d^2y}{dx^2} \right) - 2y \frac{d^3y}{dx^3}$

$\frac{d^5y}{dx^5} = -4 \frac{d^3y}{dx^3} - 4 \left(\frac{d^2y}{dx^2} \right)^2 - 2 \left(\frac{d^3y}{dx^3} \right)^2 - 2y \frac{d^4y}{dx^4}$

$\frac{d^5y}{dx^5} = -4 \frac{d^3y}{dx^3} - 2 \frac{d^2y}{dx^2} \left(\frac{d^2y}{dx^2} \right) - 2 \left(\frac{dy}{dx} \right) \frac{d^3y}{dx^3} - 2 \frac{dy}{dx} \left(\frac{d^3y}{dx^3} \right)$

$\left(-4 \frac{dy}{dx} \left(\frac{d^2y}{dx^2} \right) - 4 \left(\frac{dy}{dx} \right) \left(\frac{d^3y}{dx^3} \right) \right) - 2y \frac{d^4y}{dx^4}$

$= -4 - 6 \left(\frac{d^2y}{dx^2} \right)^2 - 4 \frac{dy}{dx} \left(\frac{d^3y}{dx^3} \right) - 2y \frac{d^4y}{dx^4}$

$a = -2, b = -4, c = -6$

$$(b) \left. \frac{dy}{dx} \right|_0 = -1$$

$$\left. \frac{d^2y}{dx^2} \right|_0 = 1 - 2 \times 1 \times (-1)$$

$$= 2$$

$$\left. \frac{d^3y}{dx^3} \right|_0 = -2(-1)(-1) - 2 \times 1 \times 2$$

$$= -2 - 4$$

$$= -6$$

$$\left. \frac{d^4y}{dx^4} \right|_0 = -4(-1) \times 2 - 2(-1)(2) - 2(-6)$$

$$= 8 + 4 + 12$$

$$= 24$$

$$\left. \frac{d^5y}{dx^5} \right|_0 = -6(2)^2 - 4(-1)(-6) - 2 \times 24$$

$$= -24 - 24 - 48$$

$$= -96$$

$$y = -x + \frac{-1}{2!} x^2 + \frac{-6}{3!} x^3 + \frac{24}{4!} x^4 + \frac{-96}{5!} x^5$$

$$= -x - x^2 - x^3 + x^4 - \frac{4}{5} x^5$$

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Examiner Comments

In part (a), this candidate scores the first 2 marks for obtaining the correct first and second derivatives. The candidate then continues to differentiate again and reaches the 5th derivative and so scores the second method mark but the final mark is lost as the 5th derivative is incorrect. Note that it is the uncrossed out work that is marked.

In part (b), the candidate obtains the correct value for the first derivative at $x = 0$ and goes on find the values of all the other required derivatives and so scores the first 2 marks. When attempting to apply the Maclaurin expansion, the “1 +” is missing and so this candidate does not score the final 2 marks in this part.

Student Response C

$$a) \frac{d^2 y}{dx^2} = 1 - 2y \frac{dy}{dx}$$

$$\frac{d^3 y}{dx^3} = -2 \left(\frac{dy}{dx} \right)^2 - 2y \frac{d^2 y}{dx^2}$$

$$\frac{d^4 y}{dx^4} = -4 \left(\frac{dy}{dx} \right) \left(\frac{d^2 y}{dx^2} \right) - 2y \frac{d^3 y}{dx^3} - 2 \frac{d^2 y}{dx^2} \frac{dy}{dx}$$

$$\frac{d^5 y}{dx^5} = -4 \left(\frac{d^2 y}{dx^2} \right)^2 - 4 \frac{dy}{dx} \frac{d^3 y}{dx^3} - 2 \frac{d^3 y}{dx^3} \frac{dy}{dx} - 2y \frac{d^4 y}{dx^4}$$

$$-2 \frac{d^3 y}{dx^3} \frac{dy}{dx} - 2 \left(\frac{d^2 y}{dx^2} \right)^2$$

$$= -2y \frac{d^4 y}{dx^4} - 8 \frac{dy}{dx} \frac{d^3 y}{dx^3} - 6 \left(\frac{dy}{dx} \right)^2$$

$$a = -2 \quad b = -8 \quad c = -6$$

b)

$$f(x) + x f'(x) + \frac{x^2}{2} f''(x) + \frac{x^3}{6} f'''(x) \dots$$

$$f'(1) = 0 - (1)^2 = -1$$

$$f'' = 1 - 2(1)(-1) = 3$$

$$f''' = -2(-1)^2 - 2(1)(3) = -8$$

$$f^{(4)} = -4(-1)(3) - 2(1)(-8) - 2(3)(-1) = 12 + 16 + 6 = 34$$

$$f^{(5)} = -2(1)(34) - 8(-1)(-8) - 6(3)^2 = -68 - 64 - 54 = -86$$

$$f(x) = 1 - x + \frac{3x^2}{2} - \frac{4x^3}{3} + \frac{17x^4}{12} - \frac{43x^5}{60}$$

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Examiner Comments

Part (a) is fully correct.

In part (b), the only error is with the value for the 5th derivative. Note that the final mark is a follow through mark and is scored by this candidate.

Exemplar Question 4

4. The parabola C has equation

$$y^2 = 16x$$

The distinct points $P(p^2, 4p)$ and $Q(q^2, 4q)$ lie on C , where $p \neq 0$, $q \neq 0$

The tangent to C at P and the tangent to C at Q meet at the point $R(-28, 6)$.

Show that the area of triangle PQR is 1331

(8)

(Total for Question 4 is 8 marks)

Mean Score 5.1 out of 8

Examiner Comments

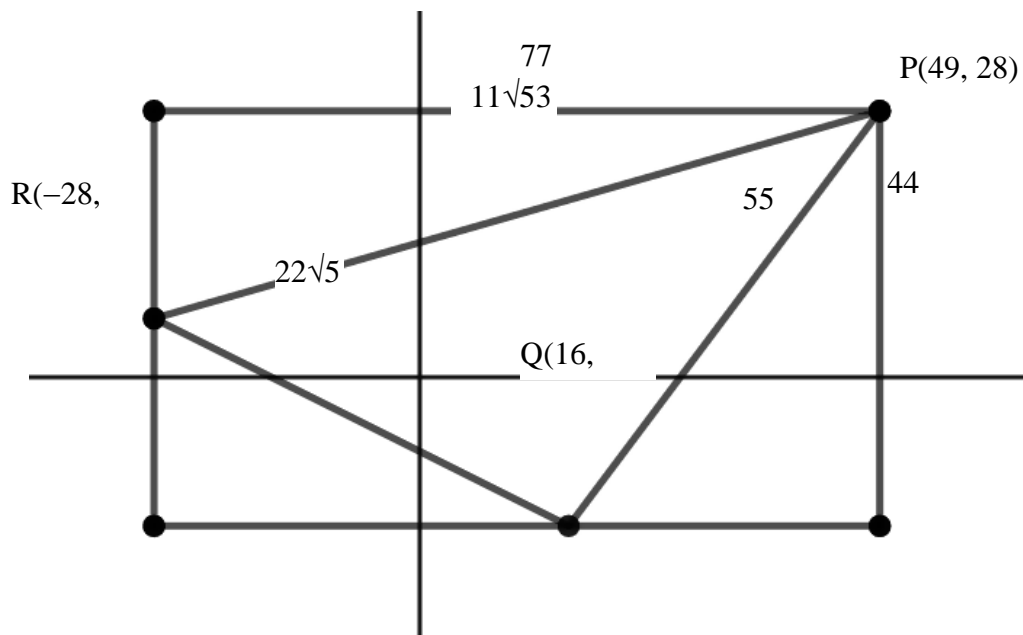
The first part of this question required candidates to produce the equation of a tangent to a parabola and then use the given point to find the coordinates of the two points where the two tangents touched the parabola. The most successful approach was to differentiate using the chain rule and then substitute in the given general P or Q coordinates to find an equation of the tangent at either P or Q . Candidates who drew a diagram realised that substituting in the given intersection point $R(-28, 6)$ would yield the p and q needed in order to obtain the specific coordinates of the points P and Q . Such attempts usually yielded the correct coordinates for those points. For others it was a case of resorting to simultaneous equations using their general tangent equations and then substituting in the coordinates of the point of intersection. However, candidates needed to work much harder in order to obtain the correct quadratic equations which yielded the values of p and q . The complexity of this algebra proved too much for some students. It was rare to see the resulting quadratic solved incorrectly.

The second part of the question involved finding the area of the triangle formed by the given point and the two points found in the first part. It was disappointing to see many candidates setting off to find this area without first doing a quick sketch to help them. Longer solutions involved using the cosine rule and then a triangle area formula. Concise, and simpler, solutions used the determinant method for finding the area of a triangle, enclosing the triangle in a rectangle and finding the area of the rectangle and subtracting the area of three right angled triangles and using one half of the modulus of the cross product of the vectors forming two of the sides of the triangle. It should be noted that the use of these more efficient ways of finding the area of a triangle were relatively rare.

Mark Scheme

Question	Scheme	Marks	AOs
4	$y^2 = 16x \Rightarrow 2y \frac{dy}{dx} = 16 \Rightarrow \frac{dy}{dx} = \frac{8}{y} = \frac{8}{4p}$ <p>Requires $\alpha y \frac{dy}{dx} = \beta \Rightarrow \frac{dy}{dx} = f(p \text{ or } q)$</p> $y^2 = 16x \Rightarrow \frac{dy}{dx} = 2x^{-\frac{1}{2}} = 2(p^2)^{-\frac{1}{2}}$ <p>Requires $\frac{dy}{dx} = \alpha x^{-\frac{1}{2}} \Rightarrow \frac{dy}{dx} = f(p \text{ or } q)$</p> $\frac{dy}{dx} = \frac{dy}{dp} \frac{dp}{dx} = \frac{4}{2p}$ <p>Requires $\frac{dy}{dx} = \text{their } \frac{dy}{dp} \div \text{their } \frac{dp}{dx} \Rightarrow \frac{dy}{dx} = f(p \text{ or } q)$</p>	M1	3.1a
	$\frac{dy}{dx} = \frac{8}{4p} \Rightarrow y - 4p = \frac{2}{p}(x - p^2) \text{ or } y - 4q = \frac{2}{q}(x - q^2)$	M1 A1	3.1a 1.1b
	Using $x = -28$ and $y = 6$, $6p = -56 + 2p^2 \Rightarrow p = \dots$	M1	3.1a
	<p>Alternative for 3rd Method mark:</p> $py = 2x + 2p^2, qy = 2x + 2q^2 \Rightarrow x = pq, y = 2(p + q)$ <p>Using $x = -28$ and $y = 6 \Rightarrow p(\text{or } q) = \dots$</p>		
	$p \text{ (or } q) = -4, 7$	A1	1.1b
	$(16, -16), (49, 28)$	A1	2.2a
	<p>Way 1</p> $\frac{1}{2} \begin{vmatrix} -28 & 16 & 49 & -28 \\ 6 & -16 & 28 & 6 \end{vmatrix} = \frac{1}{2} 448 + 448 + 294 - 96 + 784 + 784 $		
	<p>Way 2</p> $77 \times 44 - \frac{1}{2} \times 44 \times 22 - \frac{1}{2} \times 77 \times 22 - \frac{1}{2} \times 44 \times 33$		
	<p>Way 3</p> $\frac{1}{2} 22\sqrt{5} \times 11\sqrt{53} \sin \left(\cos^{-1} \left(\frac{(11\sqrt{53})^2 + (22\sqrt{5})^2 - 55^2}{2 \times 11\sqrt{53} \times 22\sqrt{5}} \right) \right)$ <p>NB angle at R is 42.5 (1dp)</p>	M1	3.1a
	<p>Way 4</p> $\frac{1}{2} 55 \times 11\sqrt{53} \sin \left(\cos^{-1} \left(\frac{(11\sqrt{53})^2 + 55^2 - (22\sqrt{5})^2}{2 \times 11\sqrt{53} \times 55} \right) \right)$ <p>NB angle at P is 37.2 (1dp)</p>		
	<p>Way 5</p> $\frac{1}{2} 55 \times 22\sqrt{5} \sin \left(\cos^{-1} \left(\frac{(22\sqrt{5})^2 + 55^2 - (11\sqrt{53})^2}{2 \times 22\sqrt{5} \times 55} \right) \right)$ <p>NB angle at Q is 100.3 (1dp)</p>		

	Way 6		
	$S = \frac{55 + 22\sqrt{5} + 11\sqrt{53}}{2} \Rightarrow A = \sqrt{S(S-55)(S-22\sqrt{5})(S-11\sqrt{53})}$		
	Way 7		
	Line PR $y - 28 = \frac{28-6}{49+28}(x-49), x=16 \Rightarrow y = \frac{130}{7}$		
	$A = \frac{1}{2} \times \frac{242}{7}(28+16) + \frac{1}{2} \times \frac{242}{7}(49-16)$		
	Way 8		
	$\frac{1}{2} RP \times QP = \frac{1}{2} \left \begin{pmatrix} 77 \\ 22 \end{pmatrix} \times \begin{pmatrix} 33 \\ 44 \end{pmatrix} \right = \frac{1}{2}(2662)$		
	For such methods, a minimum of e.g. $\frac{1}{2}(2662)$ must be seen		
	$= 1331 \text{ (units}^2\text{)*}$	A1*	1.1b
		(8)	
(8 marks)			
Notes			
<p>M1: Attempts to solve the problem by using differentiation to obtain an expression for $\frac{dy}{dx}$ in terms of p or q.</p> <p>See scheme for requirements for this mark depending on the method chosen. (Can be implied by a correct expression)</p> <p>M1: Correct straight line method to find the equation of the tangent using P or Q. If using $y = mx + c$, must reach as far as $c = \dots$</p> <p>A1: Obtains a correct general tangent at P or Q or both Note that if a correct tangent equation is quoted, the first 3 marks are available</p> <p>M1: Uses $x = -28$ and $y = 6$ with the values correctly placed in one of their tangent equations and attempts to solve the resulting 3TQ to obtain 2 values for p (or q). An alternative approach for this mark is to obtain equations for both tangents and solve simultaneously to obtain the coordinates for the intersection and then to use $x = -28$ and $y = 6$ to find values for p and q. Note that a calculator may be used for the simultaneous equations but answers must be correct for their equations if no working is shown.</p> <p>A1: Correct values A1: Deduces the correct coordinates of P and Q</p> <p>M1: Completes the problem by using a suitable complete correct method for finding the area of PQR – See examples – there will be others – in general, score M1 for a correct triangle area method for their values</p> <p>A1*: Correct area. Allow this mark even if the candidate reverts to decimals within their solution, providing all the working is correct.</p>			



Generally, using midpoints of sides is unlikely to be successful, however, the line from R to the midpoint of PQ is horizontal so this is a correct approach:

$$\text{Midpoint: } \left(\frac{49+16}{2}, \frac{28-16}{2} \right) = \left(\frac{65}{2}, 6 \right) \Rightarrow \text{Area} = \frac{1}{2} \left(\frac{65}{2} + 28 \right) \times 44 = 1331$$

Student Response A

$$y^2 = 16x$$

$$2y \frac{dy}{dx} = 16x$$

$$\frac{dy}{dx} = \frac{8x}{y}$$

$$R(-28, 6)$$

The tangent of P:

$$y - 4p = \frac{8p^2}{4p} (x - p)$$

$$y - 4p = 2p(x - p)$$

$$y = 2px - 2p^2 + 4p$$

$$6 - 4p = 2p(x - p)$$

$$6 - 4p = -16p - 2p^3$$

$$0 = -2p^3 - 12p - 6$$

→

The tangent of Q: $y - 4q = \frac{8q^2}{4q} (x - q)$

$$y - 4q = 2q(x - q)$$

$$y = 2qx - 2q^2 + 4q$$

~~Intersection P = (-28, 6)~~

$$2qx - 2q^2 + 4q = 2px - 2p^2 + 4p$$

$$0 = 4(p - q) + 2x(p - q) - 2(q^2 - p^2)$$

$$0 = 4(p - q) + 2x(p - q) + 2(p^2 - q^2)$$

When $x = -28$

$$0 = 4(p - q) - 16(p - q) + 2(p^2 - q^2)$$

$$= -12(p - q) + 2(p^2 - q^2)$$

$$26(p - q) = p^2 - q^2$$

$$\begin{cases} 6 = -16p - 2p^2 + 4p \\ 6 = -16q - 2q^2 + 4q \end{cases}$$

$$\text{A.P.A.R} = \frac{1}{2}$$

Examiner Comments

This candidate chooses to use implicit differentiation to establish the gradient at P . The differentiation attempt does not satisfy the conditions specified in the mark scheme (they have not differentiated the right hand side). The first method mark is therefore not scored but the second method mark is scored for the attempt at the tangent equation as this follows on from their incorrect gradient.

The subsequent work gains no credit as there is no attempt to find the coordinates of P or Q .

Student Response B

$$y^2 = 4(4)x$$

$$P(P^2, 4P) \quad x = P^2 \quad y = 4P$$

$$\frac{dx}{dP} = 2P \quad \frac{dy}{dP} = 4$$

$$\frac{dy}{dx} = \frac{4}{2P} = \frac{2}{P}$$

$$\therefore y - 4P = \frac{2}{P}(x - P^2)$$

$$Py - 4P^2 = 2x - 2P^2$$

$$Py - 2x = 4P^2 - 2P^2, \text{ tangent at } P$$

$$Py - 2x = 2P^2$$

$$\therefore \text{tangent at } Q, \quad Qy - 2x = 4Q^2 - 2Q^2$$

$$\hookrightarrow Qy - 2x = 2Q^2$$

tangents meet at $R(-28, 6)$

$$P(6) - 2(-28) = 2P^2$$

$$Q(6) - 2(-28) = 2Q^2$$

↓

$$6P + 56 = 2P^2 \rightarrow 2P^2 - 6P - 56 = 0$$

$$6Q + 56 = 2Q^2 \rightarrow 2Q^2 - 6Q - 56 = 0$$

$$(P-7)(P+4) = 0$$

$$P = 7, P = -4$$

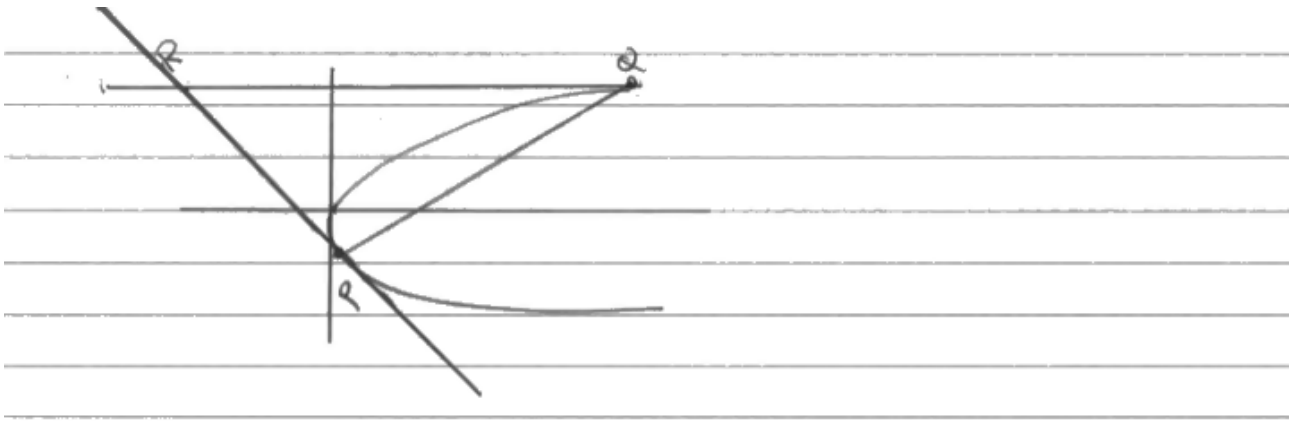
$$2Q^2 - 6Q - 56 = 0$$

$$2Q^2 - 6Q - 56 = 0$$

$$2Q^2 - 6Q - 56 = 0$$

$$(Q-7)(Q+4) = 0$$

$$Q = 7, Q = -4$$



4/8

Examiner Comments

This candidate has fully correct work leading to the correct values for p (and q) but it seems that the candidate does not know how to proceed and stops at this point.

Student Response C

101

$$4) \quad y^2 = 16x \quad P(p^2, 4p) \quad Q(q^2, 4q)$$

$$2y \frac{dy}{dx} = 16$$

$$\frac{dy}{dx} = \frac{8}{y}$$

$$y = \frac{8}{y} x$$

$$(y - 4p) = \frac{8}{y}(x - p^2)$$

So 4 Tangents are:

$$(y - 4p) = \frac{8}{y}(x - p^2) \quad y^2 - 4py = 8x - 8p^2$$

$$(y - 4q) = \frac{8}{y}(x - q^2) \quad y^2 - 4qy = 8x - 8q^2$$

$$36 - 24p = 8 - 8 \times 28 - 8p^2$$

$$8p^2 - 24p + 260 = 0$$

$$4) \quad P(p^2, 4p) \quad Q(q^2, 4q)$$

$$\frac{dy}{dx} = \frac{8}{y}$$

$$y = \frac{2}{p}x + c$$

$$y = \frac{8}{y}x + c \quad 4p = 2p + c \quad c = 2p$$

$$\text{So } y = \frac{2}{p}x + 2p$$

$$y = \frac{2}{q}x + 2q$$

$$6 = 1 - \frac{14}{p} + 2p$$

$$6p^2 - 14 = 2p^2 - 14p$$

$$2p^2 - 6p - 14 = 0$$

$$p = \frac{3 + \sqrt{37}}{2}, \quad \frac{3 - \sqrt{37}}{2}$$

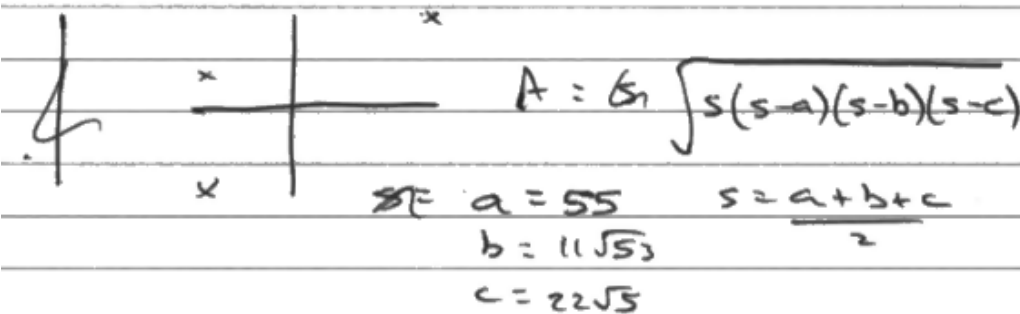
$$G = -\frac{56}{p} + 2p$$

$$2p^2 - 6p - 56 = 0 \quad p = 7, -4$$

$$q = 7, -4$$

$$\text{So } P(49, 28) \quad Q(16, -16)$$

$$R(-28, 6)$$



$$A = \frac{1}{2} \sqrt{s(s-a)(s-b)(s-c)}$$

$$\text{So } a = 55 \quad s = \frac{a+b+c}{2}$$

$$b = 11\sqrt{5}$$

$$c = 22\sqrt{5}$$

$$\sqrt{s(s-55)(s-11\sqrt{5})(s-22\sqrt{5})} = \underline{\underline{1331}}$$

8/8

Examiner Comments

This candidate has a fully correct solution. There were many alternatives seen for establishing the area of the triangle and in this case, Heron's formula was used successfully.

Exemplar Question 5

5.

$$I = \int \frac{1}{4\cos x - 3\sin x} dx \quad 0 < x < \frac{\rho}{4}$$

Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to show that

$$I = \frac{1}{5} \ln \left(\frac{2 + \tan\left(\frac{x}{2}\right)}{1 - 2 \tan\left(\frac{x}{2}\right)} \right) + k$$

where k is an arbitrary constant.

(8)

(Total for Question 5 is 8 marks)

Mean Score 5.9 out of 8

Examiner Comments

Most candidates could start to use the t formulae to express the given integral in terms of t . A minority of candidates forgot to change their dx in the integral to an expression involving dt . Candidates then proceeded to manipulate their integrand correctly to the required quadratic form. The two routes from there involved either partial fractions or completing the square. The negative ' t^2 ' term caused some problems as many candidates couldn't factorise the expression without bringing the minus out and this occasionally caused problems later on and such algebraic errors caused loss of 3 accuracy marks in the final part of the question.

When using the partial fraction method, a fraction with a denominator of $2t - 1$ was possible but few candidates could convincingly deal with changing this to $1 - 2t$ with $|2t - 1| = 1 - 2t$ seen with no justification. Others left their answer with modulus signs in thus losing the final mark since they had not reached the required answer. Other candidates expressed their denominator in the form $a^2 - x^2$ or $x^2 - a^2$ in order that a standard integral could be used. This method usually resulted in a term $\frac{1}{2} - t$ or $t - \frac{1}{2}$ in the denominator which frequently changed into the required $1 - 2t$ with no explanation so losing credit.

Mark Scheme

Question	Scheme	Marks	AOs
5	$4 \cos x - 3 \sin x = 4 \left(\frac{1-t^2}{1+t^2} \right) - 3 \left(\frac{2t}{1+t^2} \right)$	B1	1.1a
	$\frac{dt}{dx} = \frac{1+t^2}{2} \text{ or } \frac{dx}{dt} = \frac{2}{1+t^2} \text{ or } dx = \frac{2dt}{1+t^2} \text{ or } dt = \frac{1+t^2}{2} dx \text{ oe}$	B1 M1 on ePEN	2.1
	$\int \frac{1}{4 \cos x - 3 \sin x} dx = \int \frac{1}{4 \left(\frac{1-t^2}{1+t^2} \right) - 3 \left(\frac{2t}{1+t^2} \right)} \times \frac{2dt}{1+t^2}$	M1	2.1
	$= \int \frac{2}{4-4t^2-6t} (dt) \text{ or } \int \frac{1}{2-2t^2-3t} (dt) \text{ or } \int \frac{-1}{2t^2+3t-2} (dt)$ etc.	A1	1.1b
	$\frac{-2}{4t^2+6t-4} = \frac{-1}{(t+2)(2t-1)} = \frac{A}{(t+2)} + \frac{B}{(2t-1)}$ $\frac{-1}{(t+2)(2t-1)} = \frac{1}{5(t+2)} + \frac{2}{5(1-2t)}$	M1	3.1a
	$\Rightarrow I = \frac{1}{5} \int \frac{1}{(t+2)} - \frac{2}{(2t-1)} (dt) \text{ or equivalent}$	A1	1.1b
	$= \frac{1}{5} \int \frac{1}{(t+2)} - \frac{2}{(2t-1)} dt = \frac{1}{5} \ln(t+2) - \frac{1}{5} \ln(1-2t) (+k)$	A1	1.1b
	$= \frac{1}{5} \ln \left(\frac{2+t}{1-2t} \right) (+k) = \frac{1}{5} \ln \left(\frac{2+\tan(\frac{x}{2})}{1-2\tan(\frac{x}{2})} \right) + k^*$	A1*	2.1
	(8)		
Alternative for final 4 marks:			
	$= \int \frac{2}{4-4t^2-6t} (dt) = -\frac{1}{2} \int \frac{1}{t^2+\frac{3}{2}t-1} (dt) = -\frac{1}{2} \int \frac{1}{(t+\frac{3}{4})^2-\frac{25}{16}} (dt)$ or e.g. $\int \frac{1}{\frac{25}{8}-2(t+\frac{3}{4})^2} (dt)$	M1 A1	3.1a 1.1b
	$-\frac{1}{2} \times \frac{1}{2} \times \frac{4}{5} \ln \left(\frac{t+\frac{3}{4}-\frac{5}{4}}{t+\frac{3}{4}+\frac{5}{4}} \right) (+c)$	A1	1.1b
	$-\frac{1}{5} \ln \left \frac{\tan(\frac{x}{2})-\frac{1}{2}}{\tan(\frac{x}{2})+\frac{1}{2}} \right + c = \frac{1}{5} \ln \left \frac{\tan(\frac{x}{2})+2}{\tan(\frac{x}{2})-\frac{1}{2}} \right + c = \frac{1}{5} \ln \left(\frac{\tan(\frac{x}{2})+2}{\frac{1}{2}-\tan(\frac{x}{2})} \right) + c$ $= \frac{1}{5} \ln \left(\frac{2(\tan(\frac{x}{2})+2)}{1-2\tan(\frac{x}{2})} \right) + c = \frac{1}{5} \ln \left(\frac{(\tan(\frac{x}{2})+2)}{1-2\tan(\frac{x}{2})} \right) + \frac{1}{5} \ln 2 + c$ $= \frac{1}{5} \ln \left(\frac{(\tan(\frac{x}{2})+2)}{1-2\tan(\frac{x}{2})} \right) + k$	A1*	2.1
(8 marks)			

Notes

B1: Uses the **correct** formulae to express $4\cos x - 3\sin x$ in terms of t

B1(M1 on ePEN): Correct equation in terms of dx , dt and t – can be implied if seen as part of their substitution.

M1: Makes a **complete** substitution to obtain an integral in terms of t only. Allow slips with the substitution of “ dx ” but must be $dx = f(t)dt$ where $f(t) \neq 1$. This mark is also available if the

candidate makes errors when attempting to simplify $4\left(\frac{1-t^2}{1+t^2}\right) - 3\left(\frac{2t}{1+t^2}\right)$ before attempting the substitution.

A1: For obtaining a fully correct simplified integral with a constant in the numerator and a 3 term quadratic expression in the denominator. (“ dt ” not required)

M1: Realises the need to express the integrand in terms of partial fractions in order to attempt the integration. **Must have a 3 term quadratic expression in the denominator and a constant in the numerator.**

A1: Correct integral in terms of partial fractions – allow any equivalent **correct** integral. (“ dt ” not required)

A1: Fully correct integration in terms of t

A1*: Correct solution with no errors including “ $+ k$ ” (allow “ $+ c$ ”) and with the constant dealt with correctly if necessary. The denominator must also be dealt with correctly. E.g. if it appears as $2t - 1$ initially and becomes $1 - 2t$ without justification, this final mark should be withheld.

Alternative for final 4 marks:

M1: Realises the need to express the integrand in completed square form in order to attempt the integration. **Must have a 3 term quadratic expression in the denominator and a constant in the numerator.**

A1: Correct integral with the square completed – allow any equivalent **correct** integral (“ dt ” not required)

A1: Fully correct integration in terms of t

A1*: Correct solution with no errors including “ $+ k$ ” (allow “ $+ c$ ”) and with the constant dealt with correctly if necessary as shown in the scheme and with the denominator dealt with correctly if necessary.

Note that it is acceptable for the “ dt ” to appear and disappear throughout the proof as long as the intention is clear.

Student Response A

$$4\cos x - 3\sin x$$

$$= \frac{4(1-t^2)}{1+t^2} - \frac{3 \times 2t}{1+t^2}$$

$$= \frac{4-4t^2-6t^2}{1+t^2}$$

$$= \frac{4-10t^2}{1+t^2}$$

$$\therefore \frac{1}{4\cos x - 3\sin x} = \frac{1+t^2}{4-10t^2}$$

$$t = \tan\left(\frac{x}{2}\right)$$

$$\frac{dt}{dx} = \frac{1}{2} \sec^2\left(\frac{x}{2}\right) = \frac{1}{2\cos^2\left(\frac{x}{2}\right)}$$

$$dx = \frac{2 dt}{\sec^2\left(\frac{x}{2}\right)} = \frac{2 dt}{\cos^2\left(\frac{x}{2}\right)} = \frac{1}{2\cos^2\left(\frac{x}{2}\right) - 1 + 1}$$

$$= \frac{1}{\cos x + 1}$$

$$= \frac{1}{1-t^2+1+t^2}$$

$$= \frac{1+t^2}{2}$$

$$\therefore \frac{dt}{dx} = \frac{1+t^2}{2}$$

$$\therefore \frac{dx}{dt} = \frac{2}{1+t^2}$$

$$\therefore dx = \frac{2}{1+t^2} dt$$

$$4\cos x - 3\sin x$$

$$= 4\left(\frac{1-t^2}{1+t^2}\right) - 3\left(\frac{2t}{1+t^2}\right)$$

$$= \frac{4-4t^2-6t^2}{1+t^2}$$

$$= \frac{4-10t^2}{1+t^2}$$

$$\therefore \frac{1}{4\cos x - 3\sin x}$$

$$= \frac{1+t^2}{4-10t^2}$$

$$\therefore I = \int \frac{1}{4\cos x - 3\sin x} dx$$

$$= \int \frac{1+t^2}{4-10t^2} \cdot \frac{2}{1+t^2} dt$$

$$= \int \frac{2}{4-10t^2} dt.$$

$$= \int \frac{2}{(2-\sqrt{10}t)(2+\sqrt{10}t)} dt = \int \frac{\frac{1}{2} \times \frac{\sqrt{10}}{2\sqrt{10}}}{(2-\sqrt{10}t)} - \int \frac{\frac{1}{2}}{2+\sqrt{10}t}$$

$$\frac{A}{2-\sqrt{10}t} + \frac{B}{2+\sqrt{10}t} =$$

$$2A + \sqrt{10}At + 2B + \sqrt{10}tB = 2$$

$$2A + 2B = 2$$

$$A + B = 1$$

$$A = \frac{1}{2} \quad B = \frac{1}{2}$$

$$\sqrt{10}At - \sqrt{10}tB = 0$$

$$A - B = 0$$

2/8

Examiner Comments

This candidate has an incorrect expression for $\sin x$ in terms of t as they have $\frac{2t^2}{1+t^2}$ rather than $\frac{2t}{1+t^2}$. There is a correct expression for $\frac{dx}{dt}$ followed by a correct attempt to substitute into the integral and so this candidate scores B0B1M1 at the start. There is a subsequent attempt to use partial fractions but their integrand is not of the required form and no further marks are scored.

Student Response B

$$t = \tan\left(\frac{x}{2}\right) \Rightarrow \cos x = \frac{1-t^2}{1+t^2}, \sin x = \frac{2t}{1+t^2}$$

$$dx = \frac{2}{1+t^2} dt$$

$$\Rightarrow \int \frac{1}{4\cos x - 3\sin x} dx$$

$$= \int \frac{1}{4\left(\frac{1-t^2}{1+t^2}\right) - 3\left(\frac{2t}{1+t^2}\right)} \times \frac{2}{1+t^2} dt$$

$$= \int \frac{1+t^2}{4-4t^2-6t} \times \frac{2}{1+t^2} dt$$

$$= \int \frac{2}{2(2-2t^2-3t)} dt = \int \frac{1}{2-2t^2-3t} dt$$

$$= \int \frac{-1}{2t^2+3t-2} dt = \int \frac{1}{2\left(t^2+\frac{3}{2}t-1\right)} dt$$

$$= \int \frac{1}{2\left(\left(t+\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2 - 1\right)} dt$$

$$= \int \frac{1}{2\left(t + \frac{3}{4}\right)^2 - \frac{25}{16}} dt = \frac{1}{2} \int \frac{1}{\left(t + \frac{3}{4}\right)^2 - \frac{25}{16}} dt$$

$$= \frac{1}{2} \left[\frac{1}{2\sqrt{\frac{25}{16}}} \ln \left| \frac{t + \frac{3}{4} - \sqrt{\frac{25}{16}}}{t + \frac{3}{4} + \sqrt{\frac{25}{16}}} \right| \right] + k$$

$$= \frac{1}{5} \ln \left| \frac{t - \frac{1}{2}}{t + 2} \right| + k$$

$$= \frac{1}{5} \ln \left| \frac{\tan\left(\frac{x}{2}\right) - \frac{1}{2}}{\tan\left(\frac{x}{2}\right) + 2} \right| + k$$

$$= \frac{1}{5} \ln \left| \frac{2 + \tan\left(\frac{x}{2}\right)}{1 - 2\tan\left(\frac{x}{2}\right)} \right| + k$$

5/8

Examiner Comments

This candidate makes good progress initially. The first 5 marks are scored as the candidate proceeds to obtain a correct simplified integral in terms of t . The candidate then chooses to complete the square on their denominator which gains them the next method mark but in the process, the minus sign in the numerator is “lost” and so none of the subsequent accuracy marks are scored.

Student Response C

$$t = \tan \frac{x}{2} \quad \frac{dt}{dx} = \frac{1}{2} \sec^2 \frac{x}{2} = \frac{1}{2} (1 + \tan^2 \frac{x}{2})$$

$$\frac{dt}{dx} = \frac{1}{2} (1 + t^2)$$

$$\frac{2}{1+t^2} dt = dx$$

$$\cos t = \cos x = \frac{1-t^2}{1+t^2} \quad \sin x = \frac{2t}{1+t^2}$$

$$I = \int \frac{1}{\frac{4(1-t^2)}{1+t^2} - \frac{36t}{1+t^2}} \times \frac{2}{1+t^2} dt$$

$$= \int \frac{2}{4-4t^2-6t} dt = \int \frac{1}{2-2t^2-3t} = \int \frac{-1}{2t^2+3t-2}$$

$$= \int \frac{-1}{(2t-1)(t+2)} dt = \int \frac{1}{5(2t-1)} - \frac{2}{5(t+2)}$$

$$-1 = A(2t-1) + B(t+2) \quad \frac{-1}{(2t-1)(t+2)} = \frac{1}{5(t+2)} - \frac{2}{5(2t-1)}$$

$$t = \frac{1}{2} \quad -1 = \frac{5}{2} B \quad B = -\frac{2}{5}$$

$$t = -2 \quad -1 = -5A \quad A = \frac{1}{5}$$

$$\begin{aligned}
 I &= \int \frac{1}{5(t+2)} - \frac{2}{5(2t-1)} dt \\
 &= \frac{1}{5} \ln|t+2| - \frac{1}{5} \ln|2t-1| + C \\
 &= \frac{1}{5} \ln \left| \frac{t+2}{2t-1} \right| + C \\
 &= \frac{1}{5} \ln \left| \frac{2 + \tan \frac{x}{2}}{2 \tan \frac{x}{2} - 1} \right| + C \\
 &= \frac{1}{5} \ln \frac{|2 + \tan \frac{x}{2}|}{|2 \tan \frac{x}{2} - 1|} + C
 \end{aligned}$$

$0 < x < \frac{\pi}{4}$: $\tan \frac{x}{2}$ $\tan \frac{\pi}{8} = 0.4142\dots$ $\tan 0 = 0$
 $0 < \frac{x}{2} < \frac{\pi}{8}$ $0 < \tan \frac{x}{2} < 0.4142$

$$2 \tan \frac{x}{2} - 1 < 0 < 2 \tan \frac{x}{2} < 0.8284$$

$$-1 < 2 \tan \frac{x}{2} - 1 < -0.1714$$

\therefore always negative in range $0 < x < \frac{\pi}{4}$

$$2 < 2 + \tan \frac{x}{2} < 2.414 \text{ always positive in range}$$

$$I = \frac{1}{5} \ln \left(\frac{2 + \tan \frac{x}{2}}{1 - 2 \tan \frac{x}{2}} \right) + C$$

as $|2 \tan \frac{x}{2} - 1| = 1 - 2 \tan \frac{x}{2}$ for $0 < x < \frac{\pi}{4}$

Examiner Comments

This candidate presents a fully correct response. Note that the candidate uses modulus notation when integrating but then obtains the printed answer including an explanation for the reversal of the terms in the denominator. Note also that c was allowed as the constant of integration rather than k .

Exemplar Question 6

6. The concentration of a drug in the bloodstream of a patient, t hours after the drug has been administered, where $t \leq 6$, is modelled by the differential equation

$$t^2 \frac{d^2C}{dt^2} - 5t \frac{dC}{dt} + 8C = t^3 \quad (\text{I})$$

where C is measured in micrograms per litre.

- (a) Show that the transformation $t = e^x$ transforms equation (I) into the equation

$$\frac{d^2C}{dx^2} - 6 \frac{dC}{dx} + 8C = e^{3x} \quad (\text{II})$$

(5)

- (b) Hence find the general solution for the concentration C at time t hours.

(7)

Given that when $t = 6$, $C = 0$ and $\frac{dC}{dt} = -36$

- (c) find the maximum concentration of the drug in the bloodstream of the patient.

(5)

(Total for Question 6 is 17 marks)

Mean Score 11.4 out of 17

Examiner Comments

Part (a) was poorly answered by a good number of candidates, with many incorrect attempts at applying both the product and chain rule seen. It was also common to see students trying to ‘fudge’ their answer to make it fit with the printed differential equation. The most successful approach seemed to be differentiating with respect to t after obtaining $\frac{dC}{dt}$. Differentiating with respect to a different variable seemed to cause difficulties in all but the more prepared candidates.

Some very poor responses were seen asserting that, for example,

$\frac{d^2C}{dt^2} = \frac{d^2C}{dx^2} \times \frac{dx^2}{dt^2}$. However, the majority of candidates made a sound attempt at part (b) with many fully correct solutions seen. There was a sound knowledge of the technique of solving a second order linear differential equation and only the occasional slip prevented a candidate from achieving the correct form for C in terms of x . Some candidates did not transform C into a function of t which did lead to some problems whilst attempting to fit their general solution to the initial conditions of the problem.

Part (c) was also well attempted with many fully correct solutions seen. By far the most common error here was to see students substitute -36 into an equation which had been differentiated with respect to x rather than t and in some instances also using $x = 6$ rather than $\ln 6$. However, some students rectified this by using -6×36 for the value of $\frac{dC}{dx}$ at $t = 6$.

Quite a few students also failed to give appropriate units in their final answer.

Mark Scheme

Question	Scheme	Marks	AOs
6(a)	Examples: $t = e^x \Rightarrow \frac{dt}{dC} = e^x \frac{dx}{dC}$ or $\frac{dC}{dx} = t \frac{dC}{dt}$ or $\frac{dC}{dt} = e^{-x} \frac{dC}{dx}$ or $\frac{dC}{dt} = \frac{1}{t} \frac{dC}{dx}$	M1	1.1b
	E.g. $\frac{dC}{dx} = t \frac{dC}{dt} \Rightarrow \frac{d^2C}{dx^2} \times \frac{dx}{dt} = t \frac{d^2C}{dt^2} + \frac{dC}{dt}$	dM1 A1	2.1 1.1b
	$\frac{d^2C}{dx^2} \times \frac{1}{t} = t \frac{d^2C}{dt^2} + \frac{1}{t} \frac{dC}{dx} \Rightarrow t^2 \frac{d^2C}{dt^2} = \frac{d^2C}{dx^2} - \frac{dC}{dx}$ $t^2 \frac{d^2C}{dt^2} - 5t \frac{dC}{dt} + 8C = \frac{d^2C}{dx^2} - \frac{dC}{dx} - 5 \frac{dC}{dx} + 8C$	dM1	2.1
	$\frac{d^2C}{dx^2} - 6 \frac{dC}{dx} + 8C = e^{3x} *$	A1*	1.1b
		(5)	
Mark (b) and (c) together and ignore labelling			
(b)	$m^2 - 6m + 8 = 0 \Rightarrow m = 2, 4$	M1	1.1b
	$(C =) Ae^{4x} + Be^{2x}$	A1ft	1.1b
	PI is $C = ke^{3x}$	B1	2.2a
	$\frac{dC}{dx} = 3ke^{3x}, \frac{d^2C}{dx^2} = 9ke^{3x} \Rightarrow 9k - 18k + 8k = 1 \Rightarrow k = -1$	M1	1.1b
	$C = Ae^{4x} + Be^{2x} - e^{3x}$	A1	1.1b
	$t = e^x \Rightarrow C = \dots$	M1	3.4
	$C = At^4 + Bt^2 - t^3$	A1	2.2a
	(7)		
(c)	$t = 6, C = 0 \Rightarrow 1296A + 36B - 216 = 0$	M1	3.4
	$\frac{dC}{dt} = 4At^3 + 2Bt - 3t^2 \Rightarrow -36 = 864A + 12B - 108$	M1	3.4
	$A = 0, B = 6 \Rightarrow C = 6t^2 - t^3$	A1	1.1b
	$\frac{dC}{dt} = 12t - 3t^2 = 0 \Rightarrow t = 4 \Rightarrow C = \dots$	ddM1	1.1b
	$C = 6(4)^2 - (4)^3 = 32 \mu\text{gL}^{-1}$	A1	3.2a
		(5)	
(17 marks)			

Notes

(a)

M1: Uses $t = e^x$ to obtain a correct equation in terms of $\frac{dC}{dx}$, $\frac{dC}{dt}$ and t (or e^x) or their reciprocals

dM1: Differentiates again **correctly** with the product rule and chain rule in order to obtain an equation involving $\frac{d^2C}{dt^2}$ and $\frac{d^2C}{dx^2}$. **This needs to be fully correct calculus work allowing sign errors only.**

A1: Correct equation.

dM1: Shows clearly their substitution into the differential equation (or equivalent work) in order to form the new equation. **Dependent on the first method mark and dependent on having obtained two terms for the second derivative.**

Allow substitution for $\frac{dC}{dx}$ and $\frac{d^2C}{dx^2}$ into equation (II) to achieve equation (I)

A1*: Fully correct proof with no errors

(b)

M1: Forms and solves a quadratic auxiliary equation $m^2 - 6m + 8 = 0$

A1ft: Correct form for the CF for their AE solutions **which must be distinct and real**

B1: Deduces the correct form for the PI (ke^{3x})

M1: Differentiates their PI, **which is of the correct form**, and substitutes their derivatives into the DE to find “ k ”

A1: Correct GS for C in terms of x (**this must be seen explicitly unless implied by subsequent work**)

M1: Links the solution to DE (II) to the solution of the model to find the concentration at time t

A1: Deduces the correct GS for the concentration

If a correct GS is fortuitously found in (b) (e.g. from an incorrect PI form, allow full recovery in (c).

(c)

M1: Uses the conditions of the model ($t = 6$, $C = 0$) to form an equation in A and B .

***Note that **is** acceptable to use their C in terms of x for this mark as long as they use $x = \ln 6$ when $C = 0$

M1: Uses the conditions of the model $\left(t = 6, \frac{dC}{dt} = -36\right)$ to form another equation in A and B .

***Note that it is **not** acceptable to use $\frac{dC}{dx} = -36$ with $x = \ln 6$, as it is necessary to use

$$\frac{dC}{dt} = \frac{dC}{dx} \frac{dx}{dt} \text{ e.g. } -36 = (4Ae^{4\ln 6} + 2Be^{2\ln 6} - 3e^{3\ln 6}) \times e^{-\ln 6} \text{ or } -216 = 4Ae^{4\ln 6} + 2Be^{2\ln 6} - 3e^{3\ln 6}$$

A1: Correct equation connecting C with t

ddM1: Uses a suitable method to find the maximum concentration. E.g. solves $\frac{dC}{dt} = 0$ for t and

substitutes to find C . Allow a solution that solves $\frac{dC}{dx} = 0$ for x and uses this correctly to find C .

Dependent on both previous method marks.

A1: Obtains $32 \mu\text{gL}^{-1}$ using the model. Units are required but allow e.g.

- micrograms per litre
- $\mu\text{g/L}$
- $\mu\text{g/l}$
- $\mu\text{g}t^{-1}$

Student Response A

~~(a) $t = e^x$~~

~~$e^{2x} \frac{d^2c}{dt^2} - 5e^x \frac{dc}{dt} + 8c = e^{3x}$~~

~~$\frac{dt}{dx} = e^x \therefore dt = e^x dx$~~

~~$\frac{d^2t}{dx^2} = e^x$~~ ~~$\frac{d^2}{dt^2} = e^{3x} \therefore dt = e^x dx^2$~~

~~$e^{2x} \frac{d^2c}{e^x dx^2} - \frac{5e^x dc}{e^x dx} + 8c = e^{3x}$~~

~~$t = e^x \frac{dt}{dx} = e^x \frac{d^2t}{dx^2} = e^x$~~

~~$\Rightarrow e^x \frac{d^2c}{dx^2} - 5 \frac{dc}{dx} + 8c = e^{3x}$~~

~~$e^{3x} (a) t = e^x$~~

$1 = e^x \frac{dx}{dt}$

$\therefore \frac{dx}{dt} = \frac{1}{e^x} \Rightarrow \frac{d^2x}{dt^2} = -\frac{1}{e^x} \frac{dx}{dt}$

$\therefore (1) = e^{2x} \frac{d^2c}{dt^2} - 5e^x \frac{dc}{dt} + 8c = t^3$

$dt = e^x dx$ ~~$\frac{d^2x}{dt^2} = -\frac{1}{e^{3x}}$~~

~~$dt^2 = -e^{3x} dx^2$~~

$-\frac{1}{e^x} \frac{d^2c}{dx^2} - 5 \frac{dc}{dx} + 8c = e^{3x}$

$$(b) \quad \frac{d^2c}{dx^2} - 6\frac{dc}{dx} + 8c = e^{3x}$$

$$\text{C.E.} \Rightarrow m^2 - 6m + 8 = 0$$

$$(m-2)(m-4) = 0$$

$$m=2 \text{ or } m=4$$

$$\therefore c = Ae^{2x} + Be^{4x}$$

$$\text{P.I.} \Rightarrow y = de^{3x} \quad \frac{dy}{dx} = 3de^{3x} \quad \frac{d^2y}{dx^2} = 9de^{3x}$$

$$\therefore 9de^{3x} - 18de^{3x} + 8de^{3x} = -de^{3x} = e^{3x}$$

$$d = -1 \Rightarrow \therefore c = Ae^{2x} + Be^{4x} - e^{3x}$$

$$(c) \text{ when } t=6 \quad c=0$$

$$0 = Ae^{12} + Be^{24} - e^{18}$$

$$\frac{dc}{dt} = -36 = 2Ae^{2x} + 4Be^{4x} - 3e^{3x}$$

$$= 2Ae^{12} + 4Be^{24} - 3e^{18}$$

$$2Ae^{12} + 2Be^{24} - 2e^{18} = 0$$

$$-2Be^{24} + e^{18} = 36$$

$$\therefore 2B = \frac{e^{18} - 36}{e^{24}}$$

$$B = \frac{e^{18} - 36}{2e^{24}}$$

$$Ae^{12} = \frac{e^{18} - Be^{24}}{2e^{24}}$$

$$A = \frac{e^{18} - \frac{e^{18} - 36}{2}}{e^{12}} = \frac{e^{18} + 36}{2e^{12}}$$

$$\therefore \frac{dc}{dt} = \frac{e^{18} + 36}{e^{12}} e^{2x} + \frac{2(e^{18} - 36)}{e^{24}} e^{4x} - 3e^{3x}$$

$$\frac{dc}{dt} = \frac{e^{18} + 36}{e^{12}} e^{2x} + \frac{2(e^{18} - 36)}{e^{24}} e^{4x} - 3e^{3x}$$

$$\therefore \frac{dc}{dt} = \frac{e^{18} + 36}{e^{12}} e^{2x} + \frac{2(e^{18} - 36)}{e^{24}} e^{4x} - 3e^{3x}$$

$$\frac{dc}{dt} = \frac{e^{18} + 36}{e^{12}} e^{2x} + \frac{2(e^{18} - 36)}{e^{24}} e^{4x} - 3e^{3x}$$

$= 0$ is the maximum.

$$\therefore \frac{e^{18} + 36}{e^{12}} + \frac{2(e^{18} - 36)}{e^{24}} e^{2x} - 3e^x = 0$$

$$e^x = 403.429015$$

$$\text{or } e^x = 201.714507$$

~~200~~

5/17

Examiner Comments

In part (a), this candidate makes no progress in establishing the result.

In part (b) this candidate scores the first 5 marks as they obtain the correct equation connecting C and x . As with many candidates, the final 2 marks in part (b) are not awarded as an equation connecting C and t is not obtained.

In part (b) this candidate uses $C = 0$ with $x = 6$ which is incorrect. They should be using $C = 0$ with $x = \ln 6$ as they have an equation connecting C with x not C with t and so no marks are scored in this part.

Student Response B

$$(a) \frac{dC}{dt} = \frac{dC}{dx} \times \frac{dx}{dt}$$

$$t = e^x$$

$$\frac{dt}{dx} = e^x$$

$$\frac{dx}{dt} = e^{-x}$$

$$\frac{dC}{dt} = e^{-x} \frac{dC}{dx}$$

$$\Rightarrow \frac{d^2C}{dt^2} = \frac{d^2C}{dx^2} \times \frac{dx}{dt} + \frac{dC}{dx} \times \frac{d^2x}{dt^2}$$

$$\frac{dt}{dx} = e^x$$

$$\frac{d^2t}{dx^2} = e^x$$

$$\frac{dx^2}{dt^2} = -x$$

$$\frac{d^2C}{dt^2} = \frac{d^2C}{dx^2} e^{-x} + e^{-x} \frac{dC}{dx}$$

$$\rightarrow e^{2x} \left(\frac{d^2C}{dx^2} e^{-x} + e^{-x} \frac{dC}{dx} \right)$$

$$-5e^x \left(e^{-x} \frac{dC}{dt} \right) + 8C = e^{2x}$$

$$\rightarrow e^x \frac{d^2C}{dx^2} + e^x \frac{dC}{dx} - 5 \frac{dC}{dx} + 8C = e^{2x}$$

$$(b) m^2 - 6m + 8 = 0$$

$$m = 4, 2$$

$$C = Ae^{4x} + Be^{2x}$$

$$C \rightarrow \lambda e^{3x}$$

$$\frac{dC}{dx} = 3\lambda e^{3x}$$

$$\frac{d^2C}{dx^2} = 9\lambda e^{3x}$$

$$9\lambda e^{3x} - 18\lambda e^{3x} + 8\lambda e^{3x} = e^{3x}$$

$$\lambda = -1$$

$$C = Ae^{4x} + Be^{2x} - e^{3x}$$

$$(c) 6 = e^x$$

$$x = \ln 6$$

$$0 = 1296A + 36B - 216$$

$$-36 = 4A \cdot 1296 + 2B \cdot 36 - 648$$

$$5184A + 72B = 612$$

$$1296A + 36B = 216$$

$$A = \frac{5}{72}$$

$$B = \frac{7}{2}$$

$$C = \frac{5}{72}e^{4x} + \frac{7}{2}e^{2x} - e^x$$

$$\frac{dC}{dx} = \frac{5}{18}e^{4x} + 7e^{2x} - e^x = 0$$

$$x \approx$$

$$C = \frac{5}{72}e^{4(6)} + \frac{7}{2}e$$

$$\text{or } \frac{dC}{dx}$$

$$C = \frac{117}{18} \text{ mg/l}$$

Examiner Comments

In part (a), this candidate scores the first mark for obtaining a correct equation but their method for the second derivative is incorrect and so the subsequent method and accuracy mark are lost.

The third method mark is still available for substituting into differential equation (1).

As with the previous response, in part (b) this candidate scores the first 5 marks as they obtain the correct equation connecting C and x but the final 2 marks are not awarded as an equation connecting C and t is not obtained.

In part (c), the candidate correctly uses $C = 0$ with $x = \ln 6$ but when attempting $\frac{dC}{dt} = -36$, the candidate does not use the chain rule and so no more marks are available in this part.

Student Response C

a) $t = e^x$

$$\downarrow$$

$$\frac{dt}{dx} = e^x$$

$$\frac{dx}{dt} = \frac{1}{e^x}$$

$$\frac{dc}{dt} = \frac{dc}{dx} \times \frac{dx}{dt}$$

$$\frac{dc}{dx} = \frac{dc}{dt} \div \frac{dx}{dt}$$

$$\therefore \frac{dc}{dx} = \frac{1}{e^x} \frac{dc}{dt} = \frac{1}{t} \frac{dc}{dt}$$

$$\frac{d^2c}{dt^2} = -\frac{1}{t^2} \frac{dc}{dt} + \frac{d^2c}{dx^2} \frac{dx}{dt} \times \frac{1}{t}$$

$$= -\frac{1}{t^2} \frac{dc}{dt} + \frac{1}{t^2} \frac{d^2c}{dx^2}$$

$$\therefore t^2 \left[-\frac{1}{t^2} \frac{dc}{dt} + \frac{1}{t^2} \frac{d^2c}{dx^2} \right] - 5t \left[\frac{1}{t} \frac{dc}{dt} \right] + 8c = (e^x)^3$$

$$\downarrow \qquad \qquad \downarrow$$

$$-\frac{dc}{dx} + \frac{d^2c}{dx^2} - 5 \frac{dc}{dx} + 8c = e^{3x}$$

$$\therefore \frac{d^2c}{dx^2} - 6 \frac{dc}{dx} + 8c = e^{3x}$$

b) C.F.

$$m^2 - 6m + 8 = 0$$

$$\downarrow$$

$$(m-4)(m-2) = 0$$

$$\therefore m=4, m=2$$

$$C = Ae^{4x} + Be^{2x}$$

Let P.I be

$$c = \lambda e^{3x}$$

$$\frac{dc}{dx} = 3\lambda e^{3x}$$

$$\frac{d^2c}{dx^2} = 9\lambda e^{3x}$$

$$\therefore (9\lambda e^{3x}) - 6(3\lambda e^{3x}) + 8(\lambda e^{3x}) = e^{3x}$$

$$\downarrow$$

$$-\lambda e^{3x} = e^{3x}$$

$$\therefore \lambda = -1$$

$$C = Ae^{4x} + Be^{2x} - e^{3x}$$

as $t = e^x$

$$\therefore C = At^4 + Bt^2 - t^3$$

$$c) t=6, C=0, \frac{dC}{dt} = -36$$

$$\frac{dC}{dt} = 4At^3 + 2Bt - 3t^2$$

$$\begin{cases} 0 = A(6)^4 + B(6)^2 - (6)^3 \\ -36 = 4A(6)^3 + 2B(6) - 3(6)^2 \end{cases}$$

$$\begin{cases} 1296A + 36B = 216 \\ 864A + 12B = 72 \end{cases} \Rightarrow \begin{cases} A = 0 \\ B = 6 \end{cases}$$

$$\therefore C = 6t^2 - t^3$$

$$\frac{dC}{dt} = 12t - 3t^2, \frac{dC}{dt} = 0$$

$$\therefore 12t - 3t^2 = 0$$

$$t(12 - 3t) = 0$$

$$t = 0, t = 4$$

exclude when $t = 0$

\therefore maximum at $t = 4$

$$t = 4$$

$$C = 6(4)^2 - (4)^3$$

$$\underline{\underline{C = 32 \text{ micrograms per litre}}}$$

17/17

Examiner Comments

This candidate has a fully correct and well-presented solution.

Exemplar Question 7

- 7 With respect to a fixed origin O , the points A , B and C have coordinates $(3, 4, 5)$, $(10, -1, 5)$ and $(4, 7, -9)$ respectively.

The plane Π has equation $4x - 8y + z = 2$

The line segment AB meets the plane Π at the point P and the line segment BC meets the plane Π at the point Q .

- (a) Show that, to 3 significant figures, the area of quadrilateral $APQC$ is 38.5

(6)

The point D has coordinates $(k, 4, -1)$, where k is a constant.

Given that the vectors \vec{AB} , \vec{AC} and \vec{AD} form three edges of a parallelepiped of volume 226

- (b) find the possible values of the constant k .

(4)

(Total for Question 7 is 10 marks)

Mean Score 3.8 out of 10

Examiner Comments

In part (a), it was rare to see a correct strategy for the area of the quadrilateral. Candidates understood the idea of how to find the intersection of the two lines; only numerical slips prevented candidates getting the correct coordinates for the two points. The most successful candidates here drew a diagram of the situation and were then able to find the required area by splitting it up into smaller triangles and using the cross product. It was very common to see students attempt to evaluate a cross product which would produce the area of a single triangle and then write down the given answer. Another common problem, which a diagram would have avoided, was to add the areas of overlapping triangles.

In part (b) the strategy was generally well known. Errors involved the use of incorrect vectors even though the correct formula was written down. Candidates knew how the scalar triple product related to the required volume although a spurious $\frac{1}{6}$ was seen in the formula at times.

Of those candidates that correctly evaluated the scalar triple product many ignored the fact that its value could be plus or minus 2 and also the requirements of the question which asked for the values of k .

Mark Scheme

Question	Scheme	Marks	AOs
7(a)	<p>Examples:</p> <p>Area $APQC = \text{Area } ABC - \text{Area } PBQ$ Area $APQC = \text{Area } APC + \text{Area } CPQ$ Area $APQC = \text{Area } APQ + \text{Area } AQC$</p> $\text{Area } APQC = \frac{1}{2} \mathbf{AQ} \times \mathbf{PC} $	M1	3.1a
	<p>Line $AB: r = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 10-3 \\ -1-4 \\ 5-5 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ -5 \\ 0 \end{pmatrix}$</p> <p>or</p> <p>Line $BC: r = \begin{pmatrix} 10 \\ -1 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 10-4 \\ -1-7 \\ 5+9 \end{pmatrix} = \begin{pmatrix} 10 \\ -1 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 6 \\ -8 \\ 14 \end{pmatrix}$</p>	M1	3.1a
	<p>$4(3+7\lambda) - 8(4-5\lambda) + 5 = 2 \Rightarrow \lambda = \dots \Rightarrow P$ is ...</p> <p>or</p> <p>$4(10+6\mu) - 8(-1-8\mu) + 5 + 14\mu = 2 \Rightarrow \mu = \dots \Rightarrow Q$ is ...</p> <p>$\left(\text{NB } \lambda = \frac{1}{4}, \mu = -\frac{1}{2} \right)$</p>	M1	2.1
	$P(4.75, 2.75, 5)$ and $Q(7, 3, -2)$	A1	1.1b
	<p>Area $ABC = \frac{1}{2} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & -5 & 0 \\ 6 & -8 & 14 \end{vmatrix} = \frac{1}{2} \sqrt{70^2 + 98^2 + 26^2}$</p> <p>Area $PBQ = \frac{1}{2} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5.25 & -3.75 & 0 \\ 3 & -4 & 7 \end{vmatrix} = \frac{1}{2} \sqrt{26.25^2 + 36.75^2 + 9.75^2}$</p> <p>Area $APQC = \frac{1}{2} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 1 & 7 \\ 0.75 & -4.25 & 14 \end{vmatrix} = \frac{1}{2} \sqrt{43.75^2 + 61.25^2 + 16.25^2}$</p> <p>NB: Area $APQ = 7.7004$, Area $AQC = 30.8018$, Area $CPQ = 23.101$, Area $APC = 15.4008$</p>	M1	2.1
	Area $ABC - \text{Area } PBQ = 38.5^*$	A1*	1.1b
		(6)	
(b)	$\overrightarrow{AB} = \begin{pmatrix} 7 \\ -5 \\ 0 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} 1 \\ 3 \\ -14 \end{pmatrix}, \overrightarrow{AD} = \begin{pmatrix} k-3 \\ 0 \\ -6 \end{pmatrix}$	M1	3.1a

$\overrightarrow{AB} \times \overrightarrow{AC} \cdot \overrightarrow{AD} = \begin{vmatrix} 7 & -5 & 0 \\ 1 & 3 & -14 \\ k-3 & 0 & -6 \end{vmatrix} = \dots$		
$\overrightarrow{AB} \times \overrightarrow{AC} \cdot \overrightarrow{AD} = 7 \times -18 + 5(-6 + 14k - 42)$	A1	1.1b
$7 \times -18 + 5(-6 + 14k - 42) = \pm 226 \Rightarrow k = \dots$	dM1	3.1a
$k = 2 \text{ or } \frac{296}{35}$	A1	1.1b
	(4)	

(10 marks)

Notes

(a)

M1: Identifies a correct strategy to determine the area of the required quadrilateral. The attempt does **not need to be complete** for this mark so one of the statements (or intentions) in the markscheme would be sufficient.

M1: Correct attempt to find the equation of the line AB or the line BC

M1: Uses **at least one** of their lines and the equation of the given plane to determine the value of **at least one** of the parameters and hence the coordinates of P or Q

A1: **Both** coordinates correct – allow as vectors and may be implied if for example the candidate calculates the vectors e.g. AP , AQ , CP , CQ without stating the coordinates explicitly

M1: Uses all the required information to **calculate appropriate areas correctly** leading to the area of the quadrilateral. Needs to be a complete method here.

A1*: Reaches 38.5 with no errors

(b)

M1: Adopts a correct strategy by finding suitable vectors and forming the scalar triple product. This is often done in 2 steps e.g.

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 70 \\ 98 \\ 26 \end{pmatrix} \text{ or } \overrightarrow{AB} \times \overrightarrow{AD} = \begin{pmatrix} 30 \\ 42 \\ 5k-15 \end{pmatrix} \text{ or } \overrightarrow{AC} \times \overrightarrow{AD} = \begin{pmatrix} -18 \\ 48-14k \\ -3k+9 \end{pmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} \cdot \overrightarrow{AD} = \begin{pmatrix} 70 \\ 98 \\ 26 \end{pmatrix} \cdot \begin{pmatrix} k-3 \\ 0 \\ -6 \end{pmatrix} = 70k - 210 - 156$$

$$\text{or } \overrightarrow{AB} \times \overrightarrow{AD} \cdot \overrightarrow{AC} = \begin{pmatrix} 30 \\ 42 \\ 5k-15 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ -14 \end{pmatrix} = 30 + 126 - 70k + 210$$

$$\text{or } \overrightarrow{AC} \times \overrightarrow{AD} \cdot \overrightarrow{AB} = \begin{pmatrix} -18 \\ 48-14k \\ -3k+9 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ -5 \\ 0 \end{pmatrix} = -126 + 70k - 240$$

If it is not clear that the vector product is being used, at least 2 of the components should be correct.

A1: Correct expression for the triple product in terms of k (should be $\pm(70k - 366)$)

Ignore the presence or absence of “1/6” for the first 2 marks

dM1: Realises that ± 226 is possible for the value of the triple product and attempts to solve to obtain 2 values for k . **Dependent on the previous method mark.**

A1: Correct values (must be exact)

Student Response A

$$a.) \text{ area of quadrilateral} = |\vec{AP} \times \vec{AC}|$$

The line passing through A and B is given by

$$L = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ -5 \\ 0 \end{pmatrix} \text{ or}$$

Point P has co-ordinates $\begin{pmatrix} 3+7\lambda \\ 4-5\lambda \\ 5 \end{pmatrix}$ and lies on Π

$$\Rightarrow 4(3+7\lambda) - 8(4-5\lambda) + 5 = 2$$

$$28\lambda + 12 + 40\lambda - 32 + 5 = 2$$

$$68\lambda = 17 \quad \lambda = \frac{1}{4}$$

$$\text{So } \vec{AP} \text{ has eqn: } \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 7 \\ -5 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{7}{4} \\ -\frac{5}{4} \\ 0 \end{pmatrix}$$

$$\vec{AC} = \begin{pmatrix} 1 \\ 3 \\ -14 \end{pmatrix}$$

$$\vec{AP} \times \vec{AC} = \begin{vmatrix} i & j & k \\ \frac{7}{4} & -\frac{5}{4} & 0 \\ 1 & 3 & -14 \end{vmatrix}$$

$$= i\left(\frac{7}{4} \cdot \frac{5}{4} \times 14\right) - j\left(\frac{7}{4} \times -14\right) + k\left(\frac{21}{4} + \frac{5}{4}\right)$$

$$= \frac{35}{2}i + \frac{49}{2}j + \frac{26}{4}$$

~~$$|\vec{AP} \times \vec{AC}| = \sqrt{\left(\frac{35}{2}\right)^2 + \left(\frac{49}{2}\right)^2 + \left(\frac{26}{4}\right)^2} = 30.8$$~~

~~$$|\vec{AP} \times \vec{AC}| = \sqrt{\left(\frac{35}{2}\right)^2 + \left(\frac{49}{2}\right)^2 + \left(\frac{26}{4}\right)^2} = 30.8$$~~

b.) AD. $(\vec{AB} \times \vec{AC}) = 226$

$$\vec{AB} = \begin{pmatrix} 7 \\ -5 \\ 0 \end{pmatrix} \quad \vec{AC} = \begin{pmatrix} 1 \\ 3 \\ -14 \end{pmatrix}$$

$$|\vec{AB} \times \vec{AC}| = \begin{vmatrix} i & j & k \\ 7 & -5 & 0 \\ 1 & 3 & -14 \end{vmatrix} = i(5 \times 14) - j(14 \times 7) + k(21 + 5)$$

$$= 70i - 98j + 26k$$

$$|\text{AD. } \vec{AB} \times \vec{AC}| = 226$$

$$\Rightarrow 70k - 98 \times 4 - 26 = 226$$

$$\text{or } 70k - 98 \times 4 - 26 = -226$$

$$k = 9.2 \quad \text{or } k = \frac{96}{35}$$

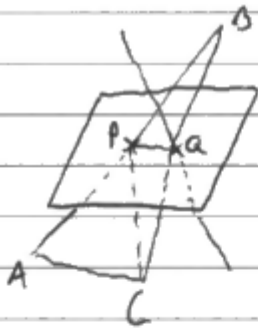
Examiner Comments

In part (a), there is no correct strategy for the required area but the candidate does adopt a correct strategy for finding the coordinates of the point P and so the 2nd and 3rd method marks are awarded.

When attempting the scalar triple product in part (b), this candidate uses the vector OD when they should be using AD . This means that the first method mark is not awarded and because the second mark is dependent, no marks are awarded in part (b)

Student Response B

7.



$$AB = \begin{vmatrix} 10 \\ -1 \\ 5 \end{vmatrix} - \begin{vmatrix} 3 \\ 4 \\ 5 \end{vmatrix} = \begin{vmatrix} 7 \\ -5 \\ 0 \end{vmatrix}$$

$$BC = \begin{vmatrix} 4 \\ 7 \\ -9 \end{vmatrix} - \begin{vmatrix} 10 \\ -1 \\ 5 \end{vmatrix} = \begin{vmatrix} -6 \\ 8 \\ -14 \end{vmatrix}$$

$$\left(\begin{vmatrix} 3 \\ 4 \\ 5 \end{vmatrix} + \lambda \begin{vmatrix} 7 \\ -5 \\ 0 \end{vmatrix} \right) \cdot \begin{vmatrix} 4 \\ -8 \\ 1 \end{vmatrix} = 12 + 28\lambda - 32 + 40\lambda + 5 = 2$$

$$68\lambda = 17 \quad \lambda = \frac{17}{68}$$

$$P = \begin{vmatrix} 17/68 \\ 11/4 \\ 5 \end{vmatrix} \quad \left(\begin{vmatrix} 4 \\ 7 \\ -9 \end{vmatrix} + \lambda \begin{vmatrix} -6 \\ 8 \\ -14 \end{vmatrix} \right) \cdot \begin{vmatrix} 4 \\ -8 \\ 1 \end{vmatrix} = 16 - 24\lambda - 56 - 64\lambda - 9 - 14\lambda = 2$$

$$-100 - 102\lambda = 51$$

$$Q = \begin{vmatrix} 7 \\ 3 \\ -2 \end{vmatrix}$$

$$\lambda = -\frac{1}{2}$$

~~APVU~~ $CA = \begin{vmatrix} 3 \\ 4 \\ 5 \end{vmatrix} - \begin{vmatrix} 4 \\ 7 \\ -9 \end{vmatrix} = \begin{vmatrix} -1 \\ -3 \\ 14 \end{vmatrix}$

~~$CA = \begin{vmatrix} 7 \\ 3 \\ -2 \end{vmatrix} - \begin{vmatrix} 4 \\ 7 \\ -9 \end{vmatrix} = \begin{vmatrix} 3 \\ -4 \\ 7 \end{vmatrix}$~~

~~$\begin{vmatrix} i & j & k \\ -1 & -3 & 14 \\ 3 & -4 & 7 \end{vmatrix} = i(35) - j(-49) + k(13) = \begin{vmatrix} 35 \\ 49 \\ 13 \end{vmatrix}$~~

$$AB = \begin{pmatrix} 10 \\ -8 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ -12 \\ 0 \end{pmatrix} \quad AC = \begin{pmatrix} 4 \\ 7 \\ -1 \end{pmatrix} - \begin{pmatrix} 10 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -6 \\ 8 \\ -14 \end{pmatrix}$$

$$AP = \begin{pmatrix} 19/4 \\ 11/4 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 5/4 \\ -3/4 \\ 0 \end{pmatrix} \quad \begin{vmatrix} i & j & k \\ 7 & -12 & 0 \\ 1 & 3 & -14 \end{vmatrix} = i \left(\frac{35}{2} \right) - j \left(-\frac{49}{2} \right) + k \left(\frac{13}{2} \right)$$

$$AC = \begin{pmatrix} 4 \\ 7 \\ -9 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -14 \end{pmatrix} = \begin{pmatrix} 35 \\ 2 \\ -49 \\ 2 \\ 13 \\ 2 \end{pmatrix}$$

$$\sqrt{\frac{35^2}{2} + \frac{49^2}{2} + \frac{13^2}{2}}$$

→ = 30.248

$$B \cdot AB = \begin{pmatrix} 7 \\ -12 \\ 0 \end{pmatrix}$$

$$PQ = \begin{pmatrix} 7 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 19/4 \\ 11/4 \\ 5 \end{pmatrix} = \begin{pmatrix} 9/4 \\ 1/4 \\ -7 \end{pmatrix}$$

$$AC = \begin{pmatrix} 1 \\ 3 \\ -14 \end{pmatrix}$$

$$AO = \begin{pmatrix} 4 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} k-3 \\ 0 \\ -6 \end{pmatrix}$$

$$\begin{vmatrix} i & j & k \\ 7 & -12 & 0 \\ 1 & 3 & -14 \end{vmatrix} = i(70) - j(-98) + k(26)$$

$$\begin{vmatrix} k-3 & 0 \\ 0 & 98 \\ -6 & 26 \end{vmatrix} = 70k - 210 - 156 = 226$$

$$70k - 366 = 226$$

$$70k = 592$$

$$k = \frac{296}{35}$$

5/10

Examiner Comments

In part (a), there is no correct strategy for the required area but the candidate does adopt a correct strategy for finding the coordinates of the points P and Q and these are found correctly and so this response scores M0M1M1A1M0A0.

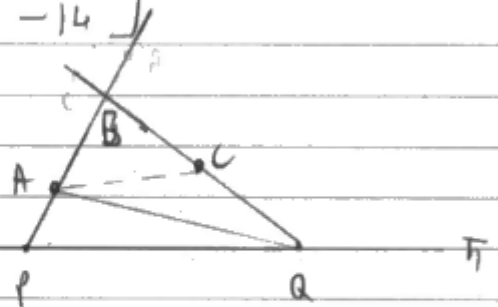
In part (b), this candidate has a correct solution apart from not considering the two possible values for the volume. This means that the first 2 marks are awarded.

Student Response C

a) $A(3, 4, 5)$ $B(10, -1, 5)$ $C(4, 7, -9)$

$$\pi \text{ is } \begin{bmatrix} 4 \\ -8 \\ 1 \end{bmatrix} = 2$$

$$\vec{AB} = \begin{bmatrix} 7 \\ -5 \\ 0 \end{bmatrix} \quad \vec{BC} = \begin{bmatrix} -6 \\ 8 \\ -14 \end{bmatrix}$$



$$\text{Line } \vec{r} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + \lambda \begin{bmatrix} 7 \\ -5 \\ 0 \end{bmatrix}$$

$$2 = \begin{bmatrix} 3+7\lambda \\ 4-5\lambda \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -8 \\ 1 \end{bmatrix}$$

$$2 = 12 + 28\lambda - 32 + 40\lambda + 5$$

$$\lambda = \frac{1}{4}$$

$$P\left(\frac{19}{4}, \frac{11}{4}, 5\right)$$

$$\text{Line } \vec{r} = \begin{bmatrix} 10 \\ -1 \\ 5 \end{bmatrix} + \mu \begin{bmatrix} -3 \\ 4 \\ -7 \end{bmatrix}$$

$$2 = \begin{bmatrix} 10-3\mu \\ -1+4\mu \\ 5-7\mu \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -8 \\ 1 \end{bmatrix}$$

$$2 = 40 - 12\mu + 8 - 32\mu + 5 - 7\mu$$

$$\mu = 1$$

$$Q(7, 3, -2)$$

$$A(3, 4, 5)$$

$$C(4, 7, -9)$$

$$P\left(\frac{19}{6}, \frac{11}{6}, 5\right) \quad Q(7, 3, -2)$$

$$\vec{AP} = \begin{bmatrix} 7/4 \\ -5/4 \\ 0 \end{bmatrix} \quad \vec{AC} = \begin{bmatrix} 1 \\ 3 \\ -14 \end{bmatrix} \quad \vec{QP} = \begin{bmatrix} -9/4 \\ -1/4 \\ 7 \end{bmatrix} \quad \vec{QC} = \begin{bmatrix} -3 \\ 4 \\ -7 \end{bmatrix}$$

$$\text{Area APO} = \frac{1}{2} |\vec{AP} \times \vec{AC}|$$

$$= \frac{1}{2} \left| \begin{bmatrix} 35/2 \\ 49/2 \\ 13/2 \end{bmatrix} \right|$$

$$= 15.4 \text{ 3v}$$

$$\text{Area QCP} = \frac{1}{2} |\vec{QP} \times \vec{QC}|$$

$$= \frac{1}{2} \left| \begin{bmatrix} -165/4 \\ -147/4 \\ -39/2 \end{bmatrix} \right|$$

$$= 23.1 \text{ 3v}$$

$$15.4 + 23.1 = 38.5 \text{ units}^2 \text{ 3v}$$

~~58~~ $P(2, 4, -1)$

$$h) \quad A(3, 4, 5) \quad B(10, -1, 5) \quad C(4, 7, -9) \quad D(k, 4, -1)$$

$$\vec{AB} = \begin{bmatrix} 7 \\ -5 \\ 0 \end{bmatrix} \quad \vec{AC} = \begin{bmatrix} 1 \\ 3 \\ -14 \end{bmatrix} \quad \vec{AD} = \begin{bmatrix} -3+k \\ 0 \\ -6 \end{bmatrix}$$

$$226 = |\vec{AB} \times \vec{AC} \cdot \vec{AD}|$$

$$226 = \left| \begin{bmatrix} 70 \\ 48 \\ 26 \end{bmatrix} \cdot \begin{bmatrix} k-3 \\ 0 \\ -6 \end{bmatrix} \right|$$

$$226 = |70k - 210 - 156|$$

$$226 = 70k - 366$$

$$226 = 366 - 70k$$

$$k = \frac{296}{70}$$

$$\text{or } k = 2$$

10/10

Examiner Comments

This candidate has a fully correct solution in both parts.

Exemplar Question 8

8 The hyperbola H has equation

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

The line l_1 is the tangent to H at the point $P(4\cosh \theta, 3\sinh \theta)$.

The line l_1 meets the x -axis at the point A .

The line l_2 is the tangent to H at the point $(4, 0)$.

The lines l_1 and l_2 meet at the point B and the midpoint of AB is the point M .

(a) Show that, as θ varies, a Cartesian equation for the locus of M is

$$y^2 = \frac{9(4-x)}{4x} \quad p < x < q$$

where p and q are values to be determined.

(11)

Let S be the focus of H that lies on the positive x -axis.

(b) Show that the distance from M to S is greater than 1

(3)

(Total for Question 8 is 14 marks)

Mean Score 4.5 out of 14

Examiner Comments

There were few candidates who did not attempt this question which would suggest that the paper was of an appropriate length.

Candidates appreciated the strategy involved in finding the x coordinate of the point of intersection with the x -axis although some attempts were unnecessarily complex resulting from rearrangements of the tangent equation before setting $y = 0$. Many could then proceed to find the equation of the required line and the midpoint. It was disappointing to see slips in finding the mid-point where the x and y co-ordinates were subtracted rather than added.

Establishing the form of the equation of the locus caused difficulty for many and solutions were seen with much unhelpful algebraic manipulation. The strategy of finding $\cosh \theta$ in terms of x and then substituting into the expression for y^2 from the mid-point was the most successful route. Working on both sides of the required expression to achieve a common expression was also used.

Generally, neither was completed with a great deal of success, many getting bogged down in the algebra. Even amongst those who were successful in this manipulation, it was common to see no reference to the values of p and q required. This was probably because many were pre-occupied with trying to obtain the equation of the locus of M . Of those who did attempt this part, $q = 4$ was seen but usually $p = 0$ was seen rather than $p = 2$.

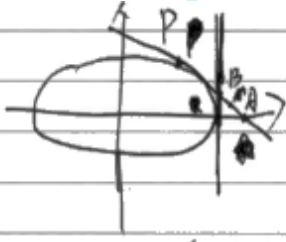
Fully correct solutions to part (b) were extremely rare. Most candidates could not progress beyond finding the coordinates of the focus. It was rare to see any use of a diagram to help to understand what was required.

Mark Scheme

Question	Scheme	Marks	AOs
8(a)	$\frac{x^2}{16} - \frac{y^2}{9} = 1 \Rightarrow \frac{x}{8} - \frac{2yy'}{9} = 0 \Rightarrow y' = \frac{9x}{16y} = \frac{36 \cosh \theta}{48 \sinh \theta}$ <p style="text-align: center;">or</p> $x = 4 \cosh \theta, y = 3 \sinh \theta \Rightarrow \frac{dy}{dx} = \frac{3 \cosh \theta}{4 \sinh \theta}$	M1	3.1a
	$y - 3 \sinh \theta = \frac{3 \cosh \theta}{4 \sinh \theta} (x - 4 \cosh \theta)$	M1	3.1a
	$y = 0 \Rightarrow x = \frac{4}{\cosh \theta}$	A1	2.2a
	line l_2 has equation $x = 4$	B1	2.2a
	$x = 4 \Rightarrow y - 3 \sinh \theta = \frac{3 \cosh \theta}{4 \sinh \theta} (4 - 4 \cosh \theta)$	M1	2.1
	$y = \frac{3 \cosh \theta - 3}{\sinh \theta}$	A1	2.2a
	$M \text{ is } \left(\frac{1}{2} \left(4 + \frac{4}{\cosh \theta} \right), \frac{1}{2} \left(\frac{3 \cosh \theta - 3}{\sinh \theta} \right) \right)$	M1	1.1b
	$x = 2 + \frac{2}{\cosh \theta} \Rightarrow \cosh \theta = \frac{2}{x-2}$ $\Rightarrow y^2 = \frac{9(\cosh \theta - 1)^2}{4 \sinh^2 \theta} = \frac{9 \left(\frac{2}{x-2} - 1 \right)^2}{4 \left(\left(\frac{2}{x-2} \right)^2 - 1 \right)}$	M1	3.1a
	$= \frac{9 \left(\frac{2}{x-2} - 1 \right)^2}{4 \left(\frac{2}{x-2} - 1 \right) \left(\frac{2}{x-2} + 1 \right)} = \frac{9 \left(\frac{2}{x-2} - 1 \right)}{4 \left(\frac{2}{x-2} + 1 \right)} = \frac{9(4-x)}{4x} *$	A1*	1.1b
	<p style="text-align: center;">Alternative for M1A1:</p> $y^2 = \frac{9(\cosh \theta - 1)^2}{4 \sinh^2 \theta} = \frac{9(\cosh \theta - 1)^2}{4(\cosh \theta - 1)(\cosh \theta + 1)} = \frac{9(\cosh \theta - 1)}{4(\cosh \theta + 1)}$ $\frac{9(4-x)}{4x} = \frac{9 \left(4 - 2 - \frac{2}{\cosh \theta} \right)}{8 + \frac{8}{\cosh \theta}} = \frac{9(\cosh \theta - 1)}{4(\cosh \theta + 1)} \Rightarrow y^2 = \frac{9(4-x)}{4x}$		
	$p = 2$ or $q = 4$	M1	3.1a
$p = 2$ and $q = 4$	A1	1.1b	
	(11)		

(b)	$b^2 = a^2(e^2 - 1) \Rightarrow 9 = 16(e^2 - 1) \Rightarrow e = \frac{5}{4}$ $\text{Focus is at } x = ae = 4 \times \frac{5}{4} = 5$	M1	1.1b
	$d > "5" - 4 = \dots$	M1	3.1a
	$d > 1^*$	A1*	1.1b
		(3)	
(14 marks)			
Notes			
(a)			
M1: Attempts to solve the problem by using differentiation to obtain an expression for $\frac{dy}{dx}$ in terms of θ . Allow this mark for $\frac{x^2}{16} - \frac{y^2}{9} = 1 \Rightarrow \alpha x - \beta yy' = 0 \Rightarrow y' = \dots$ or an attempt to differentiate x and y wrt θ and then $\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \dots$			
M1: Correct straight line method using the coordinates of P and their gradient in terms of θ Allow the results for the first 2 M marks to be "quoted", but any statements must be correct to score the marks.			
A1: Uses $y = 0$ to deduce the correct coordinates (or value of x) for the point A . Allow in any form, simplified or unsimplified (e.g. unsimplified: $x = 4 \cosh \theta - 4 \tanh \theta \sinh \theta$)			
B1: Deduces that the equation of l_2 is $x = 4$ (may be implied by $x = 4$ used to find y coordinate of B)			
M1: Realises that $x = 4$ is all that is needed for the second line and substitutes this into the first line in order to find the point B			
A1: Deduces the correct coordinates or y value for B (e.g. unsimplified $y = \frac{3}{\tanh \theta} - \frac{3 \cosh \theta}{\tanh \theta} + 3 \sinh \theta$)			
M1: Uses a correct method for the midpoint of AB (coordinates must be the right way round). This may be seen as the coordinates written separately e.g. $x = \dots, y = \dots$			
M1: Having found the midpoint, identifies a correct strategy that will enable a Cartesian equation to be found. E.g. find $\cosh \theta$ in terms of x and substitutes into y or y^2 to obtain an equation in terms of y and x only. Mark positively here, so allow the mark if the candidate makes progress in eliminating θ even if there are slips in the working.			
A1*: Obtains the printed answer with no errors Alternative for the previous 2 marks: Substitutes the coordinates of their midpoint into both sides of the given equation in an attempt to show they are equal. Again, mark positively but having made the substitution, some progress needs to be made in showing that both sides are equal. For this method there must be a minimal conclusion for the A1 e.g. tick, hence true etc. Note that these 2 marks can also be attempted by expressing the midpoint in terms of exponentials – if you are in doubt whether to award marks seek advice from your Team Leader.			
M1: For $p = 2$ or $q = 4$			
A1: For $p = 2$ and $q = 4$			
(b)			
M1: A complete method for finding the x coordinate of the focus using a correct eccentricity formula to find a value for e and then calculating $4e$			
M1: Completes the problem by subtracting 4 from the x coordinate of the focus			
A1*: Correct answer			
If you come across correct attempts using Pythagoras to prove the result send to review.			

Student Response A



c) tangent to H at ~~(4,0)~~ (4,0)
 $\Rightarrow x=4$

tangent to H at $(4 \cosh \theta, 3 \sinh \theta)$:

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

$$9x^2 - 16y^2 = 144$$

$$18x - 32y \frac{dy}{dx} = 0$$

$$32y \frac{dy}{dx} = 18x$$

$$\frac{dy}{dx} = \frac{9}{16} \frac{x}{y} = \frac{4 \cosh \theta}{3 \sinh \theta} \cdot \frac{4 \cosh \theta}{3}$$

$$= \frac{3 \cosh \theta}{4 \sinh \theta}$$

$$y - 3 \sinh \theta = \frac{3 \cosh \theta}{4 \sinh \theta} (x - 4 \cosh \theta)$$

$$4y \sinh \theta - 12 \sinh^2 \theta = 3x \cosh \theta - 12 \cosh^2 \theta$$

$$4y \sinh \theta - 3x \cosh \theta = 12 \sinh^2 \theta - 12 \cosh^2 \theta = -12$$

$$\begin{cases} 4y \sinh \theta - 3x \cosh \theta = -12 \\ x = 4 \end{cases}$$

$$4y \sinh \theta - 12 \cosh \theta = -12$$

$$y \sinh \theta = -3 + \cosh \theta$$

$$y = \frac{-3 + \cosh \theta}{\sinh \theta}$$

$$\therefore B = \left(4, \frac{-3 + \cosh \theta}{\sinh \theta} \right)$$

$$A \rightarrow 4y \sinh \theta - 3x \cosh \theta = -12$$

$$\textcircled{a} \quad y=0$$

$$3x \cosh \theta = 12 \quad 4$$

$$x = \frac{4}{\cosh \theta}$$

$$= A: \left(\frac{4}{\cosh \theta}, 0 \right)$$

$$M: \left(\frac{\frac{-3 + \cosh \theta}{\sinh \theta}}{2}, \frac{\frac{4}{\cosh \theta} + 4}{2} \right)$$

$$= \left(\frac{-6 + 2 \cosh \theta}{\sinh \theta}, \frac{3}{\cosh \theta} + 2 \right)$$

~~②~~

5/14

Examiner Comments

In part (a), this candidate uses implicit differentiation to establish the gradient of the tangent and also proceeds correctly to obtain the tangent equation. The B marks is also scored for writing down the correct equation of line 2. Note that the candidate then confuses their coordinates when attempting the midpoint and no further progress is made.

There is no creditable progress in part (b).

Student Response B

$$A. \quad \frac{x}{8} - \frac{2y}{9} \frac{dy}{dx} = 0 \quad \frac{dy}{dx} = \frac{x}{8} \times \frac{9}{2y} = \frac{9x}{16y} = \frac{36 \operatorname{cosh} \theta}{48 \sinh \theta} = \frac{3 \operatorname{cosh} \theta}{4 \sinh \theta}$$

$$\frac{3 \operatorname{cosh} \theta}{4 \sinh \theta} x - \frac{3 \operatorname{cosh}^2 \theta}{\sinh \theta} = y - 3 \sinh \theta$$

$$3x \operatorname{cosh} \theta - 12 \operatorname{cosh}^2 \theta = 4y \sinh \theta - 12 \sinh^2 \theta$$

$$4y \sinh \theta - 3x \operatorname{cosh} \theta = -12 \operatorname{cosh}^2 \theta + 12 \sinh^2 \theta = -12$$

$$L_2 = x = 4 \quad 4y \sinh \theta - 12 \operatorname{cosh} \theta = -12$$

$$4y \sinh \theta = 12(\operatorname{cosh} \theta - 1)$$

$$y \sinh \theta = 3(\operatorname{cosh} \theta - 1)$$

$$y = \frac{3(\operatorname{cosh} \theta - 1)}{\sinh \theta} \quad x = 4$$

$$B = \left(4, \frac{3 \operatorname{cosh} \theta - 3}{\sinh \theta} \right)$$

$$A = \left(\frac{4}{\operatorname{cosh} \theta}, 0 \right)$$

$$-3x \operatorname{cosh} \theta = -12$$

$$x \operatorname{cosh} \theta = 4 \quad x = \frac{4}{\operatorname{cosh} \theta}$$

$$\operatorname{cosh}^2 \theta - \sinh^2 \theta = 1$$

$$M = \left(2 + \frac{2}{\operatorname{cosh} \theta}, \frac{3 \operatorname{cosh} \theta - 3}{2 \sinh \theta} \right)$$

$$\frac{18 \operatorname{cosh} \theta - 18}{8 \operatorname{cosh} \theta + 8} = \frac{9(4-x)}{4x}$$

$$y^2 = \frac{9 \operatorname{cosh}^2 \theta + 9 - 18 \operatorname{cosh} \theta}{4 \sinh^2 \theta} =$$

$$4-x = 4 - 2 - \frac{2}{\operatorname{cosh} \theta} = 2 - \frac{2}{\operatorname{cosh} \theta}$$

$$9(4-x) = 18 - \frac{18}{\operatorname{cosh} \theta}$$

$$\frac{18 - 18}{\operatorname{cosh} \theta} = \frac{18 \operatorname{cosh} \theta - 18}{8 \operatorname{cosh} \theta + 8}$$

$$4x = 8 - \frac{8}{\operatorname{cosh} \theta}$$

$$8 - \frac{8}{\operatorname{cosh} \theta} = \frac{8 \operatorname{cosh} \theta + 8}{8 \operatorname{cosh} \theta + 8}$$

$$\frac{36-9x}{4x} = \frac{9}{x} - \frac{9}{4}$$

Brakow

$$\frac{9}{2+\frac{2}{\cosh\theta}} = \frac{9}{4}$$

$$\text{Let } \sinh\theta = x \text{ then } \cosh\theta = \sqrt{1+x^2}$$

$$9 \cosh^2\theta = 9 - 18 \cosh\theta$$

$$-3x \cosh\theta = -12 \quad x = \frac{4}{\cosh\theta}$$

$$4 \cosh^2\theta = 4$$

$$\text{Let } \sinh\theta = 0 \text{ then } \cosh\theta = 1$$

$$y = \frac{3 \cosh\theta - 3}{\sinh\theta}$$

B. Wt $3 = \frac{1}{2}(e^2-1) \quad e = \frac{\sqrt{7}}{2} \quad ae = 2\sqrt{7} \quad S = (2\sqrt{7}, 0)$

$$y^2 = \frac{9(4-2\sqrt{7})}{8\sqrt{7}} \quad \sqrt{\left(2 + \frac{2}{\cosh\theta} - 2\sqrt{7}\right)^2 + \left(\frac{3\cosh\theta-3}{2\cosh\theta}\right)^2} > 1$$

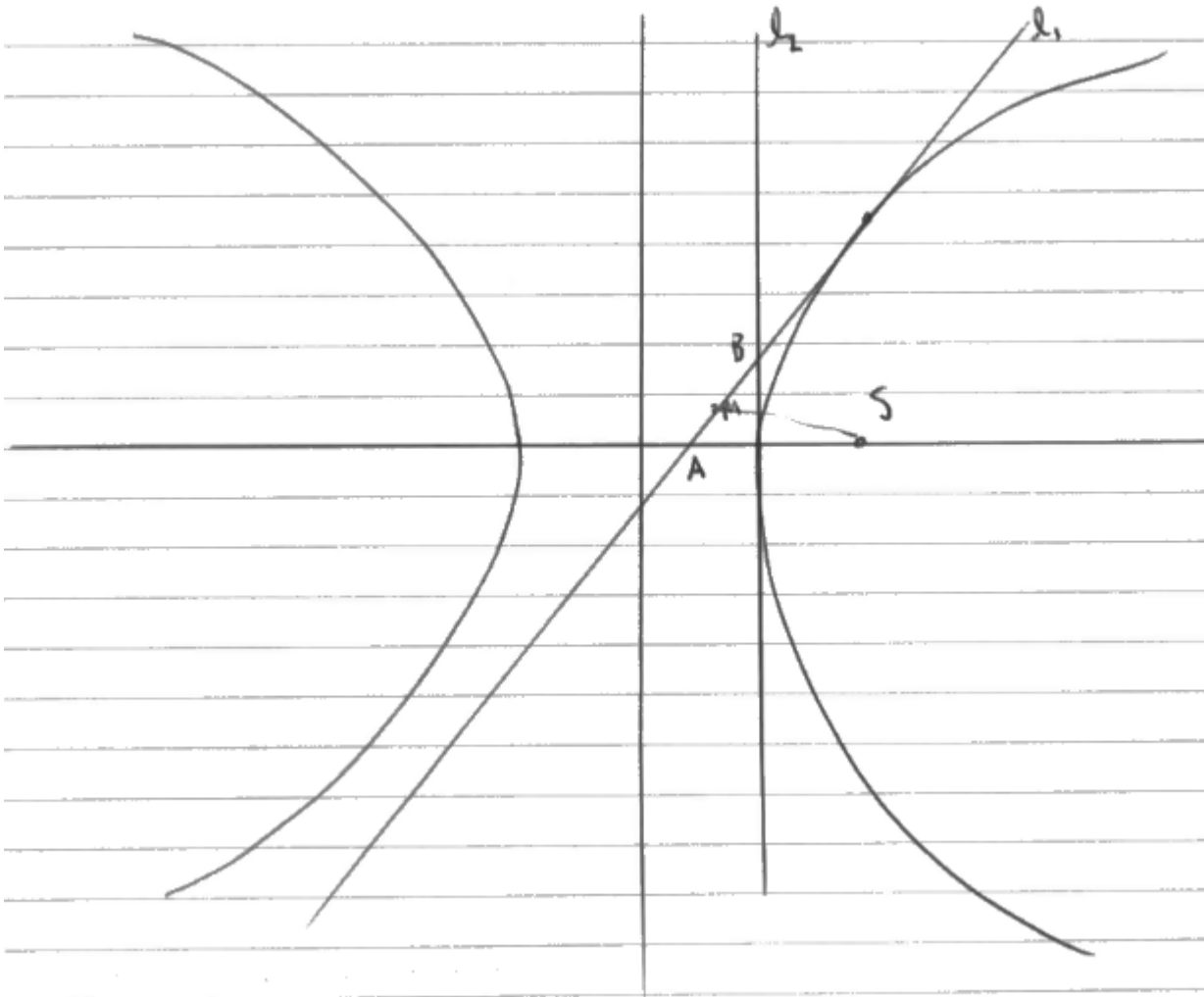
7/14

Examiner Comments

This candidate makes good progress in part (a) and obtains the correct coordinates of the point M. This is followed by an attempt to substitute into both sides of the given locus equation but it was considered that insufficient progress was made in showing that both sides were equal and as there was no reference to the values of p or q , only the first 7 marks were awarded.

In part (b), the working suggested that an incorrect eccentricity formula had been used and because the method marks were dependent, no marks were scored in this part.

Student Response C



$$a) \frac{x}{8} - \frac{2y}{9} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{11x}{8} \cdot \frac{9}{28y} = \frac{119x}{16y}$$

$$= \frac{1136 \cosh \theta}{48 \sinh \theta} = \frac{43 \cosh \theta}{4 \sinh \theta}$$

$$\Rightarrow y - 3 \sinh \theta = \frac{43 \cosh \theta}{4 \sinh \theta} (x - 4 \cosh \theta)$$

$$4 \sinh \theta y - 12 \sinh^2 \theta = \frac{43 \cosh \theta}{4 \sinh \theta} x - 12 \cosh^2 \theta$$

$$4 \sinh \theta y - 3 \cosh \theta x = -12$$

$$y=0 \Rightarrow \cosh \theta x = 4$$

$$x = \frac{4}{\cosh \theta}$$

$$\Rightarrow A = \left(\frac{4}{\cosh \theta}, 0 \right)$$

$$y x = 4 \Rightarrow \sinh \theta y - 3 \cosh \theta = -3$$

$$y = \frac{3 \cosh \theta - 3}{\sinh \theta}$$

$$\Rightarrow B = \left(4, \frac{3 \cosh \theta - 3}{\sinh \theta} \right)$$

~~$$\Rightarrow M = \left(4, \frac{3 \cosh \theta - 3}{\sinh \theta} \right)$$~~

$$\Rightarrow M = \left(2 + \frac{2}{\cosh \theta}, \frac{3 \cosh \theta - 3}{2 \sinh \theta} \right)$$

$$\Rightarrow \frac{9(4-x)}{4x} = \frac{9 \left(2 - \frac{2}{\cosh \theta} \right)}{8 + \frac{6}{\cosh \theta}}$$

$$= \frac{18 \cosh \theta - 18}{8 \cosh \theta + 8} = \frac{9 \cosh \theta - 9}{4 \cosh \theta - 4} = \frac{9}{4} \left(\frac{\cosh \theta - 1}{\cosh \theta + 1} \right)$$

$$y^2 = \left(\frac{3 \cosh \theta - 3}{2 \sinh \theta} \right)^2$$

$$= \frac{9 \cosh^2 \theta - 18 \cosh \theta + 9}{4 \sinh^2 \theta}$$

$$= \frac{9 \cosh^2 \theta - 18 \cosh \theta + 9}{4 \cosh^2 \theta - 4}$$

$$= \frac{9}{4} \left(\frac{\cosh^2 \theta - 2 \cosh \theta + 1}{\cosh^2 \theta - 1} \right)$$

Asymptote = $y = +\frac{b}{a}x$
 $= +\frac{3}{4}x$

$y = \frac{3}{4}x$ is the limiting asymptote

$\Rightarrow 0 < x < 4$

$p=0, q=4$

$$= \frac{9}{4} \left(\frac{\cosh \theta - 1}{\cosh \theta + 1} \right)$$

$$\Rightarrow y^2 = \frac{9(4-x)}{4x}$$

$$b) S = (ae, 0)$$

$$b^2 = a^2(e^2 - 1)$$

$$9 = 16(e^2 - 1)$$

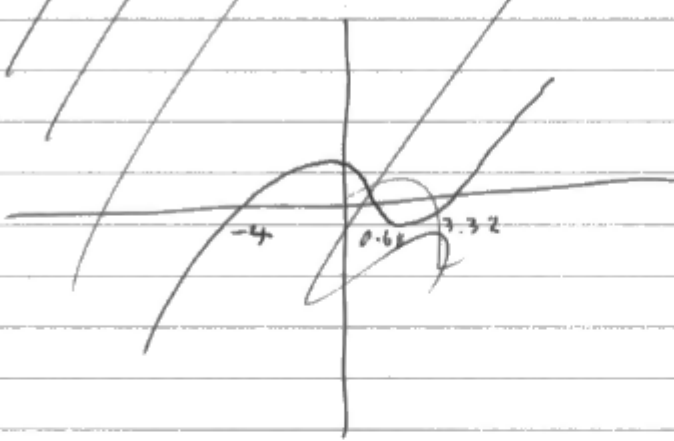
$$e = \frac{5}{4} \Rightarrow S = (5, 0), \quad \text{focus } M = (x, y)$$

$$\begin{aligned} \Rightarrow MS^2 &= y^2 + (5-x)^2 \\ &= \frac{9(4-x)}{4x} + x^2 - 10x + 25 \\ &= \frac{36 - 9x + 4x^3 - 40x + 100x}{4x} = \frac{4x^3 - 35x + 36}{4x} \end{aligned}$$

$$\text{If } MS \leq 1 \Rightarrow 4x^3 - 51x + 36 \leq 4x$$

$$4x^3 - 55x + 36 \leq 0$$

$$\Rightarrow x = -4, x = 3.32, x = 0.68$$



$$\Rightarrow MS = y^2 + (5-x)^2$$

$$= (4-x)^2 + (5-x)^2$$

MS_{\min} in given by $x=4$ for $M \rightarrow$ b) cont

limiting value $x < 4 \Rightarrow y > 0$

$$\Rightarrow \text{minimum } M = (4, 0)$$

$$MS \text{ for } (4, 0) = (5-4)^2 + (0)^2$$

$$= 1$$

$\Rightarrow MS$ is always greater than 1

13/14

Examiner Comments

This candidate has a fully correct response in both parts of the question apart from the value of p in part (a).

A Level Further Mathematics – Further Pure 2 (9FM0 4A)

Exemplar Question 1

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1. A complex number $z = x + iy$ is represented by the point P in an Argand diagram.

Given that

$$|z - 3| = 4|z + 1|$$

- (a) show that the locus of P has equation

$$15x^2 + 15y^2 + 38x + 7 = 0 \quad (2)$$

- (b) Hence find the maximum value of $|z|$

(3)

(Total for Question 1 is 5 marks)

Mean Score 3.7 out of 5

Examiner comment:

This question, testing further complex loci (4.1), was generally answered well, and provided a suitable opening to the paper, though there was a discriminating mark in part (b).

In part (a) the majority of students knew the procedure for this part and proceeded to apply the magnitude formula correctly to both sides. However some errors expanding and manipulating to the required form were seen leading to the loss of the accuracy mark. Only a small number of students forgot to square the 4, or ended up with negative y^2 terms.

The first mark of part (b) was scored by the majority of students via completing the square, usually correctly. Knowing what to do with this form proved more challenging, with many giving 16/15 as the answer at this stage. Some did state the centre and radius to gain the second mark, but went no further. Those who drew a sketch were more successful in spotting the correct method required. Only a minority went on to score the A by using their centre and radius appropriately. A few students tried to differentiate as a “maximum” was required, and such cases usually only scored the first mark.

Mark Scheme

Question	Scheme	Marks	AOs
1(a)	$(x-3)^2 + y^2 = 16((x+1)^2 + y^2)$	M1	1.1b
	$x^2 - 6x + 9 + y^2 = 16x^2 + 32x + 16 + 16y^2$ $15x^2 + 15y^2 + 38x + 7 = 0^*$	A1*	2.1
		(2)	
1(b)	$15x^2 + 15y^2 + 38x + 7 = 15\left(x \pm \frac{19}{15}\right)^2 - \dots + 15y^2 + 7 = 0$	M1	2.1
	Centre is $\left(-\frac{19}{5}, 0\right)$ and radius is $\sqrt{\left(\frac{19}{15}\right)^2 - \frac{7}{15}} \left(= \frac{16}{15}\right)$	M1	2.2a
	$\max z = \frac{16}{15} + \frac{19}{15} = \frac{7}{3}$	A1	3.1a
		(3)	
(5 marks)			
Notes			
<p>(a)</p> <p>M1: Obtains an equation in terms of x and y using the given information. Allow if the 4 is not squared, but i^2 must have been dealt with correctly (ie positive y^2 terms). Condone invisible brackets for the M mark.</p> <p>A1*: Expands and simplifies and obtains a circle equation correctly. Accept terms in different order but must include $=0$. No errors seen, so bracketing errors in solution are A0.</p> <p>(b)</p> <p>M1: Completes the square on the x term achieving $A\left(x \pm \frac{19}{15}\right)^2 - B$, or uses other appropriate method in order to attempt the radius and/or centre of the circle. Award if correct x coordinate of centre or radius is found.</p> <p>M1: Deduces both centre and radius for their completed square form, either seen used in work clearly as centre and radius, stated or labelled on a diagram, not just embedded within the equation. This is implied by the correct calculation being carried out for their centre and radius.</p> <p>A1cso: Realises the need to add distance of centre from origin to radius to achieve the correct answer. Must come from correct work.</p> <p>Note that completing the square as $\left(x - \frac{19}{15}\right)^2 - \dots$ can score a maximum M1M1A0</p>			

Student Response A

$$z = 2 + iy$$

$$|2 + iy - 3| = 4|z + iy + 1|$$

$$|(2-3) + iy| = 4|(z+1) + iy|$$

$$\left[|(2-3) + iy| \right]^2 = \left[4|(z+1) + iy| \right]^2$$

$$(2-3)^2 + (iy)^2 = 4^2(x+1)^2 + (4iy)^2$$

$$(2-3)^2 + y^2 = 16(x+1)^2 + y^2$$

$$(2-3)^2 + y^2 = 16(x+1)^2 + 16y^2$$

$$2^2 - 6x + 9 = 16(x^2 + 2x + 1) + y^2$$

$$2^2 - 6x + 9 = 16x^2 + 32x + 16 + y^2$$

$$15x^2 + 38x + 7 = 0, \text{ as required.}$$

$$x^2 - 6x + 9 + y^2 = 16x^2 + 32x + 16 + 16y^2$$

$$15x^2 + 15y^2 + 38x + 7 = 0$$

$$\text{as required.}$$

b)

$$15x^2 + 38x + 15y^2 = -7$$

1/5

Examiner comment:

An equation in x and y is obtained by an attempt at the modulus of each side is made in part (a) to score the method mark. The incorrect line with $(iy)^2$ is not penalised here due to the recovery in the next line, which implies the i has been dealt with correctly. However, the accuracy is not scored due to there being errors in the proof – such as the erroneous line noted above, and also this and the following line have “missing brackets” with the y^2 term not being multiplied by 16 at this stage.

No progress is made in part (b).

Student Response B

$$\begin{aligned}
 \text{a)} \quad & |(x-3) + yi| = 4 |(x+1) + yi| \\
 & (x-3)^2 + y^2 = 16 [(x+1)^2 + y^2] \\
 & x^2 - 6x + 9 + y^2 = 16x^2 + 32x + 16 + 16y^2 \\
 & 15x^2 + 15y^2 + 38x + 7 = 0 \\
 \\
 \text{b)} \quad & 15 \left[\left(x + \frac{19}{15}\right)^2 - \frac{261}{225} \right] + 15y^2 + 7 = 0 \\
 & 15 \left(x + \frac{19}{15}\right)^2 - \frac{256}{15} + 15y^2 = 0 \\
 & 15 \left(x + \frac{19}{15}\right)^2 + 15y^2 = \frac{256}{15} \\
 & \left(x + \frac{19}{15}\right)^2 + y^2 = \frac{256}{225}
 \end{aligned}$$

3/5

Examiner comment:

The proof in part (a) is fully correct with sufficient steps shown and no errors seen, scoring both marks for part (a).

The student completes the square correctly in part (b), but does not extract the centre and radius or go on to find the maximum value of $|z|$, so only the first method mark is scored in part (b).

Student Response C

$$\begin{aligned} \text{(a)} \quad |z-3| &= 4|z+1| \\ |x+yi-3| &= 4|x+yi+1| \\ |(x-3)+yi| &= 4|(x+1)+yi| \\ (x-3)^2 + y^2 &= 16[(x+1)^2 + y^2] \\ x^2 - 6x + 9 + y^2 &= 16[x^2 + 2x + 1 + y^2] \\ x^2 - 6x + 9 + y^2 &= 16x^2 + 32x + 16 + 16y^2 \\ 0 &= 15x^2 + 38x + 7 + 15y^2 \\ 0 &= 15x^2 + 15y^2 + 38x + 7 \end{aligned}$$

Q.E.D

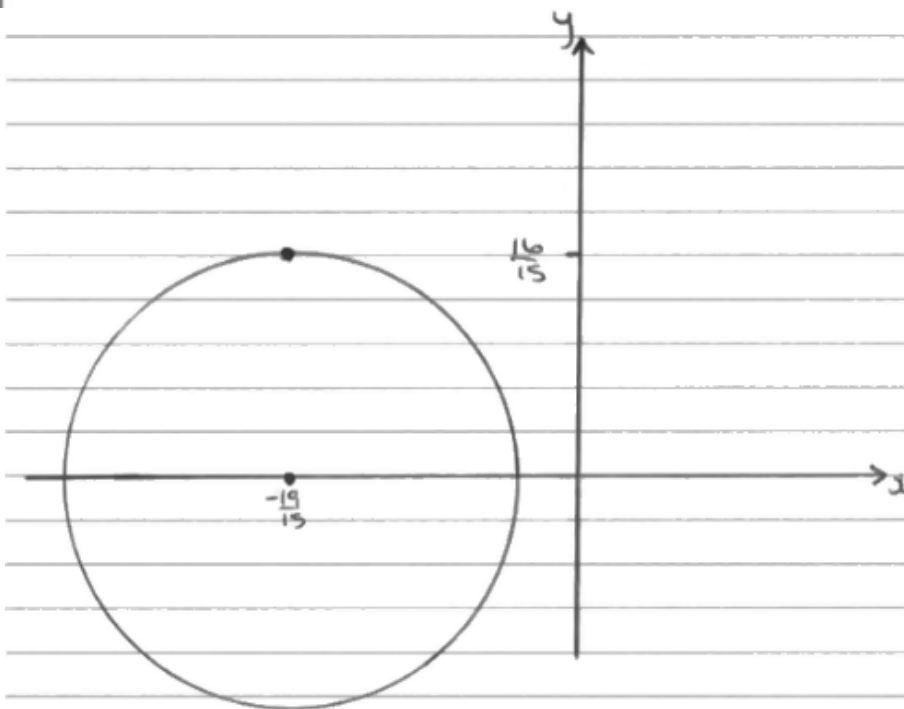
$$\begin{aligned} \text{(b)} \quad 0 &= 15x^2 + 15y^2 + 38x + 7 \\ 0 &= 15\left(x^2 + \frac{38}{15}x\right) + 15y^2 + 7 \end{aligned}$$

$$0 = x^2 + \frac{38}{15}x + y^2 + \frac{7}{15}$$

$$0 = \left(x + \frac{19}{15}\right)^2 - \frac{361}{225} + y^2 + \frac{7}{15}$$

$$\frac{166}{225} = \left(x + \frac{19}{15}\right)^2 + y^2$$

centre is $\left(-\frac{19}{15}, 0\right)$, radius is $\sqrt{\frac{166}{15}}$ (1.06)



maximum value of $|z| = \frac{16}{15}$

4/5

Examiner comment:

The proof in part (a) is fully correct with sufficient steps shown and no errors seen, scoring both marks for part (a).

The student completes the square correctly in part (b), and extracts the centre and radius, clearly stated at the bottom of the first page. Thus the two method marks are earned. However, they do not successfully link the information to find the maximum value of $|z|$ as required, instead giving the maximum y value of z (ie the radius in this case).

Exemplar Question 2

2. The matrix \mathbf{A} is given by

$$\mathbf{A} = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$$

- (a) Show that 2 is a repeated eigenvalue of \mathbf{A} and find the other eigenvalue. (5)
- (b) Hence find three non-parallel eigenvectors of \mathbf{A} . (4)
- (c) Find a matrix \mathbf{P} such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix. (2)

(Total for Question 2 is 11 marks)

Mean Score 8.2 out of 11

Examiner comment:

This question on further matrix algebra (3.1,3.2) was well answered by students, although a lack of attention to details lost some marks. The methods required were shown to be well understood with accuracy in part (b) being the main cause for lost marks.

For part (a) most students understood how to find eigenvalues and successfully found the characteristic polynomial before attempting to factorise. There were only a small number of cases of incorrect factorisation, some of the algebraic manipulation being inaccurate and resulting in 3 distinct eigenvalues, such as 2, 4 and 6.

As part (a) was a “show” question, evidence was needed that there were repeated roots, not simply an assertion that such was the case, so some lost marks because they didn't specify either that 2 was a repeated eigenvalue or the other eigenvalue being 8.

In part (b) errors mainly arose from bad algebra, particularly for the eigenvector related to eigenvalue 8, but the correct method for finding eigenvectors was shown by most. There were many correct alternatives found for the eigenvectors for $\lambda = 2$ and attaining three correct eigenvectors \mathbf{A} for was the most common outcome for this part.

Again, part (c) was answered well by students, who used their column vectors to produce a suitable matrix \mathbf{P} . Due to the follow through mark here, that meant most students were able to score both marks in this part. A few students normalised their eigenvectors first, the process of diagonalising a symmetric matrix using \mathbf{P} and its transpose, which still gains the marks here. Such a procedure was not necessary and indicates that some students are unaware of difference in the two cases.

Of those who did not achieve full marks in part (c), this was mainly due to only having found one eigenvector for the eigenvalue 2. In such cases either use of the zero vector as one column, or a repeated column or simply leaving part (c) unanswered resulted. There were a few cases where the two eigenvectors for $\lambda = 2$ were multiples, which could score only one mark in part (c).

Mark Scheme

Question	Scheme	Marks	AOs
2(a)	$ \mathbf{A} - \lambda\mathbf{I} = \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = (6-\lambda)[...] - (-2)[...] + 2[...] = \dots$	M1	1.1b
	$(6-\lambda)((3-\lambda)^2 - 1) + 2(2(\lambda-3)+2) + 2(2-2(3-\lambda)) (=0)$ $(\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0)$	A1	1.1b
	$= (\lambda-2)(\lambda^2 + \dots\lambda + \dots)$	M1	2.1
	$= (\lambda-2)(\lambda^2 - 10\lambda + 16) = (\lambda-2)^2(\lambda-8) \Rightarrow \lambda = 2 \text{ is a repeated eigenvalue *}$	A1*	2.2a
	$\lambda = 8$	B1	1.1b
		(5)	
(b)	$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \mathbf{v} = 2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ or } \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \mathbf{v} = 8 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \mathbf{v} = \dots$	M1	1.1b
	Obtains any multiple of $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ for $\lambda = 8$	A1	1.1b
	Obtains any (non-zero) multiple or linear combination of $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ for $\lambda = 2$	A1	1.1b
	Obtains a different linear combination or (non-zero) multiple of different vector from $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ for $\lambda = 2$	A1	3.1a
		(4)	
(c)	Forms a matrix with their eigenvectors as columns	M1	1.2
	E.g. $\begin{pmatrix} -1 & 1 & 2 \\ 0 & 2 & -1 \\ 2 & 0 & 1 \end{pmatrix}$	A1ft	1.1b
		(2)	
(11 marks)			

Notes

(a)

M1: Attempts to expand the determinant to find the characteristic polynomial.

Note: other methods of expanding the determinant are possible. If unsure send to review.

A1: Correct expansion need not be simplified. (Need not see set equal to zero) Allow recovery of missing brackets if indicated by later working.

M1: Attempts to take out a factor of $(\lambda - 2)$ of their equation (may first expand to cubic or may spot the factor and take out without full expansion). E.g

$$\begin{aligned} (6-\lambda)((3-\lambda)^2-1)+2(2(\lambda-3)+2)+2(2-2(3-\lambda)) &= (6-\lambda)(4-\lambda)(2-\lambda)+4(\lambda-2)+4(\lambda-2) \\ &= (\lambda-2)((6-\lambda)(4-\lambda)+4+4) \end{aligned}$$

This is for a method that will allow λ to be shown as a repeated eigenvalue, so just stating two solutions is not sufficient, factorisation must be seen.

A1*: Obtains a correct factor of $(\lambda - 2)^2$ and deduces that 2 is a repeated eigenvalue. Must see **statement** about 2 being repeated. (Just listing 2 twice is not sufficient.)

B1: (Note this is A1 on ePEN) Obtains and identifies 8 as the other eigenvalue (B0 if not identified in (a) but full marks can be scored in (b) and (c) for use of 8 as eigenvalue)

(b)

M1: Uses a correct method to find at least one eigenvector

A1: Obtains one correct eigenvector for $\lambda = 8$

A1: Obtains one correct eigenvector for $\lambda = 2$

A1: Obtains two correct linearly independent eigenvectors for $\lambda = 2$

Note some other common eigenvectors for $\lambda = 2$ are $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$

(c)

M1: Forms a matrix with their three **different non-zero** eigenvectors as columns or with their normalised (or any scaled version) of their eigenvectors.

A1ft: Correct matrix with the eigenvectors (normalised/scaled) as columns in any order (follow through their three **different** vectors which are **not multiples** of any other)

Examiner comment:

Note there are some alternatives to expanding the determinant in part (a). Any correct method scores the method mark.

Note other eigenvectors for $\lambda = 2$ are possible, the list given in the scheme is not exhaustive but showed the most common ones seen.

Student Response A

a) ~~not~~

$$A - \lambda = \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}$$

$$\det(A - \lambda) = (6-\lambda) \begin{bmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{bmatrix} - (-2) \begin{bmatrix} -2 & -1 \\ 2 & 3-\lambda \end{bmatrix} + 2 \begin{bmatrix} -2 & 3-\lambda \\ 2 & -1 \end{bmatrix}$$

$$= (6-\lambda) [(3-\lambda)(3-\lambda) - (-1)(-1)] + 2 [-2(3-\lambda) - (2)(-1)] + 2 [(2)(-1) - (3-\lambda)(2)]$$

$$= (6-\lambda) [\lambda^2 - 6\lambda + 8] + 2 [-6 + 2\lambda + 2] + 2 [2 - 6 + 2\lambda]$$

~~$$= (6-\lambda) [\lambda^2 - 6\lambda + 8] + 2 [2\lambda - 4] + 2 [2\lambda - 4]$$~~

$$= (6-\lambda) [\lambda^2 - 6\lambda + 8] + 2 [-4 + 2\lambda] + 2 [-4 + 2\lambda]$$

$$= 6\lambda^2 - 36\lambda + 48 - \lambda^3 + 6\lambda^2 - 8\lambda - 8 + 4\lambda - 8 + 4\lambda$$

$$= -\lambda^3$$

$$= -\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$$

$$-\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0 \quad \text{since } \lambda = 2 \text{ is known}$$

$$\lambda = 2 \text{ is a root.}$$

~~$$\lambda = 2, \lambda = 8$$~~

$$\lambda = 2, \lambda = 2, \lambda = 8$$

$$\begin{array}{r}
 \lambda^2 + 5\lambda - 26 \\
 \lambda^3 + 12\lambda^2 - 36\lambda + 32 \\
 \hline
 -\lambda^2 + 2\lambda^2 + 36\lambda - 32 \\
 \hline
 10\lambda^2 - 36\lambda \\
 -10\lambda^2 + 10\lambda \\
 \hline
 -26\lambda + 32
 \end{array}$$

$$\begin{array}{r}
 \lambda^2 + 5\lambda - 26 \\
 \lambda^3 + 12\lambda^2 - 36\lambda + 32 \\
 \hline
 -\lambda^2 + 2\lambda^2 + 36\lambda - 32 \\
 \hline
 10\lambda^2 - 36\lambda \\
 -10\lambda^2 + 10\lambda \\
 \hline
 -26\lambda + 32
 \end{array}$$

$$\text{b) } \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$$

$$\begin{array}{l}
 6x - 2y + 2z = 4 \quad \rightarrow 6x - 2y \quad \textcircled{1} \\
 -2x + 3y - z = 10 \quad \rightarrow -2x + 3y - z = 10 \quad \textcircled{2} \\
 2x - y + 3z = 14 \quad \rightarrow 2x - y + 3z = 14 \quad \textcircled{3}
 \end{array}$$

$$\textcircled{1} \rightarrow 4x - 2y + 2z = 0$$

$$\textcircled{2} \rightarrow -2x + y - z = 10 \Rightarrow z = -2x + y - 10$$

$$\textcircled{3} \rightarrow 2x - y + 3z = 14 \Rightarrow z = -2x + y + \frac{14}{3}$$

$$\underline{-2x + y = -2x + y}$$

$$\text{into } \textcircled{1} \rightarrow 4x - 2y + 4x + 2y = 0 \Rightarrow 8x = 0 \Rightarrow x = 0$$

Student Response B

$$4) A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix}.$$

$$\det = \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0.$$

$$(6-\lambda)(3-\lambda)^2 - 6 + \lambda + 4 - 12 + 4\lambda + 4 - 12 + 4\lambda = 0.$$

$$(6-\lambda)(3-\lambda)^2 - 22 + 9\lambda = 0.$$

$$(\lambda^2 - 6\lambda + 9)(6-\lambda) - 22 + 9\lambda = 0$$

$$-\lambda^3 + 6\lambda^2 - 9\lambda + 6\lambda^2 - 36\lambda + 54 - 22 + 9\lambda = 0$$

$$-\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$$

$$(\lambda-8)(\lambda-2)(\lambda-2) = 0.$$

So 2 is a repeated eigenvalue of A.

b) $\lambda = 8$

$$A = \begin{pmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{pmatrix}$$

$$\begin{pmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8x \\ 8y \\ 8z \end{pmatrix}$$

$$-2x - 2y + 2z = 8x$$

$$\left\{ \begin{array}{l} -x - y + z = 4x \\ -2x - 5y - z = 8y \\ 2x - y - 5z = 8z \end{array} \right.$$

$$-2x - 5y - z = 8y$$

$$2x - y - 5z = 8z$$

$$-6y - 6z = 8y + 8z$$

$$y = -z$$

$$z = 1 \quad y = -1 \quad x = 2$$

$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$\lambda = 2$

$$A = \begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

$$4x - 2y + 2z = 2x$$

$$-2x + y - z = 2y$$

$$2x - y + z = 2z$$

$$2y + 2z = 0$$

$$y = -z$$

$$z = 1 \quad y = -1 \quad x = 0$$

$$\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\text{or } \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

c) $P = \begin{pmatrix} 2 & 0 & 0 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

Examiner comment:

In part (a) there is a correct attempt to expand the determinant of $\mathbf{A} - \lambda \mathbf{I}$ to form the characteristic polynomial. Note this method uses $|c_1 \cdot (c_2 \times c_3)|$, which is a correct method to expand the determinant. The polynomial is then correctly factorised to show a factor of $(\lambda - 2)$ for the second method mark, and two factors of $(\lambda - 2)$ are seen with statement that 2 is a repeated eigenvalue to gain the second A mark. There is no clear statement or identification in part (a) that the other eigenvalue is $\lambda = 8$, so the B mark is not awarded. Just seeing a factor $(\lambda - 8)$ is insufficient.

A correct method to find at least one eigenvector is seen in part (b), leading to a correct eigenvector for $\lambda = 8$, scoring the first two marks. However, neither eigenvector for $\lambda = 2$ is correct, so the second and third A marks are not earned.

In part (c) the method mark is awarded since the three **different** eigenvectors found in part (b) are used as columns of their \mathbf{P} . Note that they must be different to gain the method mark. The accuracy mark cannot be awarded here, however, as the two vectors for $\lambda = 2$ are multiples of each other.

Student Response C

$$a) A - \lambda I = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix}$$

$$(6-\lambda) \begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} + 2 \begin{vmatrix} -2 & -1 \\ 2 & 3-\lambda \end{vmatrix} + 2 \begin{vmatrix} -2 & 3-\lambda \\ 2 & -1 \end{vmatrix}$$

$$\begin{aligned} &= (6-\lambda)((3-\lambda)(3-\lambda)-1) + 2(-6+2\lambda+2) + 2(2-2(3-\lambda)) \\ &= (6-\lambda)(9-3\lambda-3\lambda+\lambda^2-1) + 2(-4+2\lambda) + 2(2-6+2\lambda) \\ &= (6-\lambda)(\lambda^2-6\lambda+8) - 8+4\lambda - 8+4\lambda \\ &= 6\lambda^2-36\lambda+48-\lambda^3+6\lambda^2-8\lambda - 8+4\lambda - 8+4\lambda \\ &= -\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0 \\ &\quad \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0 \end{aligned}$$

$$(\lambda-8)(\lambda-2)^2 = 0 \quad \dots \quad 2 \text{ is a repeated eigen value.}$$

$$\lambda = 8, \lambda = 2$$

—#

$$b) \quad Ax = \lambda x$$

$$\lambda = 2 \quad \textcircled{1} \quad \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$6x - 2y + 2z = 2x \quad -2x + 3y - z = 2y$$

$$4x - 2y + 2z = 0 \quad -2x + 2y - z = 0$$

$$2x - y + z = 0$$

Let $z = 0$ $2x = y$ $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

—H

$$\lambda = 2 \quad \textcircled{2} \quad \text{Let } x = 0 \quad 2z = -y \quad \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

—H

$$\lambda = 8 \quad \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 8 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$6x - 2y + 2z = 8x \quad -2x + 3y - z = 8y \quad 2x - y + 3z = 8z$$

$$-2x - 2y + 2z = 0 \quad -2x - 5y - z = 0 \quad 2x - y - 5z = 0$$

$$x + y - z = 0 \quad 2x + 5y + z = 0$$

Let $y = z = 0$

$$\begin{matrix} x + y - z & 2x + 2y - 2z = 0 \\ 2x + 5y + z & 2x + 5y + z = 0 \\ 2x - y - 5z & 2x - y - 5z = 0 \end{matrix} \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

—H

$6y - 6z = 0$
 $y = z$

$$c) \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix} \quad P = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & -2 & 1 \end{pmatrix}$$

—H

Examiner comment:

Part (a) is fully correct, with the determinant of $\mathbf{A} - \lambda\mathbf{I}$ correctly expanded to find the characteristic polynomial (note “=0” is not needed) and a factor of $(\lambda - 2)$ extracted. The correct factorisation following a correct equation is accepted for the second M. The factorisation, with $(\lambda - 2)^2$, is seen and it is stated 2 is a repeated eigenvalue so the second A is earned, and $\lambda=8$ is stated as the other value for the B mark.

A correct strategy for finding at least one eigenvector is seen in part (b), and two correct, non-multiple eigenvectors for $\lambda=2$ are produced. However, the eigenvector for $\lambda = 8$ is incorrect due to an error in solving the equations. This loses the first accuracy mark in part (b) (scoring M1A0A1A1).

The eigenvectors found are used correctly to form a suitable matrix \mathbf{P} in part (c), and as none is a multiple of any other the follow through accuracy mark is also awarded. (Note that matrix \mathbf{P} is singular since $(0,1,1)$ is another eigenvector for $\lambda = 2$, but since this is not immediately obvious, it is not penalised.)

Exemplar Question 3

3. The number of visits to a website, in any particular month, is modelled as the number of visits received in the previous month plus k times the number of visits received in the month before that, where k is a positive constant.

Given that V_n is the number of visits to the website in month n ,

- (a) write down a general recurrence relation for V_{n+2} in terms of V_{n+1} , V_n and k .

(1)

For a particular website you are given that

- $k = 0.24$
- In month 1, there were 65 visits to the website.
- In month 2, there were 71 visits to the website.

- (b) Show that

$$V_n = 50(1.2)^n - 25(-0.2)^n$$

(5)

This model predicts that the number of visits to this website will exceed one million for the first time in month N .

- (c) Find the value of N .

(2)

(Total for Question 3 is 8 marks)

Mean Score 5.2 out of 8

Examiner comment:

This question tested a new topic on the specification in recurrence relations (6.1,6.2), but the methods required for solving were shown by most. The main hindrance for this question was a lack of showing method to achieve a given solution in part (b).

Part (a) was accessible to most student who were able to write down the correct relation.

In part (b) the process of forming and solving the auxiliary equation before setting up and solving equations to find the coefficients was demonstrated well by most candidates. However, the question was a “show” and with the given answer on the page, students need to ensure that sufficient working is shown. An over reliance on calculator technology meant that many students simply wrote down the answers 1.2 and -0.2 to the auxiliary equation, and the values of a and b from the simultaneous equations – but these were given in the question. Particularly for the latter, evidence was required as to where these values came from and so many student lost marks for a failure to show method.

Where students had given an incorrect answer to (a), they either made little progress in (b) when their solutions to the auxiliary equation were not what was expected, or else managed to find the correct solutions for their incorrect equation – as the answers were on the page.

For part (c) very few students used the envisaged method as they did not realise that $(-0.2)^N$ is negligible for large N . However, this was not a given answer and so in this part use of calculator method were acceptable and many gave $N = 55$ as the answer from little or no working, or as a restart after failed attempts to solve a log equation with $(-0.2)^N$ included.

Mark Scheme

Question	Scheme	Marks	AOs
3(a)	$V_{n+2} = V_{n+1} + kV_n$	B1	3.3
		(1)	
(b)	$\lambda^2 - \lambda - 0.24 = 0 \Rightarrow \lambda = \dots(1.2, -0.2)$	M1	1.1b
	$V_n = a(1.2)^n + b(-0.2)^n$	A1	2.2a
	$65 = a(1.2)^1 + b(-0.2)^1$ and $71 = a(1.2)^2 + b(-0.2)^2$	B1ft	3.4
	E.g. $\left. \begin{array}{l} 78 = 1.44a - 0.24b \\ 71 = 1.44a + 0.04b \end{array} \right\} \Rightarrow 7 = -0.28b \Rightarrow b = \dots$	M1	2.1
	$a = 50, b = -25 \Rightarrow V_n = 50(1.2)^n - 25(-0.2)^n$ *	A1*	1.1b
		(5)	
(c)	$50(1.2)^N > 10^6 \Rightarrow N = \dots$	M1	3.1b
	$\Rightarrow N = 55$ i.e. month 55	A1	3.2a
		(2)	
(8 marks)			
Notes			
<p>(a) B1: A correct expression for the model using the information given</p> <p>(b) M1: Forms and solves the auxiliary equation for their answer to (a) with $k = 0.24$ A1: The correct closed form deduced from their solutions. This must be consistent with their equation. Note the answer is given so check carefully. This is not a follow through mark. B1ft: Applies initial conditions to their general equation – correct two equations for their general form with $V_1 = 65$ and $V_2 = 71$ M1: Attempts to solve their equations showing a correct method, reaching a value for at least one variable. It is a show that question and answers are on the paper, so method is needed. Look for one equation multiplied through to give same coefficients before attempting eliminating or substitution. If a matrix system is used the inverse must be found, not just solutions stated. A1*: Correct expression formed following suitable working with no errors seen. With fractions instead of decimals is fine.</p> <p>(c) M1: Selects a suitable method to solve the problem. For example, realises that in the model, $(-0.2)^n$ is negligible for large n and so attempts to solve e.g. $50(1.2)^N = 10^6$, or tries at least one value either side of $N = 55$ as a process of trial and improvement, or uses a calculator/graphical approach – implied by a value of $N = 55$ or $N = 54$ stated. A1: $N = 55$. The correct answer will imply both marks for this part. Ignore erroneous working if correct answer is stated as a restart.</p>			

3(b) Alt	$V_n = 50(1.2)^n - 25(-0.2)^n, V_1 = 65, V_2 = 71$		
	$V_1 = 50 \times 1.2 - 25 \times -0.2 = 60 + 5 = 65$ $V_2 = 50 \times (1.2)^2 - 25 \times (-0.2)^2 = 72 - 1 = 71$	M1	1.1b
	Hence true for $n = 1$ and $n = 2$ Assume true for $n = k$ and $n = k + 1$ (for some $k > 0$)	A1	2.2a
	$V_{k+2} = V_{k+1} + 0.24V_k$ $= 50(1.2)^{k+1} - 25(-0.2)^{k+1} + 0.24(50(1.2)^k - 25(-0.2)^k)$	B1ft	3.4
	$= \frac{50}{1.2}(1.2)^{k+2} - \frac{25}{-0.2}(-0.2)^{k+2} + \frac{12}{1.2^2}(1.2)^{k+2} - \frac{6}{(-0.2)^2}(-0.2)^{k+2}$ $= \frac{125}{3}(1.2)^{k+2} + 125(-0.2)^{k+2} + \frac{25}{3}(1.2)^{k+2} - 150(-0.2)^{k+2} = \dots$	M1	2.1
	So $V_{k+2} = 50(1.2)^{k+2} - 25(-0.2)^{k+2}$ Hence true for $n = k + 2$. So the result is true for $n = 1$ and $n = 2$, and if true for $n = k$ and $n = k + 1$ then it is true for $n = k + 2$. Hence by mathematical induction, for all $n \in \mathbb{N}$ $V_n = 50(1.2)^n - 25(-0.2)^n$ *	A1*	1.1b
		(5)	

Notes

M1: Substitutes into equation for $n = 1$ and $n = 2$ to verify true for these cases.
 A1: Deduces true for base cases and makes a **correct** assumption statement. This must include two successive cases assumed true, so e.g. as in scheme, or with $k - 2$ and $k - 1$ etc, or may assume true for all (integers) $k \leq n$. But do not allow if assumed true for just k .
 B1ft: Substitutes the formula for k and $k + 1$ (or their successive values) into the recurrence formula, follow through their equation from part (a).
 M1: Rearranges to the form $a(1.2)^{(k+2)} + b(-0.2)^{k+2}$
 A1*: Correct work leading to the correct equation for V_{k+2} and makes suitable inductive conclusion, including the ideas of “true for $n = 1$ and $n = 2$ ”, “if true for $n = k$ and $n = k + 1$ then true for $n = k + 2$ ” and “hence true for all integers”.

Examiner comment:

There were only a very small minority of students who tried to prove the formula using proof by induction, and this was rarely fully successful.

Student Response A

(1)

For a particular website you are given that

- $k = 0.24$
- In month 1, there were 65 visits to the website.
- In month 2, there were 71 visits to the website.

$$\left(x - \frac{6}{5}\right)\left(x + \frac{1}{5}\right)$$

$$x^2 + \frac{1}{5}x - \frac{6}{5}x - \frac{6}{25}$$

(b) Show that

$$V_n = 50(1.2)^n - 25(-0.2)^n$$

(5)

This model predicts that the number of visits to this website will exceed one million for the first time in month N .

10000

(c) Find the value of N .

(2)

~~a. $V_n = V_{n+1} + V_{n+2} + k$~~

~~b. $V_n = V_{n+1} + V_{n+2} + 0.24$ $V_1 = 65$ $V_2 = 71$
 $r^2 - r - 1 = 0$~~

~~$r = \frac{1 \pm \sqrt{5}}{2}$ $r = \frac{1 - \sqrt{5}}{2}$~~

~~∴ $V_n = \left(\frac{1 + \sqrt{5}}{2}\right)^n A + \left(\frac{1 - \sqrt{5}}{2}\right)^n B$~~

~~P.I. $V_n = \lambda$~~

~~$\lambda = \lambda + \lambda + 0.24$~~

~~$-\lambda = 0.24$~~

~~$\lambda = -0.24$~~

~~∴ $V_n = \left(\frac{1 + \sqrt{5}}{2}\right)^n A + \left(\frac{1 - \sqrt{5}}{2}\right)^n B - 0.24$~~

~~$n=1$ $V_1 = 65$ $n=2$ $V_2 = 71$~~

~~$2 \times 65 = \left(\frac{1 + \sqrt{5}}{2}\right)A + \left(\frac{1 - \sqrt{5}}{2}\right)B - 0.24$ $\left(\frac{1 + \sqrt{5}}{2}\right)A + \left(\frac{1 - \sqrt{5}}{2}\right)B = 130.24$~~

~~$284.24 = \left(\frac{1 + \sqrt{5}}{2}\right)^2 A + \left(\frac{1 - \sqrt{5}}{2}\right)^2 B$~~

~~$A =$~~

~~$B =$~~

a $V_n = V_{n+1} + k V_{n+2}$

b AS $k = 0.24$

$$V_n = V_{n+1} = 0.24 V_{n+2}$$

$$v^2 - v - 0.24 = 0$$

$$v = \frac{1}{5} \quad v = -\frac{4}{5}$$

$$\text{So } V_n = \left(\frac{1}{5}\right)^n A + \left(-\frac{4}{5}\right)^n B$$

$$\text{As } n=1 \quad V_n = 65 \quad n=2 \quad V_n = 71$$

$$65 = \frac{1}{5}A - \frac{4}{5}B \quad A = 50$$

$$71 = \frac{1}{5}A + \left(-\frac{4}{5}\right)^2 B \quad B = -25$$

$$\text{So } V_n = 50\left(\frac{1}{5}\right)^n - 25\left(-\frac{4}{5}\right)^n$$

c $50\left(\frac{1}{5}\right)^n - 25\left(-\frac{4}{5}\right)^n > 10000$

$$2\left(\frac{1}{5}\right)^n - \left(-\frac{4}{5}\right)^n > 400 \quad n=47$$

$$2n \ln \frac{1}{5} + n \ln \frac{5}{4} > 400$$

$$3n > 2400$$

$$2n \ln 2 \times 5$$

$$n > \frac{1}{3} \left(\frac{2n \times 400}{2n \ln 2 \times 5} \right)$$

$$n > 46.6$$

$$\text{So } N = 47$$

1/8

Examiner comment:

The correct recurrence relation is not given in part (a), so no mark is awarded.

In part (b) the auxiliary equation written does not follow from the answer given to part (a) so the method and accuracy for setting up and solving the equation are not awarded. The required values are given in the question, so need to be correctly derived for the marks. Correct equations for $n = 1$ and $n = 2$ with $V_1 = 65$ and $V_2 = 71$ using their general form are given so the B follow through mark is given. But no working is shown for solving the equations so the final method is not awarded.

No correct method is seen in part (c), there is incorrect log work in the third line, the correct answer is not reached.

Student Response B

$$(a) V_{n+2} = V_{n+1} + kV_n$$

$$(b) V_{n+2} = V_{n+1} + 0.24V_n.$$

~~Find~~ Character

$$r^2 - r - 0.24 = 0$$

$$r = \frac{6}{5} \text{ or } -\frac{1}{5}.$$

$$\text{Hence } V_n = A\left(\frac{6}{5}\right)^n + B\left(-\frac{1}{5}\right)^n.$$

$$V_1 = \frac{6}{5}A - \frac{1}{5}B = 65$$

$$V_2 = \frac{36}{25}A + \frac{1}{25}B = 71$$

$$\Rightarrow \begin{cases} A = 50 \\ B = -75 \end{cases}$$

$$V_n = 50\left(\frac{6}{5}\right)^n - 75\left(-\frac{1}{5}\right)^n$$

$$= 50(1.2)^n - 75(-0.2)^n.$$

$$(c) V_N > 1,000,000.$$

$$50(1.2)^N - 75(-0.2)^N > 1,000,000$$

$$2(1.2)^N - (-0.2)^N > 40000.$$

$$2\left(\frac{6}{5}\right)^N - \left(-\frac{1}{5}\right)^N > 40000.$$

$$\text{When } N = 41$$

$$V_n = 88186.$$

$$\cancel{2\left(\frac{6}{5}\right)^N > 40000 + \left(-\frac{1}{5}\right)^N}.$$

$$N = 42$$

$$V_n = 105823.$$

As N can only be integer, so.

$$N = 42$$

4/8

Examiner comment:

The correct recurrence relation is given in part (a) to score the first mark.

This was a common answer to part (b) where little working is shown. For a “show that” question, candidates ought not to rely on calculator technology to give answers without supporting working. The auxiliary equation is correctly formed and the two roots are stated. Though these are given in the question, since they follow a correct auxiliary equation, the first method mark is awarded. The correct general form being stated then earns the accuracy mark following it. Two correct equations for $n = 1$ and $n = 2$ using the given information are written down, but at this stage the student just writes down the two values for A and B with no justification. These values were given in the question, and so the method mark for showing these are the correct values cannot be awarded. At this stage clear evidence of method is required. So the last two marks of part (b) are lost for such responses.

There is an attempt at trial and improvement in part (c), but due to an error when dividing through by 25, there is never a value above 54 trialled and so no marks are gained in this part.

Student Response C

$$a_1 \quad V_{n+2} = V_{n+1} + kV_n$$

$$b_1 \quad \begin{aligned} k &= 0.24 \\ V_1 &= 65 \\ V_2 &= 71 \end{aligned}$$

$$V_{n+2} - V_{n+1} - 0.24V_n = 0$$

$$\begin{aligned} r^2 - r - 0.24 &= 0 \\ (r + 0.2)(r - 1.2) &= 0 \\ r &= 1.2; -0.2 \end{aligned}$$

$$\therefore V_n = A(1.2)^n + B(-0.2)^n$$

$$V_1 = 1.2A - 0.2B = 65$$

$$V_2 = 1.44A + 0.04B = 71$$

$$5V_2 = 7.2A + 0.2B = 355$$

$$\begin{aligned} \therefore 8.4A &= 420 \\ A &= 50 \end{aligned}$$

$$\begin{aligned} \therefore 1.2(50) - 0.2B &= 65 \\ 60 - 0.2B &= 65 \\ -5 &= 0.2B \\ B &= -25 \end{aligned}$$

$$\therefore V_n = 50(1.2)^n - 25(-0.2)^n$$

$$c_1 \quad 1000000 = 50(1.2)^N - 25(-0.2)^N$$

~~$$1000000 = 50(-6)^N(-0.2)^N - 25(-0.2)^N$$~~

~~$$40000 = 2(-6)^N(-0.2)^N - (-0.2)^N$$~~

~~$$40000 = (-0.2)^N (2(-6)^N - 1)$$~~

$$\text{As } N \rightarrow \infty \Rightarrow (-0.2)^N \rightarrow 0$$

$$\therefore 1000000 = 50(1.2)^N$$

$$20000 = 1.2^N$$

$$N = \log_{1.2} 20000 = 54.32 = 54$$

$$\therefore \underline{N = 54}$$

Examiner comment:

The correct recurrence relation is given in part (a) to gain the B mark.

The correct auxiliary equation is formed and solved in part (b), with the factorisation shown, leading to the correct general form, which scores the first two marks. Correct equations for $n = 1$ and $n = 2$ using the information given are written down and the student shows sufficient working in solving these two equations, so the B mark and the method mark are shown. There were no errors in the proof so the final A is also awarded.

In part (c) this student has recognised that $(-0.2)^n$ tends to zero, and used this to enable them to solve an equation in N . However, $N = 54$ is chosen instead of $N = 55$, so the accuracy mark is lost.

Exemplar Question 4

4. (i) Use Fermat's Little Theorem to find the least positive residue of 6^{542} modulo 13 (5)
- (ii) Seven students, Alan, Brenda, Charles, Devindra, Enid, Felix and Graham, are attending a concert and will sit in a particular row of 7 seats. Find the number of ways they can be seated if
- (a) there are no restrictions where they sit in the row, (1)
- (b) Alan, Enid, Felix and Graham sit together, (2)
- (c) Brenda sits at one end of the row and Graham sits at the other end of the row, (2)
- (d) Charles and Devindra do not sit together. (2)

(Total for Question 4 is 12 marks)

Mean Score 8.4 out of 12

Examiner comment:

This question tested some number theory topics (5.4, 5.7).

For part (i) the students who used Fermat's Little Theorem in the form $6^{12} \equiv 1 \pmod{13}$ usually obtained full marks had more success than those who used the form $6^{13} \equiv 6 \pmod{13}$. Confusing the two versions and forming $6^{13} \equiv 1 \pmod{13}$, or other incorrect indices, were seen in a small number of cases, but it was pleasing to see that most were able to recall the correct formula.

The process of reducing the power using its remainder modulo 12 or 13 (as appropriate) was usually carried out well, and most were able to reach a least positive residue, though some left the answer as 36. Attempts via $6^{13} \equiv 6 \pmod{13}$ tended to be cumbersome, with many more steps required, and thus were more prone to error.

For part (ii) the vast majority of students answers (a) correctly, and part (c) was also correct more often than not, with at least the method mark gained by most. Parts (b) and (d) proved the most challenging parts of this question. For (b) most students recognised $4!$ had some part to play, but many left it at that, treating the friends as a block but not considering the order in the block. Only a minority successfully completed this part. Part (d) was also found to be difficult by many students, though many did gain the method for attempting to subtract something from 5040, but the attempts at what to subtract were usually incorrect.

It was good to see some students attempting to explain their reasoning through diagrams - students who did this generally obtained good marks in this question and were able to think through some interesting other approaches to the question yielding the correct answer.

Mark Scheme

Question	Scheme	Marks	AOs
4(i)	$6^{13-1} \equiv 1 \pmod{13}$ or $6^{13} \equiv 6 \pmod{13}$	B1	1.2
	Attempts $542 = 45 \times 12 + 2$ or $542 = 41 \times 13 + 9$ (seen or implied)	M1	1.1b
	$6^{542} = (6^{12})^{45} \times 6^2$ or $6^{542} = (6^{13})^{41} \times 6^9$	A1	1.1b
	$\equiv 1 \times 6^2 \equiv \dots \pmod{13}$ or $\equiv 6^{41} \times 6^9 \equiv (6^{13})^3 \times 6^2 \times 6^9 \equiv 6^3 \times 6^2 \times 6^9 \equiv 6^{13} \times 6 \equiv 6^2 \equiv \dots \pmod{13}$	M1	1.1b
	$\equiv 10 \pmod{13}$	A1	1.1b
		(5)	
(ii)(a)	$7! = 5040$	B1	1.1b
		(1)	
(b)	$4! \times 4! = 576$	M1 A1	3.1b 1.1b
		(2)	
(c)	$5! \times 2! = 240$	M1 A1	3.1b 1.1b
		(2)	
(d)	$7! - 6! \times 2! = 3600$ or $5! \times (2 \times 5 + 2 \times 4 + 2 \times 4 + 4) = 3600$	M1 A1	3.1b 1.1b
		(2)	
(12 marks)			
Notes			
<p>(i) B1: Recalls Fermat’s Little Theorem correctly. May be implied in their work. M1: Attempts 542 in the form $12a + b$. Score if an attempt at $(6^{12})^{45}$ or similar is seen with attempt to combine with another term. Allow M1 for attempt “$542 = \text{their } 12 \times a + b$” A1: Uses their a and b to write 6^{542} correctly in terms of 6^{12} or 6^{13} (must be one of these two) M1: Completes the process to find the residue (more convoluted roots are possible but look for a complete process to reach a residue using their attempt at Fermat’s Little Theorem at least once). A1: Correct residue. Allow if the “45” was incorrect so long as the remainder was 2.</p> <p>(ii)(a) B1: Correct value. Accept as $7!$ for this part but must be evaluated in the remaining parts.</p> <p>(b) M1: Evidence that the 4 students have been considered as one unit among many and sees the problem as permutations of 4 items. Score for $4! \times k$ where $k \neq 1$ A1: Correct value</p> <p>(c) M1: Realises that the other 5 students can sit in any position – evidenced by sight of $5!$ (in (c)) A1: Correct value</p> <p>(d) M1: A correct strategy applied. E.g. interprets the situation as the answer to part (ii)(a) minus the ways that they can sit together So score for $7! - \dots$ or $5040 - \dots$ where \dots is not zero. Alternatively, considers the different positions Devindra can sit for each different position for Charles, and with $5!$ positions for the rest. Look for $5! \times (\text{sum of 7 terms})$ oe. A1: Correct value</p>			

Examiner comment:

Some well explained alternative methods not using the subtraction rule were seen in part (ii)(d).

Student Response A

$$6^{547} \text{ modulo } 13$$

$$6^{11} \equiv 1 \pmod{13}$$

$$\frac{547}{11} = 49.727 \rightarrow 49$$

$$11 \times 49 = 539$$

$$6^{539} \times 6^3 = 6^{542}$$

~~$$6^3 \times 6^{539} = 6^{542}$$~~

$$6^3 \times (6^{11})^{49} = 6^3 \times 1 \pmod{13}$$

$$= 216 \pmod{13}$$

$$= 8 \pmod{13}$$

positive least residue
of ~~6~~ $6^{542} \pmod{13}$ is 8

ii a) $7! = 5040$

b) $7! \div 4! = 210$

$$1 \times 5 \times 5 \times 5 \times 5 \times 5 \times 1 =$$

c) $7! \div 2! = 2520$

d) $\frac{7!}{2!} = 2520$

3/12

Examiner comment:

An incorrect statement of Fermat's Little Theorem is made at the start, so the B mark is not awarded. The incorrect power of 11, instead of 12, is used in an attempt to apply the division algorithm on 542 with 11, with $542=49 \times 11 + 3$ formed within the working. This gains the method mark, since it is an attempt to use the power from their (incorrect) version of Fermat's Little Theorem. The accuracy is not awarded, though, since it is not followed through on their power and requires either 12 or 13 to have been used to break down the expression from 6^{542} . The student does proceed to reduce to a value between 1 and 13 (a least positive residue) and so the second method mark is also earned for a total of 2 marks in part (i).

The correct number of permutations is given in (ii) part (a) for the first mark, but there are no further marks earned in part (ii) thereafter. Although $4!$ is seen in part (b), it is not multiplied by something, rather it is used as a denominator, so the method is not earned. There is no sight of $5!$ anywhere in part (c) so the method is not awarded. A division, rather than a subtraction, is attempted in part (d), and it is not a correct alternative method.

Student Response B

$$(i) 6^{12} = 1 \pmod{13} \quad (\text{By Fermat's little theorem})$$

$$6^{542} = (6^{12})^{45} \times 6^2$$

$$6^{542} = 6^2 \pmod{13}$$

$$6^{542} = 36 \pmod{13}$$

$$6^{542} = 10 \pmod{13}$$

$$(ii) (a) \underline{7! = 5040}$$

(b) A E F G , B , C , D

$$\underline{4! = 24}$$

$$(c) \underline{5! = 120}$$

(d) CD, A, B, E, F, G

$$6! = 720$$

$$7! = 5040$$

$$\underline{5040 - 720 = 4320}$$

8/12

Examiner comment:

The correct statement of Fermat's Little Theorem in the first line is sufficient to gain the initial B mark. The next method and accuracy are both gained in the second line, where 542 is split correctly as $12 \times 45 + 2$ within the power, and the expression for 6^{542} is correctly formed. Fermat's Little Theorem is then applied and the student reaches a value between 1 and 13 inclusive to gain the second M, and the answer is correct, so full marks are gained in part (i).

In part (ii), the correct value is stated for (a) to gain the B mark. For part (b) the student has stated $4!$, but this is not sufficient for the method as the scheme requires the need to multiply the $4!$ by something is required, so no marks are gain in this part. However, for part (c) a recognition that $5!$ is required in some way was the requirement, so the method is earned, though the answer is incorrect. The answer to (d) was a common response, realising the need to subtract from $7!$, but failing to identify the correct value to subtract. Use of $7! - 6!$ was common, though various other incorrect difference were seen.

Student Response C

$$c1) \quad 6^{542} = 6^{45(12)+2} = 6^{45(12)} \times 6^2 \equiv 36 \pmod{13} \equiv 10 \pmod{13}$$

$$2) \quad a) \quad 7! = 5040$$

$$b) \quad 2 \times 4! = \cancel{2} \times 24 = 48$$

$$c) \quad 2 \times 5! = 240$$

$$d) \quad \text{Charles and Devindra sit together} = 6! = 720$$

$$G \text{ and } D \text{ do not sit together} = 7! - 6! = 4320$$

10/12

Examiner comment:

A very concise solution to the question. Part (i) is fully correct. The first B mark is implied by the correct use of Fermat's Little Theorem in the working when $6^{45(12)}$ is replaced by 1. The correct linear decomposition of 542 in terms of $12a + b$ is seen in the power in the second term of the line, and scores the method, with the accuracy gained in the next term when the split is correctly made, and from 36, the student proceeds to reduced to the correct residue of 10 modulo 13.

In part (ii), the correct value for $7!$ or 5040 is given in (a) for the first mark. Part (b) scores the method mark for an answer that is $4!$ times something, but the multiple used is incorrect so the accuracy is lost. Part (c) is correct for both marks. A correct strategy using the subtraction rule is made in part (d) for the method, but the value subtracted is incorrect as the student has forgotten Charles and Devindra could sit either way round.

Exemplar Question 5

5.

$$I_n = \int \operatorname{cosec}^n x \, dx \quad n \in \mathbb{Z}$$

(a) Prove that, for $n \geq 2$

$$I_n = \frac{n-2}{n-1} I_{n-2} - \frac{\operatorname{cosec}^{n-2} x \cot x}{n-1}$$

(4)

(b) Hence show that

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \operatorname{cosec}^6 x \, dx = \frac{56}{135} \sqrt{3}$$

(4)

(Total for Question 5 is 8 marks)**Mean Score 4.0 out of 8****Examiner comment:**

Reduction formulae (2.1) is a familiar and expected topic, but the proof of a reduction formula such as this one is still a challenge for many. Those who had practised similar questions to know the split required did well, while others found it difficult to make much progress.

Many students were able to answer part (a) using the main method shown on the mark scheme, splitting $\operatorname{cosec}^n x$ into $\operatorname{cosec}^{n-2} x$ and $\operatorname{cosec}^2 x$, and correctly integrating with the appropriate choice of u and v . Those who attempted the alternative ways were fewer in number and mainly unsuccessful. Most of the algebra seen was correct, but some weaker students could not start this part or made errors at an early stage in their integration.

Good students made their final method in part (b) very clear, using a complete series of reduction formulae and substituting explicitly at the end to obtain 3 terms which summed to the given answer. A few students made arithmetic slips or misused the reduction formula, failing to substitute values for n correctly.

Both parts had given answers (“prove that” and “show that”) and once again students need to remember to be clear in all steps of working to produce a convincing solution in such case. Many lost marks in part (b) in particular for arriving at the given answer without sufficient evidence of the substitution into their formula having been made.

Mark Scheme

Question	Scheme	Marks	AOs
5(a)	$I_n = \int \operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x \, dx$ $u = \operatorname{cosec}^{n-2} x, \frac{dv}{dx} = \operatorname{cosec}^2 x, \int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx$	M1	2.1
	$I_n = -\operatorname{cosec}^{n-2} x \cot x - (n-2) \int \operatorname{cosec}^{n-3} x (-\operatorname{cosec} x \cot x) (-\cot x) \, dx$	A1	1.1b
	$I_n = -\operatorname{cosec}^{n-2} x \cot x - (n-2) \int \operatorname{cosec}^{n-2} x \cot^2 x \, dx$		
	$I_n = -\operatorname{cosec}^{n-2} x \cot x - (n-2) \int \operatorname{cosec}^{n-2} x (\operatorname{cosec}^2 x - 1) \, dx$ $I_n = -\operatorname{cosec}^{n-2} x \cot x - (n-2) I_n + (n-2) I_{n-2}$	dM1	1.1b
	$(n-1) I_n = -\operatorname{cosec}^{n-2} x \cot x + (n-2) I_{n-2}$		
	$I_n = \frac{(n-2)}{n-1} I_{n-2} - \frac{\operatorname{cosec}^{n-2} x \cot x}{n-1} *$	A1*	2.1
		(4)	
(b)	$I_6 = \frac{4}{5} I_4 - \frac{\operatorname{cosec}^4 x \cot x}{5} \text{ or } [I_6]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{4}{5} [I_4]_{\frac{\pi}{3}}^{\frac{\pi}{2}} - \left[\frac{\operatorname{cosec}^4 x \cot x}{5} \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$	M1	1.1b
	$= \frac{4}{5} \left(\frac{2}{3} I_2 - \frac{\operatorname{cosec}^2 x \cot x}{3} \right) - \frac{\operatorname{cosec}^4 x \cot x}{5} \text{ or with limits etc}$	M1	1.1b
	$[I_6]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{8}{15} [-\cot x]_{\frac{\pi}{3}}^{\frac{\pi}{2}} - \left[\frac{4 \operatorname{cosec}^2 x \cot x}{15} \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} - \left[\frac{\operatorname{cosec}^4 x \cot x}{5} \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$	M1	2.1
	$= \frac{8}{15} \left(\frac{\sqrt{3}}{3} \right) + \frac{16\sqrt{3}}{135} + \frac{16\sqrt{3}}{135} = \frac{56}{135} \sqrt{3} *$	A1*	2.2a
		(4)	
(8 marks)			
Notes			
<p>(a) For Alt 1 and any other similar approaches marking follows the same pattern M1: Splits the integrand into the product as shown and begins the process of integration by parts For Alt 2 this requires applying the expression for $\cot^2 x$ in terms of $\operatorname{cosec}^2 x$, splitting the integral and setting up the process for integration by parts on the composite term. A1: Correct expression (for Alt 2 it is for a correct application of parts on their second term) dM1: Depends on previous M. Applies $\cot^2 x = \pm 1 \pm \operatorname{cosec}^2 x$ and introduces I_n and I_{n-2} (For Alt 2 this is for complete substitution for I_n and I_{n-2}) A1*: Completes the proof by making I_n the subject with no errors seen (but condone minor notational slips). For Alt 1 a clear statement of replacing n by $n - 2$ or should be made.</p> <p>(b) M1: Begins process of application of reduction to find I_6 in terms of I_4 (need not evaluate terms) or deduces the value of I_2 (Alt 1) M1: Uses the reduction formula correctly to find I_4 in terms of I_2 (need not be evaluated yet). M1: A fully correct method using the reduction formula correctly to reach a value for I_6. Substitutions must be shown for the non-zero terms but accept decimals/trig functions for the M. A1*: Reaches the printed answer with no errors, relevant working shown and trig terms evaluated</p>			

5(a) ALT 1	$I_n = \int \operatorname{cosec}^{n+1} x \sin x \, dx$ (Allow with $n \pm 1$ in power for M's) $u = \operatorname{cosec}^{n+1} x, \frac{dv}{dx} = \sin x, \int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx$	M1	2.1
	$I_n = -\operatorname{cosec}^{n+1} x \cos x - (n+1) \int \operatorname{cosec}^n x (-\operatorname{cosec} x \cot x)(-\cos x) \, dx$	A1	1.1b
	$I_n = -\operatorname{cosec}^n x \cot x - (n+1) \int \operatorname{cosec}^n x \cot^2 x \, dx$		
	$I_n = -\operatorname{cosec}^n x \cot x - (n+1) \int \operatorname{cosec}^n x (\operatorname{cosec}^2 x - 1) \, dx$ $I_n = -\operatorname{cosec}^n x \cot x - (n+1) I_{n+2} + (n+1) I_n$	dM1	1.1b
	$(n+1) I_{n+2} = -\operatorname{cosec}^n x \cot x + n I_n$		
	replacing n by $n-2$ gives $I_n = \frac{(n-2)}{n-1} I_{n-2} - \frac{\operatorname{cosec}^{n-2} x \cot x}{n-1} *$	A1*	2.1
		(4)	
5(a) ALT 2	$I_n = \int \operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x \, dx = \int \operatorname{cosec}^{n-2} x (1 + \cot^2 x) \, dx$ $= \int \operatorname{cosec}^{n-2} x \, dx + \left(\int \operatorname{cosec}^{n-2} x \cot x \right) (\cot x) \, dx$ $u = \cot x, \frac{dv}{dx} = \operatorname{cosec}^{n-2} x \cot x, \int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx$	M1	2.1
	$I_n = I_{n-2} + \cot x \left(-\frac{\operatorname{cosec}^{n-2} x}{n-2} \right) - \int \left(-\frac{\operatorname{cosec}^{n-2} x}{n-2} \right) (-\operatorname{cosec}^2 x) \, dx$	A1	1.1b
	$(n-2) I_n = (n-2) I_{n-2} - \operatorname{cosec}^{n-2} x \cot x - \int \operatorname{cosec}^n x \, dx$		
	$(n-2) I_n = (n-2) I_{n-2} - \operatorname{cosec}^{n-2} x \cot x - I_n$	dM1	1.1b
	$(n-1) I_n = -\operatorname{cosec}^{n-2} x \cot x + (n-2) I_{n-2}$		
	$I_n = \frac{(n-2)}{n-1} I_{n-2} - \frac{\operatorname{cosec}^{n-2} x \cot x}{n-1} *$	A1*	2.1
		(4)	
(b) ALT	$I_2 = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \operatorname{cosec}^2 x \, dx = [-\cot x]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{\sqrt{3}}{3}$	M1	2.2a
	$I_4 = \frac{2}{3} I_2 - \left[\frac{\operatorname{cosec}^2 x \cot x}{3} \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{2}{9} \sqrt{3} + \frac{4}{27} \sqrt{3}$	M1	1.1b
	$I_6 = \frac{4}{5} I_4 - \left[\frac{\operatorname{cosec}^4 x \cot x}{5} \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{4}{5} \left(\frac{4}{27} \sqrt{3} + \frac{2}{9} \sqrt{3} \right) + \frac{16}{135} \sqrt{3}$	M1	2.1
	$= \frac{56}{135} \sqrt{3} *$	A1*	1.1b
			(4)

Student Response A

$$a) I_n = \int \operatorname{cosec}^n x = \int \operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x$$

$$u = \operatorname{cosec}^{n-1} x \qquad v' = \operatorname{cosec} x$$

$$u' = (n-1) \operatorname{cosec}^{n-2} x \times \operatorname{cosec} x \cot x$$

$$= (n-1) \operatorname{cosec}^{n-1} x \cot x$$

$$\text{or } \operatorname{csc}^2 x - \cot^2 x = 1$$

$$\cot^2 x + 1 = \operatorname{cosec}^2 x$$

$$I_n = \int \operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x = \int \operatorname{cosec}^{n-2} x \cot^2 x + \int \operatorname{cosec}^{n-2} x$$

$$I_n = I_{n-2} + \int \operatorname{cosec}^{n-2} x \cot^2 x$$

$$u = \operatorname{cosec}^{n-2} x$$

$$u' = (n-2) \operatorname{cosec}^{n-3} x \cot x$$

$$v' = \cot^2 x$$

$$\int \operatorname{cosec}^2 \cot^2 x \operatorname{cosec}^n$$

$$(n-1)I_n = (n-2)I_{n-2} - \operatorname{cosec}^{n-2} x \cot x$$

$$I_n$$

$$1) \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \operatorname{cosec}^6 x$$

$$I_6 = \frac{4}{5} I_4 - \frac{\operatorname{cosec}^4 x \cot x}{5}$$

$$= -\frac{\operatorname{cosec}^4 x \cot x}{5} + \frac{4}{5} \left[\frac{2}{3} I_2 - \frac{\operatorname{cosec}^2 x \cot x}{3} \right]$$

$$= -\frac{\operatorname{cosec}^4 x \cot x}{5} - \frac{4}{15} \operatorname{cosec}^2 x \cot x + \frac{8}{15} I_2$$

$$= -\frac{\operatorname{cosec}^4 x \cot x}{5} - \frac{4}{15} \operatorname{cosec}^2 x \cot x + \frac{8}{15} [-\cot x]$$

$$= \left[-\frac{\operatorname{cosec}^4 x \cot x}{5} - \frac{4}{15} \operatorname{cosec}^2 x \cot x - \frac{8}{15} \cot x \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$\frac{1}{3} - \left(-\frac{2}{15} - \frac{4}{15} - \frac{8\sqrt{3}}{45} \right)$$

$$= \left(-\frac{\frac{1}{5}}{\frac{16}{9} \left(\frac{\sqrt{3}}{3} \right)} - \frac{\frac{4}{15}}{\frac{2}{3} \left(\frac{\sqrt{3}}{3} \right)} + \frac{\frac{8}{15}}{\sqrt{3}} \right)$$

$$= \frac{56}{135} \sqrt{3}$$

3/8

Examiner comment:

The student makes no attempt to apply integration by parts on a correctly split integrand. The first attempt has an incorrect split and make little progress, and while a correct split of the integrand into $\operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x$ is made in a second attempt, with replacement of $\operatorname{cosec}^2 x$ by $1 + \cot^2 x$, there is no attempt to apply integration by parts to this. The second page sees an attempt to start with the given result and rearrange, but no progress is made of any note, so no marks are scored in part (a).

As is usual, even when no marks are scored in part (a) there is still scope for progress of applying the result in part (b). A correct process of reduction is used to first find I_6 in terms of I_4 and then in terms of I_2 recursively, scoring the first two method marks. A further correct application of the reduction is then made to achieve a result in terms of trigonometric terms only and an attempt at substituting the limits is seen. There was deemed enough evidence of an attempt to apply the correct limits here, the confusion is with the trigonometric ratios, not the values. But as the penultimate line is incorrect (some of the trig ratios are in the wrong places in their expression) the final accuracy is withheld.

Student Response B

$$\begin{aligned}
 a) I_n &= \int \operatorname{cosec}^n x \operatorname{cosec}^{(n-2)} x \\
 I_n &= \int \operatorname{cosec}^{(n-2)} x \operatorname{cosec}^2 x \, dx \\
 &= \int \operatorname{cosec}^{(n-2)} x (1 + \cot^2 x) \, dx \\
 &= \int \operatorname{cosec}^{n-2} x \, dx + \int \operatorname{cosec}^{(n-2)} x \cot^2 x \, dx \\
 &= I_{n-2} + \frac{1}{n-1} \operatorname{cosec}^{(n-1)} x \\
 &= I_{n-2} + \frac{1}{n-1} \operatorname{cosec}^{(n-1)} x \cot x \quad \begin{array}{l} \frac{dx}{dx} = \operatorname{cosec}^{(n-2)} \cot x \\ \frac{dx}{dx} = -\operatorname{cosec}^2 x \quad v = \frac{1}{n-2} \operatorname{cosec}^{(n-2)} \end{array} \\
 &= I_{n-2} - \frac{\operatorname{cosec}^{(n-2)} x \cot x}{n-2} + \frac{1}{n-2} \int \operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x \\
 &= I_{n-2} - \frac{\operatorname{cosec}^{(n-2)} x \cot x}{n-2} - \frac{1}{n-2} I_n \\
 2I_n &= (n-2)I_{n-2} - \operatorname{cosec}^{(n-2)} x \cot x
 \end{aligned}$$

$$\begin{aligned}
 & \text{b) } \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \operatorname{cosec}^6 x \, dx \\
 &= \left[I_6 \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\
 &= \left[\frac{4}{5} I_4 - \frac{\operatorname{cosec}^4 x \cot x}{5} \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\
 &= \left[\frac{4}{5} \left(\frac{2}{3} I_2 - \frac{\operatorname{cosec}^2 x \cot x}{3} \right) - \frac{\operatorname{cosec}^4 x \cot x}{5} \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\
 &= \left[\frac{4}{5} \left(\frac{2}{3} \left(\cancel{I_0} - \cot x \right) - \frac{\operatorname{cosec}^2 x \cot x}{3} \right) - \frac{\operatorname{cosec}^4 x \cot x}{5} \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\
 &= \left[\frac{4}{5} \left(\frac{2}{3} (x - \cot x) - \frac{\operatorname{cosec}^2 x \cot x}{3} \right) - \frac{\operatorname{cosec}^4 x \cot x}{5} \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\
 &= \left[\frac{4}{5} \left(\frac{2x}{3} - \frac{2}{3} \cot x - \frac{\operatorname{cosec}^2 x \cot x}{3} \right) - \frac{\operatorname{cosec}^4 x \cot x}{5} \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\
 &= \left[\frac{8}{15} x - \frac{8}{15} \cot x - \frac{4 \operatorname{cosec}^2 x \cot x}{15} - \frac{\operatorname{cosec}^4 x \cot x}{5} \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\
 &= \left(\frac{8\pi}{30} - \frac{4}{15} - \frac{3}{15} \right) - \left(\frac{8\pi}{45} - \frac{8\sqrt{3}}{45} - \frac{16\sqrt{3}}{135} - \frac{16\sqrt{3}}{135} \right) \\
 &=
 \end{aligned}$$

5/8

Examiner comment:

This student used alternative 2 from the mark scheme in part (a). A correct split of the integrand into $\operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x$ is made, and then $\operatorname{cosec}^2 x$ is replaced by $1 + \cot^2 x$ before a correct attempt at integration by parts is made. Note that this method requires the process of applying part to be reached before any credit is given. In this case the process is carried out correctly, so the first two marks are scored for the correct result of applying integration by parts to the reformed integral. Having achieved this the next method is awarded for writing all the integrals in their expression in terms of I_n and I_{n-2} but the student does not go on to reach the printed answer and so the final accuracy mark cannot be awarded, scoring 3 marks in part (a).

A correct process of reduction is used in part (b) to first find I_6 in terms of I_4 and then in terms of I_2 recursively, scoring the first two method marks. An incorrect application of the reduction in finding I_2 in terms of I_0 loses the final method and accuracy. For the final M, the reduction needs to have been correct as well as seeing the substitution of limits.

Student Response C

$$\begin{aligned}
 \text{a) } I_n &= \int \operatorname{cosec}^n x \, dx = \int \operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x \, dx & u &= \operatorname{cosec}^{n-2} x \\
 & & \frac{du}{dx} &= (n-2) \operatorname{cosec}^{n-3} x \cdot \frac{-\operatorname{cosec} x}{\cot x} \\
 &= -\operatorname{cosec}^{n-2} x \cot x & \frac{dv}{dx} &= \operatorname{cosec}^2 x \\
 & & v &= -\cot x \\
 &= \int (n-2) \operatorname{cosec}^{n-3} x \operatorname{cosec} x \cot^2 x \, dx \\
 &= -\operatorname{cosec}^{n-2} x \cot x - (n-2) \int \operatorname{cosec}^{n-2} x (1 - \operatorname{cosec}^2 x) \, dx \\
 &= -\operatorname{cosec}^{n-2} x \cot x + (n-2) \int \operatorname{cosec}^{n-2} x \, dx \\
 &= -\operatorname{cosec}^{n-2} x \cot x + (n-2) I_{n-2} \\
 \therefore (n-1) I_n &= -\operatorname{cosec}^{n-2} x \cot x + (n-2) I_{n-2} \\
 I_n &= \frac{-\operatorname{cosec}^{n-2} x \cot x}{n-1} + \frac{(n-2) I_{n-2}}{n-1}
 \end{aligned}$$

$$\begin{aligned}
 \text{b) } \int_0^{\pi/3} \operatorname{cosec}^6 x \, dx &= \frac{4}{5} \int_0^{\pi/3} \operatorname{cosec}^4 x \, dx - \frac{\operatorname{cosec}^4 x \cot x}{5} \\
 &= \frac{4}{5} \left(\frac{2}{3} \int_0^{\pi/3} \operatorname{cosec}^2 x \, dx - \frac{\operatorname{cosec}^2 x \cot x}{3} \right) - \frac{\operatorname{cosec}^4 x \cot x}{5} \\
 &= \frac{4}{5} \left(\frac{2}{3} (-\cot x) - \frac{\operatorname{cosec}^2 x \cot x}{3} \right) - \frac{\operatorname{cosec}^4 x \cot x}{5}
 \end{aligned}$$

$$= \frac{4}{5} \left(\frac{2}{3} (-\cot x) - \frac{\operatorname{cosec}^2 x \cot x}{3} \right) - \frac{\operatorname{cosec}^4 x \cot x}{5}$$

$$\begin{aligned}
 \therefore \int_0^{\pi/3} \operatorname{cosec}^6 x \, dx &= \left[\frac{4}{5} \left(\frac{2}{3} (-\cot x) - \frac{\operatorname{cosec}^2 x \cot x}{3} \right) - \frac{\operatorname{cosec}^4 x \cot x}{5} \right]_0^{\pi/3} \\
 &= \frac{4}{5} (0) - \left(\frac{4}{5} \left(\frac{2}{3} \left(\frac{-\sqrt{3}}{3} \right) - \frac{4\sqrt{3}}{27} \right) - \frac{16\sqrt{3}}{135} \right) \\
 &= \frac{8\sqrt{3}}{27} + \frac{16\sqrt{3}}{135} = \frac{40\sqrt{3} + 16\sqrt{3}}{135} = \frac{56\sqrt{3}}{135}
 \end{aligned}$$

7/8

Examiner comment:

A correct split of the integrand into $\operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x$ followed by a correct application of integration by parts the correct way scores the first two marks. The student then applies an incorrect identity $\cot^2 x = 1 - \operatorname{cosec}^2 x$ before introducing I_n 's to the equation, but as an identity of the form $\cot^2 x = \pm \operatorname{cosec}^2 x \pm 1$ is used, this is sufficient for the method, but the accuracy mark is lost due to the error. Note that the student changes signs later in the proof to achieve the correct identity at the end, so work needs to be checked carefully.

Part (b) is fully correct with sufficient working and substitution of limits seen and no errors in the working, so scores full marks.

Exemplar Question 6

6. (i) A binary operation $*$ is defined on positive real numbers by

$$a * b = a + b + ab$$

Prove that the operation $*$ is associative.

(4)

- (ii) The set $G = \{1, 2, 3, 4, 5, 6\}$ forms a group under the operation of multiplication modulo 7

(a) Show that G is cyclic.

(2)

The set $H = \{1, 5, 7, 11, 13, 17\}$ forms a group under the operation of multiplication modulo 18

(b) List all the subgroups of H .

(3)

(c) Describe an isomorphism between G and H .

(3)

(Total for Question 6 is 12 marks)

Mean Score 6.5 out of 12

Examiner comment:

This question tested some aspects of group theory (section 1 of the specification), one of the new topics for the new specification.

In part (i) the most common score profile was M1 M1 A0 A1. The general outline of the proof was attempted well, but many lacked the rigour required as they did not achieve suitable expressions for the expansions to verify both yielded the same. The definition of associativity was understood by nearly all of the students sitting the paper, but a small number of them gave a numerical example of it working for once case, rather than a proof. A single example will be required in order to show an operation is not associative, but is not sufficient to prove an operation is.

Part (ii) is where the most of the difficulty in this question lay. For part (a) many students realised the group was cyclic since 3 (only a small number used 5) was a generator, but some merely stated this was so because its order is 6 and did not show how it generated the rest of the group. Those who listed the powers of 3 (or 5) would usually gain both marks for completing the argument that the group was cyclic. Others assumed that writing out the group tables was proof of being cyclic.

Identifying the subgroups in part (b) produced a variety of response, but usually scoring two of the available marks. The subgroup $\{1, 17\}$ was usually identified correctly, but $\{1, 7, 13\}$ was sometimes missed, as were one or other of $\{1\}$ and H itself. many also identified spurious subgroups such as $\{1, 5\}$, $\{1, 7\}$, $\{1, 13\}$ and so on, assuming 1 and any element formed a subgroup. Nevertheless, many were able to identify the three correct subgroups and no others.

Part (c) proved provided the most challenge for students in this question. Most commonly either 3 or 0 marks were scored on this part as students either knew what to do and got it correct, or did not even know where to start. Many would just match up elements in ascending order, which did not gain access to marks. Others attempted to explain what an isomorphism is in general terms rather than identify a specific isomorphism for the group.

There were some good attempts at defining mappings algebraically by identifying that powers of a generator needed to map to powers of a generator. Students attempting such an approach usually scored well.

Partially correct attempts at this approach tended to stem from attempting to match elements by order, which would give correct pairings $1 \rightarrow 1$ and $6 \rightarrow 17$ and that $\{2, 4\} \rightarrow \{7, 13\}$ and $\{3, 5\} \rightarrow \{5, 11\}$, but would then choose incorrect pairings for the latter two elements for their pairings for $\{2, 4\}$. Some did not make a selection at all, but left the set-wise mappings, which was not sufficient information to identify an isomorphism.

Mark Scheme

Question	Scheme	Marks	AOs														
6(i)	$(a*b)*c = (a+b+ab)*c = a+b+ab+c+(a+b+ab)c$	M1	2.1														
	$a*(b*c) = a*(b+c+bc) = a+b+c+bc+a(b+c+bc)$	M1	2.1														
	$\underline{a+b+ab+c+(a+b+ab)c = a+b+c+bc+ab+ac+abc}$ $\underline{= a+b+c+bc+a(b+c+bc)}$	A1	2.2a														
	so $(a*b)*c = a*(b*c)$ which means * is associative	A1	2.4														
		(4)															
(ii)(a)	$3^2 = 2 \quad 3^3 = 6 \quad 3^4 = 4 \quad 3^5 = 5 \quad 3^6 = 1$ or $5^2 = 4 \quad 5^3 = 6 \quad 5^4 = 2 \quad 5^5 = 3 \quad 5^6 = 1$	M1	2.1														
	Or special case for M1A0 if powers not shown: 3 has order 6 so generates the group																
	3 (or 5) has order 6 and so generates the group so G is cyclic	A1	2.4														
		(2)															
(b)	$\{1\}, H$	B1	1.1b														
	$\{1, 17\}$ or $\{1, 7, 13\}$	M1	1.1b														
	$\{1, 17\}$ and $\{1, 7, 13\}$ (and no others)	A1	1.1b														
		(3)															
(c)	<table border="1" style="display: inline-table; margin-right: 20px;"> <tr><td>G</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td></tr> <tr><td>H</td><td>1</td><td>7</td><td>5</td><td>13</td><td>11</td><td>17</td></tr> </table>	G	1	2	3	4	5	6	H	1	7	5	13	11	17	M1	3.1a
	G	1	2	3	4	5	6										
	H	1	7	5	13	11	17										
	or																
	<table border="1" style="display: inline-table;"> <tr><td>G</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td></tr> <tr><td>H</td><td>1</td><td>13</td><td>11</td><td>7</td><td>5</td><td>17</td></tr> </table>	G	1	2	3	4	5	6	H	1	13	11	7	5	17	A1	1.1b
G	1	2	3	4	5	6											
H	1	13	11	7	5	17											
	A1	1.1b															
	(3)																

(12 marks)

Notes

(i)
M1: Begins proof by correctly expanding $(a*b)*c$ **or** $a*(b*c)$ to an expression in a, b and c . Note they may expand as $(a*b)*c = (a*b)+c+(a*b)c = a+b+ab+c+(a+b+ab)c$ which is equally fine.
M1: Makes progress towards the required result by attempting to expand both $(a*b)*c$ **and** $a*(b*c)$, but be generous with the attempts for this method. May achieve this by working from left to right, so look for arriving at the other expression through a chain of equalities.
A1: For both underlined expressions (but accept eg. $c(a+b+ab)$ for $(a+b+ab)c$) and a correct expansion seen for each independently or part of a chain as shown. The expansion may have terms in different orders.
A1: Explains that $(a*b)*c = a*(b*c)$ means that * is associative. Depends on both M marks and a correct expression having been found.
(ii)(a)
M1: Demonstrates understanding of the term cyclic by either attempting all the powers of 3 or 5.

Accept for this a statement $\langle 3 \rangle = \{3, 2, 6, 4, 5, 1\}$ which shows the elements list in order of powers.

A1: Must have evaluated all powers of 3 or 5 correctly and explains why the group is cyclic.

Accept as 3 generates the group, or as 3 has the same order of G as reason. Must refer to cyclic in conclusion.

Special case: Allow M1A0 for a correct explanation of why G is cyclic if the order of 3 (or 5) is stated as 6 without justification – but must include reference to either being a generator or having the same order as G .

(b) (You may ignore references to the operation for this part)

B1: Identifies $\{1\}$ and H as subgroups

M1: Identifies $\{1, 17\}$ **or** $\{1, 7, 13\}$ as a subgroup

A1: Identifies $\{1, 17\}$ **and** $\{1, 7, 13\}$ as subgroups and no others

(c)

M1: Attempts to identify an isomorphism between the groups – may be implied by

- identifying at least 2 correct non-identity pairings or
- by attempting to rearrange group tables to have the same structure, or
- by attempting to map powers of a generator to powers of a generator.g
 $(\text{their } 3)^k \rightarrow (\text{their } 5)^k$ or
- by matching of non-trivial proper subgroups to each other.

A1: Identifies 4 correct pairings, or sets up a mapping with one correct generator

A1: All pairings correct, or sets up a mapping with generators of each group correct, eg. $3^k \rightarrow 5^k$

Student Response A

i, Let $a = 1$ $b = 2$

$a * b = 1 * 2 = 1 + 2 + 1 * 2 = 3 + 2 = 5$

~~$a * b =$~~

$b * a = 2 * 1 = 2 + 1 + 2 * 1 = 3 + 2 = 5$

ii, $\langle a \rangle = \{1, 2, 3, 4, 5, 6\}$

$\langle 1 \rangle = \{1\}$

$\langle 2 \rangle = \{2, 4, 1\}$

$\langle 3 \rangle = \{3, 2, 6, 4, 5, 1\}$

$\langle 4 \rangle = \{4, 2, 1\}$

$\langle 5 \rangle = \{5, 4, 6, 2, 3, 1\}$

$\langle 6 \rangle = \{6, 1\}$

X_7	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

3 and 5 are generators of the group, therefore it is a cyclic group.

b, $H = \{1, 5, 7, 11, 13, 17\}$

$\{1, 7, 13\}$

$\{0\}$

$\{1, 17\}$

$\{1\}$

X_{11}	1	5	7	11	13	17
1	1	5	7	11	13	17
5	5	7	17	1	11	13
7	7	17	13	5	1	11
11	11	1	5	13	17	7
13	13	11	1	17	7	5
17	17	13	11	7	5	1

c, $F(6) = 17$

Examiner comment:

No marks are awarded for part (i). There is no attempt at all to expand either side of the associativity axiom equation and so neither method mark is available. The student appear to be attempt to show commutativity rather than associativity and only checks one instance, rather than attempts a general proof.

In part (ii)(a) the method is awarded as the student has correctly identified the powers of 3, evidence by the correct order of elements in forming the subgroup $\langle 3 \rangle = \{3, 2, 6, 4, 5, 1\}$. Also $\langle 5 \rangle$ is correct, but the method is already gained, and so the other subgroups and group table are superfluous work here. A suitable conclusion that 3 (and 5) are generators and hence the group is cyclic gains the A mark.

For part (ii)(b) the student has listed all four correct subgroups but has also included a spurious extra one in $\{0\}$ (and 0 is not even an element of the group) and so the B mark for H and $\{1\}$ and method for at least one of the other two are awarded but the accuracy is withheld due to the incorrect extra one.

No marks are given in (ii)(c) as at least two non-identity pairings are needed, but only one pairing is given.

Student Response B

$$\begin{aligned} \text{c) } (a * b) * c &= (a + b + ab) * c \\ &= a + b + ab + c + ac + bc + abc \\ &= a + b + c + ab + bc + ca + abc \end{aligned}$$

$$\begin{aligned} a + (b * c) &= a + (b + c + bc) \\ &= a + b + c + ab + bc + ab + c + abc \\ &= a + b + c + ab + bc + ca + abc \end{aligned}$$

$\therefore (a + b) * c = a + (b + c) \therefore$ associative.

c) x_7

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

$\therefore G$ is a group as it is closed, has inverses, has an identity (1) and is associative.

Every element of G has order 7 so G is cyclic.

b) ~~Let~~

$$\begin{aligned} &(\{1, x_{18}\}), (\{1, 5, 7, 11, 13, 17\}, x_{18}), (\{1, 7, 13\}, x_{18}) \\ &(\{1, 17\}, x_{18}) \end{aligned}$$

c) $f: G \rightarrow H$

$$\begin{aligned} \therefore f(1) &= 1 \\ f(2) &= 5 \\ f(3) &= 7 \\ f(4) &= 11 \\ f(5) &= 13 \\ f(6) &= 17 \end{aligned}$$

as $G = \langle \{1, 2, 3, 4, 5, 6\}, x_7 \rangle$ & $H = \langle \{1, 5, 7, 11, 13, 17\}, x_{18} \rangle$

Examiner comment:

Once again a correct overall procedure for part (i) is seen, with correct expansions of both sides of the associativity axiom attempted and an appropriate conclusion stating both sides are equal being given. But the second stage of expansion jumps to a full expansion, rather than showing clear how the operation applies at this stage, and so the first of the accuracy marks is not awarded, scoring 3 marks of the four in this part.

For part (ii), in part (a) the student starts by writing out the full group table, but this gains no marks as it does not show why G is cyclic. The student has checked subgroup axioms rather than a criteria for being cyclic. There is an erroneous claim that every element of G has order 7, but even here there is no explanation as to why this would make the group cyclic, as there is no reference to the order of G .

The correct four subgroups of H and no extras are listed in part (b), and so all three marks are scored. In part (c) the student attempts to list pairings but has kept elements in ascending order which means only 1 mapping to 1 and 6 to 17 are correct pairings, but to access the method at least two correct non-identity pairings are required, so no marks are scored for this part.

Student Response C

$$(i) (a * b) * c = (a + b + ab) * c$$

$$= a + b + ab + c + ca + cb + cab$$

$$a * (b * c) = a * (b + c + bc)$$

$$= a + b + c + a + ab + ac + abc$$

$$= a + b + ab + c + ca + cb + cab$$

$$\therefore (a * b) * c = a * (b * c) \quad \therefore \text{associative}$$

ii)

~~12 + 3~~ consider element 3

$$3 \equiv_7 3 \quad 3^2 \equiv_7 2 \quad 3^3 \equiv_7 6 \quad 3^4 \equiv_7 4$$

$$3^5 \equiv_7 5 \quad 3^6 \equiv_7 1$$

\therefore element $\langle 3 \rangle$ generates G

$\therefore G$ is cyclic

$$\begin{aligned}
 b) \quad & 111 = 1 \\
 & 151 = 6 \\
 & 171 = 3 \qquad 13 \\
 & 1111 = 6 \\
 & 1131 = 3 \\
 & 1171 = 2
 \end{aligned}$$

H itself is a subgroup (not proper)

$\{1\}$ only identity

$\{1, 7, 13\}$

$\{1, 17\}$

c) 3 generates G
5 generates H

$\therefore f: G \rightarrow H$ such that $f(3^k) = 5^k$

11/12

Examiner comment:

A common responses to part (i), the correct two expansions are attempted, both correct, and a suitable conclusion is made, but there is insufficient working for the expansions for the proof. To satisfy the scheme clear application of the operation needs to be seen on both expansion, but the full expansion here written out following the first stage of expanding. This was a usual response and scores 3 out of four marks as noted above.

For part (ii) we see a fully correct answer. In (a) the powers of 3 are all evaluated and a conclusion referencing G being cyclic as 3 generates it is given. Incorrect use of the brackets in $\langle 3 \rangle$ is condoned. All four subgroups of H and no extras are listed in part (b). The answer to part (c) identifies the isomorphism as a mapping of powers of one generator to power of a generator in the image group, with two correct generators identified and so describes the isomorphism correctly for all three marks.

Exemplar Question 7

7. A transformation from the z -plane to the w -plane is given by

$$w = \frac{3iz - 2}{z + i} \quad z \neq -i$$

- (a) Show that the circle C with equation $|z + i| = 1$ in the z -plane is mapped to a circle D in the w -plane, giving a Cartesian equation for D . (4)
- (b) Sketch C and D on Argand diagrams. (2)

(Total for Question 7 is 6 marks)

Mean Score 3.7 out of 6

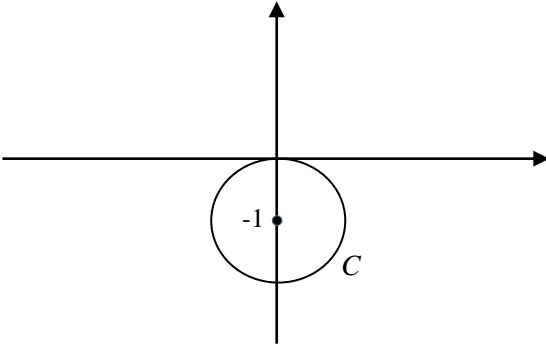
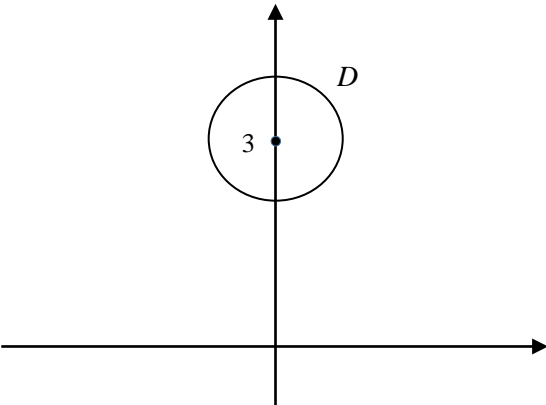
Examiner comment:

This question tested further complex number works, specifically elementary transformations of the complex plane (4.2).

Students who used the main method exemplified in the scheme in part (a) were mainly successful in reaching the correct circle equation, although some did not then express it in Cartesian form. Attempts using the other methods were less common and far less successful, as the complications of algebra led almost inevitably to errors.

For part (b) a number of students were unable to place their circles correctly on the Argand diagram, placing their centres on the real axis not the imaginary axis, or above the real axis when it should have been below. Some omitted this part entirely - usually if they had made little progress in part (a), although circle C did not depend on any work in (a) and should have been an easily accessible mark. The sketching of simple loci such as these ones should be providing students some accessible marks at this stage of the paper, so it was surprising to see such difficulty with them.

Mark Scheme

Question	Scheme	Marks	AOs
7(a)	$w = \frac{3iz - 2}{z + i} \Rightarrow z = \frac{2 + wi}{3i - w}$	M1	2.1
	$ z + i = 1 \Rightarrow \left \frac{2 + wi}{3i - w} + i \right = 1$	M1	2.1
	$\left \frac{2 + wi - 3 - iw}{3i - w} \right = 1 \Rightarrow \left \frac{1}{w - 3i} \right = 1$	A1	1.1b
	$ w - 3i = 1 \Rightarrow u^2 + (v - 3)^2 = 1$	A1	2.2a
		(4)	
(b)		B1	1.1b
		B1ft	1.1b
		(2)	

(6 marks)

Notes

- (a)
 M1: Attempts to solve the problem by attempting to make z the subject of the formula
 M1: Uses the given locus to obtain an equation in w
 A1: Obtains a correct simplified equation in terms of w
 A1: Deduces the correct form of the equation (allow x , y or u , v etc.)
- (b)
 B1: The circle C correctly positioned, passing through the origin coordinates of centre labelled. Accept as coordinates or marked on axes.
 B1ft: Their D correctly positioned with the centre correctly labelled (accept as coordinates or marked on axes).
 Accept both drawn on the same diagram.
 Allow S.C. B1B0 for two circles in correct respective positions but with no labelling.

7(a) ALT 1	$w = \frac{3iz - 2}{z + i} \Rightarrow w(z + i) = 3i(z + i) + 3 - 2$ Attempts to isolate $z + i$ terms	M1	2.1
	$(z + i)(w - 3i) = 1 \Rightarrow (z + i)(w - 3i) = 1 \Rightarrow (w - 3i) = 1$ Gathers $z + i$ terms and applies $ (z + i) = 1$	M1	2.1
	As main scheme	A1	1.1b
	As main scheme	A1	2.2a
7(a) ALT 2	$w = \frac{3iz - 2}{z + i} \Rightarrow z = \frac{2 + wi}{3i - w}$ as main scheme	M1	2.1
	$x + yi = \frac{2 - v + ui}{-u - (v - 3)i} \times \frac{-u + (v - 3)i}{-u + (v - 3)i} = \dots = \frac{u - (u^2 + v^2 - 5v + 6)i}{u^2 + (v - 3)^2}$ $\Rightarrow \left(\frac{u}{u^2 + (v - 3)^2} \right)^2 + \left(\frac{-(u^2 + v^2 - 5v + 6)}{u^2 + (v - 3)^2} + 1 \right)^2 = 1$ Applies Cartesian coordinates to both sides, extracts x and y terms and attempts to apply $x^2 + (y + 1)^2 = 1$	M1	2.1
	$\left(\frac{u}{u^2 + (v - 3)^2} \right)^2 + \left(\frac{3 - v}{u^2 + (v - 3)^2} \right)^2 = 1$ Correct expression with $y + 1$ term combined and simplified.	A1	1.1b
	$\Rightarrow u^2 + (v - 3)^2 = 1$	A1	2.2a
7(a) ALT 3	$u + iv = \frac{3i(x + iy) - 2}{x + iy + i} \times \frac{x - (y + 1)i}{x - (y + 1)i} = \frac{f(x, y) + g(x, y)i}{x^2 + (y + 1)^2}$ Applies Cartesian coordinates to expression and use complex conjugate of denominator to reach Cartesian form.	M1	2.1
	$x^2 + (y + 1)^2 = 1 \Rightarrow u + iv = x + (3x^2 + 3y^2 + 5y + 2)i$ $\Rightarrow u = x \text{ and } v = 3x^2 + 3(y + 1)^2 - y - 1 = a + by$ Uses $x^2 + (y + 1)^2 = 1$ in their equation and extract u and v as linear terms in x and y	M1	2.1
	$u = x \text{ and } v = 2 - y$ Correct u and v	A1	1.1b
	$\Rightarrow u^2 + (2 - v + 1)^2 = 1 \Rightarrow u^2 + (v - 3)^2 = 1$ Uses $x^2 + (y + 1)^2 = 1$ again to find correct equation.	A1	2.2a

Student Response A

$$\begin{aligned}
 w &= \frac{3iz - 2}{z + i} & |z + i| &= 1 \\
 & & x^2 + (y+1)^2 &= 1 \\
 w(z+i) &= 3iz - 2 \\
 wz + wi &= 3iz - 2 \\
 z(w - 3i) &= -2 - wi \\
 z &= \frac{-2 - wi}{w - 3i} \\
 z &= \frac{-2 - (u+vi)i}{u + (v-3)i} \\
 &= \frac{(-2 - ui + v) [u - \cancel{vi} + 3i]}{u^2 + (v-3)^2} \\
 &= \frac{-2u + 2v - 6i - ui - vi + 3u + uv - v^2 + 3vi}{u^2 + (v-3)^2} \\
 &= \frac{-2u + 2v - 6i - ui - vi + 3u + uv - v^2 + 3vi}{u^2 + (v-3)^2} \\
 &= \frac{u}{u^2 + (v-3)^2} + \frac{2vi - ui - vi + 3vi - 6i}{u^2 + (v-3)^2} \\
 \therefore x &= \frac{u}{u^2 + (v-3)^2} \\
 & \text{and } y = \frac{5vi - ui - vi - 6i}{u^2 + (v-3)^2} \\
 \frac{u^2}{[u^2 + (v-3)^2]^2} + \frac{5vi - ui - vi - 6i + u^2 + v^2 - 6v + 9}{[u^2 + (v-3)^2]^2} &= 1 \\
 u^2 + 3 - v &= [u^2 + (v-3)^2]^2
 \end{aligned}$$

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Examiner comment:

This response is an example of an attempt at Way 2 on the scheme, which was a fairly common approach. Usually such approaches do not go far, scoring only the first method, but in this response the first two method marks were gained. The first mark follows as per the main scheme, with z being made the subject of the equation of the transformation. The second method mark requires significant progress via this method, with first the Cartesian coordinates being applied to both sides of the equation (implied by identification of x and y late in the work for the left hand side), secondly the use of the complex conjugate on the right hand side to produce an expression in Cartesian form, thirdly the real and imaginary parts being extracted and finally use of these in the Cartesian equation for circle C to produce an equation in u and v only. In this case the numerator of the imaginary part has not been squared when applying the circle equation, so the accuracy marks are lost, but there has been clear evidence of the attempt to carry out each of the steps for the method.

No attempt was made at part (b).

Student Response B

(a) $w(z+i) = 3iz - 2$ ~~$w = 3iz - 2$~~

$wz + wi = 3iz - 2$ ~~$w = 3iz - 2$~~

$wz - 3iz + wi = -2$ ~~$w = 3iz - 2$~~

$z(w - 3i) = -2 - wi$

$z = \frac{-2 - wi}{w - 3i}$

$|z+i| = \left| \frac{-2 - wi}{w - 3i} + i \right|$

$= \left| \frac{-2 - wi + iw - 3i^2}{w - 3i} \right|$

$= \left| \frac{1}{w - 3i} \right| = 1$

$\therefore |w - 3i| = 1$

$|w - 1 - 3i|$

$|u + iv| = |1 - 3i|$ $|u + i(v+3)| = 1$

~~$|u + i(v+3)| = 1$~~

(b)

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Examiner comment:

Using the method of the main scheme, the student correct rearranges the given equation for w to make z the subject and proceeds to apply the equation of the given locus, scoring the two method marks. The equation is simplified correctly with the equation $|1 / (w - 3i)| = 1$ being sufficient to gain the first accuracy mark (although it is correctly simplified further in the following step). However, in this response a Cartesian equation is never produced for the image circle and so the final accuracy mark in part (a) is not awarded.

In part (b) both circles are drawn on the same diagram, which was accepted, although it would be advisable for the student to have made clear which is circle C and which is circle D . There is no correct circle C drawn, the centre is on the negative x axis instead of negative y axis. But circle D is correct with centre at 3 on the positive y axis and entirely in the upper half plane. So 1 mark is scored in part (b).

Student Response C

(a) $W(z+i) = 3iz-2.$

$Wz+iW = 3iz-2.$

$Wz-3iz = -iW-2$

$z(W-3i) = -iW-2.$

$z = \frac{iW+2}{3i-W}.$

$\left| \frac{iW+2}{3i-W} + i \right| = 1.$

$\left| \frac{iW+2-3-iW}{3i-W} \right| = 1.$

$|-1| = |3i-W|.$

$|3i-W| = 1.$

$|3i-(u+iv)| = 1.$

$|-u+i(3-v)| = 1.$

$u^2 + (3-v)^2 = 1.$

$u^2 + ((-1)(v-3))^2 = 1.$

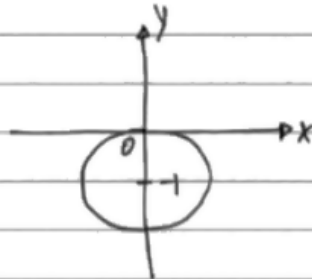
$u^2 + (v-3)^2 = 1.$

(b) $|z+i| = 1.$

$|x+iy+i| = 1.$

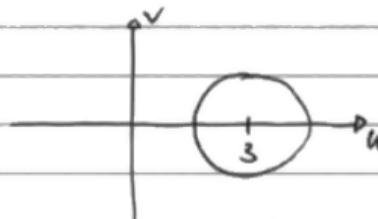
$x^2 + (y+1)^2 = 1.$

Centre $(0, -1)$, $r=1.$



$u^2 + (v-3)^2 = 1.$

Centre $(0, 3)$, $r=1$



Examiner comment:

This student uses the main scheme in part (a) and correctly rearranges the given equation to make z the subject. The given locus equation is then applied to their (correct) z and simplified as required to give $|3i - w| = 1$. Though some students stopped at this stage, scoring only the first three marks, in this case the student does produce a correct Cartesian equation for the image circle in the w -plane and so scores full marks in part (a).

A correct sketch of circle C , with centre labelled at -1 on the y axis and passing through the origin, is given in part (b), but the sketch for circle D is incorrect, with centre placed on the positive x axis instead of positive y axis. As the equation in part (a) was correct, this is not a correct follow through, so the mark cannot be awarded. (Note that if an answer $(u - 3)^2 + v^2 = 1$ has been given in error in (a), then an accuracy mark would have been lost in part (a), but the follow through could have been awarded here.)

Exemplar Question 8

8.

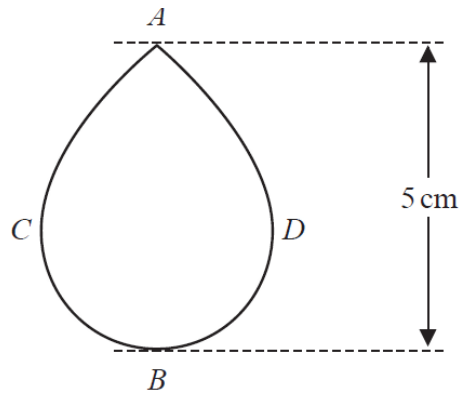


Figure 1

Figure 1 shows the vertical cross section of a child's spinning top. The point A is vertically above the point B and the height of the spinning top is 5 cm.

The line CD is perpendicular to AB such that CD is the maximum width of the spinning top.

The spinning top is modelled as the solid of revolution created when part of the curve with polar equation

$$r^2 = 25 \cos 2\theta$$

is rotated through 2π radians about the initial line.

(a) Show that, according to the model, the surface area of the spinning top is

$$k\pi(2 - \sqrt{2}) \text{ cm}^2$$

where k is a constant to be determined.

(7)

(b) Show that, according to the model, the length CD is $\frac{5\sqrt{2}}{2}$ cm

(6)

(Total for Question 8 is 13 marks)

Mean Score 4.3 out of 13

Examiner comment:

This question tested the students ability to apply the surface of revolution in polar form (2.2) to a situation in context. It was found very demanding, particularly part (b), but being the final question it was possible that timing was an issue for some students at this point. There were some good, complete answers seen, so well prepared students were able to complete the entire paper, but others may have spent more time than necessary on earlier questions leaving them short on time for this one. It was often difficult to follow the reasoning of students in their attempts at this question and generally those students who included a diagram of the correct curve obtained more marks than those responses with no explanation.

The question did test the problem solving skills of students, with the orientation of the diagram not directly relating to the curve, and this did give difficulty to many, with a number of candidates assuming rotation around the y -axis was required, rather than about the x -axis.

For part (a) most were able to correctly find $\left(\frac{dr}{d\theta}\right)^2$ and then substitute this and r^2 into a correct surface area formula (about either axis) and score the first three marks. However, achieving a correct simplified form proved much more of a challenge, with errors in algebra meaning attempts often ground to a halt. But there were also some very well executed solutions simplifying this form to a simple form to integrate. In such cases incorrect selection of limits prevented access to the final two marks. Those rotating about the y -axis in particular had trouble identifying correct limits, since the curve is not defined between the expected ones (though it is doubtful any students appreciated this).

Successful attempts at part (b) were rare, though some excellent solutions were seen, showing the understanding of what was required and carrying the processes correctly. Few who identified the correct method made errors in algebra. Some students used a maximisation process on the length of CD via completing the square on the trigonometric function rather than differentiation. It was also important that the students demonstrated that they were finding the length CD , and not the value of r , since these had the same value, and some students proceed only as far as r .

However, the majority of students did not realise what was required and either did not attempt this part or set off in a wrong direction. Many assumed the arc length was required and consequently attempted another integration. Many again mixed up directions and attempted to find $\frac{dx}{d\theta}$, for which partial credit could be scored, but in most cases they did not proceed far enough to do so.

Mark Scheme

Question	Scheme	Marks	AOs
8(a)	$SA = 2\pi \int r \sin \theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = 2\pi \int r \sin \theta \sqrt{25 \cos 2\theta + \dots} d\theta$ Or $SA = 2\pi \int r \cos \theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = 2\pi \int r \cos \theta \sqrt{25 \cos 2\theta + \dots} d\theta$	M1	2.1
	$r^2 = 25 \cos 2\theta \Rightarrow 2r \frac{dr}{d\theta} = k \sin 2\theta$ Or $r = 5 \cos^{\frac{1}{2}} 2\theta \Rightarrow \frac{dr}{d\theta} = A \cos^{-\frac{1}{2}} 2\theta \times B \sin 2\theta \text{ (oe)}$	M1	2.1
	$2r \frac{dr}{d\theta} = -50 \sin 2\theta \text{ or } \frac{dr}{d\theta} = \frac{-50 \sin 2\theta}{2r}$ Or $\frac{dr}{d\theta} = \frac{5}{2} \cos^{-\frac{1}{2}} 2\theta \times -2 \sin 2\theta \text{ (oe)}$	A1	1.1b
	$SA = 2\pi \int 5\sqrt{\cos 2\theta} \sin \theta \sqrt{25 \cos 2\theta + \frac{25 \sin^2 2\theta}{\cos 2\theta}} d\theta$ $= 2\pi \int 5\sqrt{\cos 2\theta} \sin \theta \frac{5}{\sqrt{\cos 2\theta}} d\theta = k\pi \int \sin \theta d\theta$ (may use $\cos \theta$)	M1	2.1
	$= 50\pi \int \sin \theta d\theta \text{ or } 50\pi \int \cos \theta d\theta$	A1	1.1b
	$= 50\pi \int_0^{\frac{\pi}{4}} \sin \theta d\theta = 50\pi [-\cos \theta]_0^{\frac{\pi}{4}}$	M1	3.4
	$= 25\pi(2 - \sqrt{2}) \text{ (cm}^2\text{)}$	A1	2.2a
		(7)	
(b)	Adopts the correct strategy by: Attempting $\frac{dy}{d\theta}$, finding θ when $\frac{dy}{d\theta} = 0$ and using their value of θ to find CD	M1	3.1a
	$y = r \sin \theta = 5\sqrt{\cos 2\theta} \sin \theta$ $\Rightarrow \frac{dy}{d\theta} = -\frac{5 \sin 2\theta \sin \theta}{\sqrt{\cos 2\theta}} + 5\sqrt{\cos 2\theta} \cos \theta$	M1	1.1b
	$\frac{dy}{d\theta} = 0 \Rightarrow 5 \cos \theta - 20 \cos \theta \sin^2 \theta = 0 \Rightarrow \theta = \dots$	M1	2.1
	E.g. $\sin^2 \theta = \frac{1}{4} \Rightarrow \theta = \frac{\pi}{6}$ or $\cos 3\theta = 0 \Rightarrow 3\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{6}$	A1	1.1b
	$CD = 2r \sin \frac{\pi}{6} = 2 \times 5 \times \sqrt{\cos \frac{\pi}{3}} \times \frac{1}{2}; = \frac{5\sqrt{2}}{2} \text{ (cm)}^*$	M1; A1*	3.4 2.1
		(6)	
(13 marks)			

Notes

(a)

M1: Applies the surface area formula about the x or y axis with substitution of at least the r^2 and attempt at $\left(\frac{dr}{d\theta}\right)^2$ as shown in scheme. May be completed in stages, so allow if correct formula quoted and the relevant “pieces” are found. The 2π may be recovered later but must be present at some stage.

M1: Attempts to find an expression in $\frac{dr}{d\theta}$ via implicit differentiation or first square rooting and then using the chain rule.

A1: Correct expression for or in $\frac{dr}{d\theta}$ need not be simplified.

M1: Makes a complete substitution into the SA formula and applies appropriate trigonometric identities to simplify to the form $k\pi \int \sin \theta d\theta$ or $k\pi \int \cos \theta d\theta$ as appropriate for their method.

A1: Obtains a correct simplified integral.

M1: Uses the model with appropriate limits to determine the surface area of the top using their integral.

Note that for rotation around the x axis appropriate limits will likely be 0 and $\frac{\pi}{4}$ but may be $-\frac{\pi}{4}$ and 0.

For rotation about the y axis allow this mark for limits $\frac{\pi}{4}$ and $\frac{\pi}{2}$ (note that the curve is not strictly defined for these limits).

A1: Correct expression with no errors. Must come from a correct integral – so if rotation about the y axis is used they must have made clear reference to using $r^2 = -25 \cos 2\theta$ as the curve.

(b)

M1: A complete method for finding CD . Need to see the maximum identified and the length of CD calculated (watch out as r is the same value as CD so check they are finding CD)

M1: Uses the product rule correctly to differentiate $r \sin \theta$ – expect the correct form

$$\alpha \sin 2\theta \sin \theta (\cos 2\theta)^{-\frac{1}{2}} + \beta \cos \theta (\cos 2\theta)^{\frac{1}{2}}$$

M1: Makes progress by setting their derivative (which may be $\frac{dy}{d\theta}$ or $\frac{dx}{d\theta}$) equal to 0 and proceeding via correct trig work to reach a value for θ . Various routes are possible e.g.

$$\frac{dy}{d\theta} = 0 \Rightarrow 5 \cos \theta - 20 \cos \theta \sin^2 \theta = 0 \Rightarrow \cos \theta (1 - 4 \sin^2 \theta) = 0 \Rightarrow \sin \theta = \dots \Rightarrow \theta = \dots$$

$$\frac{dy}{d\theta} = 0 \Rightarrow \cos 2\theta \cos \theta - \sin 2\theta \sin \theta = 0 \Rightarrow \cos 3\theta = 0 \Rightarrow 3\theta = \dots \Rightarrow \theta = \dots$$

$$\frac{dy}{d\theta} = 0 \Rightarrow \tan 2\theta = \frac{1}{\tan \theta} \Rightarrow \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{1}{\tan \theta} \Rightarrow \tan^2 \theta = \dots \Rightarrow \theta = \dots$$

A1: Correct value for θ from correct working – derivative must have been correct. May be implied by correct sin and cosine values used in formulae. SC Award for $\frac{\pi}{3}$ if using $x = r \cos \theta$

M1: Uses their value of θ in the model to find CD , ie $CD = 2 \times 5 \sqrt{\cos^2 2\theta} \times \sin \theta$

Allow for use of $2r \cos \theta$ for attempts stemming from $\frac{dx}{d\theta} = 0$

A1*:cso Correct proof

NB for $x = r \cos \theta$ used, a maximum M0M0M1A1M1A0 can be gained unless $r^2 = -25 \cos 2\theta$ is used, in which case full marks is possible.

ALT for (b):

M1: for correct overall strategy of finding expression for width and maximising (via any valid method, e.g completion of square, calculus).

M1: $CD = 2r \sin \theta = 10\sqrt{\cos 2\theta \sin^2 \theta} = 10\sqrt{\sin^2 \theta - 2\sin^4 \theta}$ forms trig expression inside square root.

M1A1: $\sin^2 \theta - 2\sin^4 \theta = -2\left(\sin^4 \theta - \frac{1}{2}\sin^2 \theta\right) = -2\left(\left(\sin^2 \theta - \frac{1}{4}\right)^2 - \frac{1}{16}\right)$ completes the square.

Alternatively may optimise via calculus – score for full method leading to a value for θ

M1: Hence max value for CD is $10 \times \sqrt{\frac{1}{8}}$. Correct method to achieved CD .

A1: Correct proof.

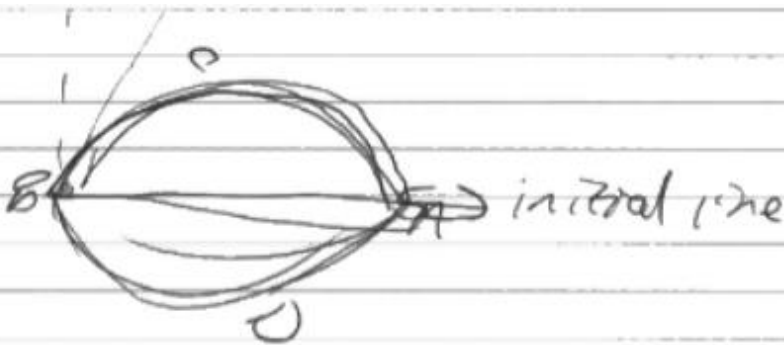
Student Response A

$$1) \quad r^2 = 25 \cos 2\theta$$

$$s = 2\pi \int r \sin \theta \sqrt{r^2 + \frac{dr}{d\theta}}$$

$$r^2 = 5 \cos 2\theta^2$$

$$\frac{dr}{d\theta} = \frac{5 \times 2 \sin 2\theta \times \cos 2\theta^2}{\sqrt{\cos 2\theta}} = \frac{5 \sin 2\theta}{\sqrt{\cos 2\theta}}$$



$r = \sqrt{25 \cos 2\theta}$			
θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
r	5	0	

$$4\pi \int_0^{\frac{\pi}{4}} \frac{\sqrt{25 \cos 2\theta} \sin \theta \sqrt{25 \cos 2\theta + \frac{5 \sin 2\theta}{\sqrt{\cos 2\theta}}}}{5 \sqrt{\cos 2\theta}} d\theta$$

$$20\pi \int_0^{\frac{\pi}{4}} \sqrt{\cos 2\theta} \sin \theta \sqrt{25 \cos 2\theta + \frac{5 \sin 2\theta}{\sqrt{\cos 2\theta}}} d\theta$$

$$20\pi \int_0^{\frac{\pi}{4}} \sin \theta \sqrt{25 \cos 2\theta - 5 \sin 2\theta} d\theta$$

$$20\pi \int_0^{\frac{\pi}{4}} \sin \theta \sqrt{25 - 25 \sin^2 \theta - 5 \sin 2\theta} d\theta$$

$$20\pi \int_0^{\frac{\pi}{4}} \sin \theta \sqrt{\quad} d\theta$$

$$CD = \frac{5\sqrt{2}}{2}$$

$$r^2 = 25 \cos 2\theta$$

$$r = 5(\cos 2\theta)^{\frac{1}{2}}$$

$$y = r \sin \theta$$

$$y = 5(\cos 2\theta)^{\frac{1}{2}} \sin \theta$$

$$\frac{dy}{d\theta} = \frac{5}{2} \times (-2 \sin 2\theta)^{\frac{1}{2}} \times \sin \theta + 5(\cos 2\theta)^{\frac{1}{2}} \cos \theta = 0$$

$$\frac{5}{2} \times (-5 \sin 2\theta) (\cos 2\theta)^{\frac{1}{2}} + 5(\cos 2\theta)^{\frac{1}{2}} \cos \theta = 0$$

$$(\cos 2\theta)^{\frac{1}{2}} (-5 \sin 2\theta + 5 \cos \theta) = 0$$

$$5 \times 2 \cos \theta \sin \theta + 5 \cos \theta = 0$$

$$-10 \sin \theta = -5$$

$$\sin \theta = \frac{1}{2}$$

$$\theta = \frac{1}{6}\pi$$

$$y = r \sin \theta$$

$$y = 5(\cos \frac{\pi}{3})^{\frac{1}{2}} \times \sin \frac{1}{6}\pi$$

$$\frac{5\sqrt{2}}{2} \times \frac{1}{2} = \frac{5\sqrt{2}}{4}$$

$$y \times 2 = CD = \frac{5\sqrt{2}}{2}$$

Examiner comment:

An incorrect formula for the surface area is quoted at the start of part (a), so the first method mark cannot be awarded. There is a correct attempt at $dr/d\theta$ for the next two marks, but due to the incorrect formula, with the $\frac{dr}{d\theta}$ not squared, the student is unable to simplify to a correct form for the next method mark and following accuracy. Correct limits have been identified and an attempt to apply these to the integral could have scored the final method in part (a), but the student has given up part way through, and does not find an integral so does not score this method. Thus only two marks are earned in part (a).

A correct overall strategy is applied in part (b), with suitable derivative $\left(\frac{dy}{d\theta}\right)$ attempted, an angle found by equating the derivative to zero and solving, and attempt at maximum width made (doubling the value for $r\sin\theta$ at the angle). This gains the first method mark. However, a derivative of the correct form is not achieved (the power of the $\cos 2\theta$ in the first term is incorrect) and so the second method is not awarded. The derivative is incorrect and the following trig work is incorrect in solving the equation ($a \sin \theta$ term disappears) which means the next method is not available, and although a correct angle is reached it is fortuitous and the accuracy marking is withheld accordingly. The angle is used in a correct formula to find CD , though, so the final method mark is awarded, but the accuracy cannot be given as it requires a fully correct solution. Thus a total of 2 marks was awarded for part (b).

Student Response B

$$(a) \int_0^{2\pi} r \cos \theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

$$r^2 = 25 \cos 2\theta$$

$$2r \frac{dr}{d\theta} = -50 \sin 2\theta$$

$$\frac{dr}{d\theta} = \frac{-25 \sin 2\theta}{r}$$

$$\text{Now } \int 25 \cos 2\theta + \left(\frac{-25 \sin 2\theta}{5 \sqrt{\cos 2\theta}} \right) d\theta =$$

Question 8 continued

$$\begin{aligned} & \sqrt{\frac{25 \cos 2\theta + \frac{625 \sin^2 2\theta}{25 \cos 2\theta}}{25 \cos 2\theta}} \\ &= \sqrt{\frac{625 \cos^2 2\theta + 625 \sin^2 2\theta}{25 \cos 2\theta}} \\ &= \frac{\cancel{625}}{\sqrt{25 \cos 2\theta}} \\ &= \sqrt{\frac{25}{\cos 2\theta}} = \frac{5}{\sqrt{\cos 2\theta}}. \end{aligned}$$

Now $r \cos \theta = 5 \sqrt{\cos 2\theta} \cos \theta$

$$\begin{aligned} S &= \int_0^{\frac{\pi}{4}} 25 \cos \theta \, d\theta \\ &= 25 \times 2\pi \left[\sin \theta \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ &= 25 \times 2\pi \left[1 - \frac{\sqrt{2}}{2} \right] \\ &= \frac{25 \times 2\pi [2 - \sqrt{2}]}{2} \\ &= 25\pi [2 - \sqrt{2}]. \end{aligned}$$

$\therefore k = 25.$

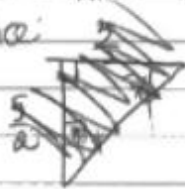
(b) Well θ Point D is a an angle of $\frac{\pi}{4}$.

Given $AB \perp CD$

$AD \perp CD$

Intersection lies on the mid point.

hence:



Consider $\theta = \frac{\pi}{4}$.

We have

$$\begin{aligned} r^2 &= \frac{25}{2} \\ r &= \pm \sqrt{\frac{25}{2}} \\ &= \pm \frac{5}{\sqrt{2}} \text{ (we reject)} \end{aligned}$$

$$= \frac{5\sqrt{2}}{2} \therefore r = \frac{5\sqrt{2}}{2}$$

Examiner comment:

Part (a) is attempted very well, with the surface area formula about the y -axis used. This was a fairly common approach due to the way the diagram was drawn in the question, though the formula for rotation about the y -axis is not given in the formula booklet. However, strictly the method requires a different curve, $r^2 = -25\cos 2\theta$ to be used, with different limits – something unlikely to be appreciated and the final mark is lost as a result, but with much of the work the same as the main scheme the first 6 marks were allowed under this method with the original curve as long as correct limits were identified.

The first method mark for this response was awarded for the work on page 1 where the formula is quoted and correct "pieces" found. The attempt at an expression in $\frac{dr}{d\theta}$ via implicit differentiation scores the next method and accuracy, as it is correct, as per the main scheme. A complete substitution into the formula is then made and trigonometric identities applied to simplify to the form required. The student reaches a correct simplified integral for this method and then attempts the integration applying the "acceptable" limits $\frac{\pi}{4}$ to $\frac{\pi}{2}$ for this method, and as such scores the first 6 marks in this part. The final accuracy cannot be scored here as $r^2 = -25\cos 2\theta$ has not been used.

There is no credit worthy work in part (b). There is no context or substantiation for what the student means by "point D is an angle of $\frac{\pi}{4}$ ", and a guess is made at $\theta = \frac{\pi}{12}$, which is incorrect as this would lead to a value of $25\sqrt{3}/2$ for r^2 . The final answer given is a value for r and not $2r\sin\theta$ so the student does not appear to have the correct method in mind.

Student Response C

$$r^2 = 25 \cos 2\theta$$



is rotated through 2π radians about the initial line.

(a) Show that, according to the model, the surface area of the spinning top is

$$k\pi(2 - \sqrt{2}) \text{ cm}^2$$

where k is a constant to be determined.

(7)

(b) Show that, according to the model, the length CD is $\frac{5\sqrt{2}}{2}$ cm.

(6)

$$r^2 = 25 \cos 2\theta$$

$$a) \quad S = 2\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r \cos \theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

$$r = 5\sqrt{\cos 2\theta}$$

$$\frac{dr}{d\theta} = -5(\cos 2\theta)^{-\frac{1}{2}} \sin 2\theta$$

$$= 2\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 5\sqrt{\cos 2\theta} \cos \theta \sqrt{25 \cos 2\theta + 25 \frac{\sin^2 2\theta}{\cos 2\theta}} d\theta$$

Question 8 continued

$$= 2\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 5 \cos \theta \sqrt{25 \cos^2 2\theta + 25 \sin^2 2\theta} d\theta$$

$$= 2\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 5 \cos \theta \sqrt{25} d\theta = 50\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos \theta d\theta$$

$$= 50\pi [\sin \theta]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = 50\pi \sqrt{2} \text{ m}^2.$$

$$= \underline{\underline{(50 + 50\sqrt{2})\pi(2 - \sqrt{2}) \text{ m}^2}}$$

~~$$\frac{dr}{d\theta} = 5(\cos 2\theta)^{-\frac{1}{2}} \sin 2\theta = 0$$~~

~~$$\frac{dr}{d\theta} = 5(\cos 2\theta)^{-\frac{1}{2}} \sin 2\theta = 0$$~~

~~$$\frac{dr}{d\theta} = 0 \Rightarrow \frac{d}{d\theta}(r \cos \theta) = \frac{dr}{d\theta} \cos \theta - r \sin \theta = 0$$~~

~~$$-5(\cos 2\theta)^{-\frac{1}{2}} \sin 2\theta \sin \theta + 5\sqrt{\cos 2\theta} \cos \theta = 0$$~~

~~$$-5 \sin 2\theta \sin \theta + 5 \cos \theta \cos 2\theta = 0$$~~

~~$$5 \cos(2\theta + \theta) = 0$$~~

~~$$\cos 3\theta = 0$$~~

~~$$3\theta = \frac{3\pi}{2}, \frac{\pi}{2}$$~~

~~$$\theta = \frac{\pi}{2}, \frac{\pi}{6}$$~~

~~$$\therefore r = 5\sqrt{\cos 2\theta} =$$~~

~~$$-5(\cos 2\theta)^{-\frac{1}{2}} \sin 2\theta \cos \theta - 5\sqrt{\cos 2\theta} \sin \theta = 0$$~~

~~$$-5 \sin 2\theta \cos \theta - 5 \cos 2\theta \sin \theta = 0$$~~

~~$$\sin 3\theta = 0$$~~

~~$$3\theta = 0, \pi, 2\pi, -\pi, -2\pi$$~~

~~$$\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, -\frac{\pi}{3}, -\frac{2\pi}{3}$$~~

(part b on next page)

$$\begin{aligned}
 \text{b) } \frac{dy}{d\theta} = 0 &= \frac{d}{d\theta}(r \sin \theta) = \frac{dr}{d\theta} \sin \theta + r \cos \theta \\
 &= -5 \sqrt{\cos 2\theta} \sin 2\theta \sin \theta + 5 \sqrt{\cos 2\theta} \cos \theta \\
 \Rightarrow 0 &= -5 \sin 2\theta \sin \theta + 5 \cos 2\theta \cos \theta \\
 &= 5 \cos(2\theta + \theta) \\
 \Leftrightarrow \cos 3\theta &= 0 \\
 3\theta &= \frac{\pi}{2}, \frac{3\pi}{2}, -\frac{\pi}{2}, -\frac{3\pi}{2} \\
 \Rightarrow \theta &= \frac{\pi}{6}, -\frac{\pi}{6} \\
 \therefore CD &= \sin \frac{\pi}{6} 5 \sqrt{\cos \frac{\pi}{3}} + \sin \frac{\pi}{6} 5 \sqrt{\cos \left(\frac{\pi}{3}\right)} \\
 &= \frac{5\sqrt{2}}{2} \text{ cm}
 \end{aligned}$$

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Examiner comment:

A correct overall approach is used in part (a), with the surface area formula quoted correctly and an attempt to use it, with attempts at r^2 and $\left(\frac{dr}{d\theta}\right)^2$ both substituted (note that seeing these identified along with a correct formula is sufficient for the first method mark). The attempt at $\left(\frac{dr}{d\theta}\right)^2$ is correct for the second method and first accuracy – in this case obtained by first finding an expression for r and then differentiating. The full substitution into the surface area formula is made and correct trigonometric work leads to an expression of the required form for the third method mark, with the accuracy also gained as all the work to this point as been correct, so the correct simplified integral has been found. The final method and accuracy are not awarded as the limits used are incorrect. For integration about the x axis the limits should be 0 and $\pi/4$, but the lower limit used here is $-\frac{\pi}{4}$ and hence the final answer is incorrect.

Part (b) is fully correct via the main scheme approach. A complete method to find CD is carried out, with suitable derivative attempted, angle found and attempt at maximum width made. The product rule has been used to differentiate $r \sin \theta$ with respect to θ , and the result set to zero followed by correct trigonometric work to reach the correct value for θ from correct working. The value of θ is then used to find CD in a correct proof, with no errors seen.