Pearson Edexcel Level 3 Advanced GCE in Further Mathematics (9FM0)

Sample Assessment Materials Model Answers – Core Pure Mathematics

First teaching from September 2017
First certification from June 2019

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Contents

Introduction

This booklet has been produced to support mathematics teachers delivering the new Pearson Edexcel Level 3 Advanced GCE in Mathematics (9FM0) specification for first teaching from September 2017.

This booklet looks at Sample Assessment Materials for A level Further Mathematics qualification, specifically at core pure mathematics questions, and is intended to offer model solutions with different methods explored.

Content of Core Pure Mathematics

Proof	**Construct proofs using mathematical induction.** **Contexts include sums of series, divisibility and powers of matrices.**
Complex numbers	**Solve any quadratic equation with real coefficients.** **Solve cubic or quartic equations with real coefficients.** **Add, subtract, multiply and divide complex numbers in the form $x + \mathrm{i}y$ with x and y real.** **Understand and use the terms 'real part' and 'imaginary part'.** **Understand and use the complex conjugate.** **Know that non-real roots of polynomial equations with real coefficients occur in conjugate pairs.** **Use and interpret Argand diagrams.** **Convert between the Cartesian form and the modulus-argument form of a complex number.** **Knowledge of radians.** **Multiply and divide complex numbers in modulus argument form.** **Knowledge of radians and compound angle formulae is assumed.** **Construct and interpret simple loci in the argand diagram such as $z - a > r$ and arg $(z - a) = \theta$** Understand de Moivre's theorem and use it to find multiple angle formulae and sums of series. Know and use the definition $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$ and the form $z = r\mathrm{e}^{\mathrm{i}\theta}$ Find the n distinct nth roots of $r\mathrm{e}^{\mathrm{i}\theta}$ for $r \neq 0$ and know that they form the vertices of a regular n-gon in the Argand diagram. Use complex roots of unity to solve geometric problems.
Matrices	**Add, subtract and multiply conformable matrices.** **Multiply a matrix by a scalar.** **Understand and use zero and identity matrices.** **Use matrices to represent linear transformations in 2-D.** **Successive transformations.** **Single transformations in 3-D.** **Find invariant points and lines for a linear transformation.** **Calculate determinants of: 2 × 2 and 3 × 3 matrices and interpret as scale factors, including the effect on orientation.** **Understand and use singular and non-singular matrices.** **Properties of inverse matrices.** **Calculate and use the inverse of non-singular 2 × 2 matrices and 3 × 3 matrices.**

	Solve three linear simultaneous equations in three variables by use of the inverse matrix. **Interpret geometrically the solution and failure of solution of three simultaneous linear equations.**
Further algebra and functions	**Understand and use the relationship between roots and coefficients of polynomial equations up to quartic equations.** **Form a polynomial equation whose roots are a linear transformation of the roots of a given polynomial equation (of at least cubic degree).** **Understand and use formulae for the sums of integers, squares and cubes and use these to sum other series.** Understand and use the method of differences for summation of series including use of partial fractions. Find the Maclaurin series of a function including the general term. Recognise and use the Maclaurin series for e^x, $\ln(1+x)$, $\sin x$, $\cos x$ and $(1+x)^n$, and be aware of the range of values of x for which they are valid (proof not required).
Further calculus	**Derive formulae for and calculate volumes of revolution.** Evaluate improper integrals where either the integrand is undefined at a value in the range of integration or the range of integration extends to infinity. Understand and evaluate the mean value of a function. Integrate using partial fractions. Differentiate inverse trigonometric functions. Integrate functions of the form $\left(a^2 - x^2\right)^{-\frac{1}{2}}$ and $\left(a^2 - x^2\right)^{-1}$ and be able to choose trigonometric substitutions to integrate associated functions.
Further vectors	**Understand and use the vector and Cartesian forms of an equation of a straight line in 3-D.** **Understand and use the vector and Cartesian forms of the equation of a plane.** **Calculate the scalar product and use it to express the equation of a plane, and to calculate the angle between two lines, the angle between two planes and the angle between a line and a plane.** **Check whether vectors are perpendicular by using the scalar product.** **Find the intersection of a line and a plane.** **Calculate the perpendicular distance between two lines, from a point to a line and from a point to a plane.**
Polar coordinates	Understand and use polar coordinates and be able to convert between polar and Cartesian coordinates. Sketch curves with r given as a function of θ, including use of trigonometric functions. Find the area enclosed by a polar curve.
Hyperbolic functions	Understand the definitions of hyperbolic functions $\sinh x$, $\cosh x$ and $\tanh x$, including their domains and ranges, and be able to sketch their graphs. Differentiate and integrate hyperbolic functions. Understand and be able to use the definitions of the inverse hyperbolic functions and their domains and ranges. Derive and use the logarithmic forms of the inverse hyperbolic functions. Integrate functions of the form $\left(a^2 + x^2\right)^{-\frac{1}{2}}$ and $\left(a^2 - x^2\right)^{-\frac{1}{2}}$ and be able to choose substitutions to integrate associated functions.
Differential equations	Find and use an integrating factor to solve differential equations of form $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$ and recognise when it is appropriate to do so.

	Find both general and particular solutions to differential equations. Use differential equations in modelling in kinematics and in other contexts. Solve differential equations of form $y'' + ay' + by = 0$ where a and b are constants by using the auxiliary equation. Solve differential equations of form $y'' + a y' + b y = \mathrm{f}(x)$ where a and b are constants by solving the homogeneous case and adding a particular integral to the complementary function (in cases where $\mathrm{f}(x)$ is a polynomial, exponential or trigonometric function). Understand and use the relationship between the cases when the discriminant of the auxiliary equation is positive, zero and negative and the form of solution of the differential equation. Solve the equation for simple harmonic motion $\ddot{x} = \omega^2 x$ and relate the solution to the motion. Model damped oscillations using second order differential equations and interpret their solutions. Analyse and interpret models of situations with one independent variable and two dependent variables as a pair of coupled first order simultaneous equations and be able to solve them, for example predator-prey models.

Core Pure Mathematics 1

Question 1

Prove that

$$\sum_{r=1}^{n} \frac{1}{(r+1)(r+3)} = \frac{n(an+b)}{12(n+2)(n+3)},$$

where a and b are constants to be found.

(5)

Split into partial fractions:

$$\frac{1}{(r+1)(r+3)} = \frac{A}{r+1} + \frac{B}{r+3}$$

$1 = A(r + 3) + B(r + 1)$

$r = -1$: $A = \frac{1}{2}$

$r = -3$: $B = -\frac{1}{2}$ M1

Method of differences:

$$\sum_{r=1}^{n}\left(\frac{1}{2(r+1)} - \frac{1}{2(r+3)}\right) = \frac{1}{2\times 2} - \frac{1}{2\times 4}$$

$$+ \frac{1}{2\times 3} - \frac{1}{2\times 5}$$

$$+ \frac{1}{2\times 4} - \frac{1}{2\times 6}$$

$$+ \ldots$$

$$+ \frac{1}{2(n-1)} - \frac{1}{2(n+1)}$$

$$+ \frac{1}{2n} - \frac{1}{2(n+2)}$$

$$+ \frac{1}{2(n+1)} - \frac{1}{2(n+3)}$$

M1

$$= \frac{1}{4} - \frac{1}{6} - \frac{1}{2(n+2)} - \frac{1}{2(n+3)}$$

A1

$$= \frac{5}{12} - \frac{6}{12(n+2)} - \frac{6}{12(n+3)}$$

$$= \frac{5(n+2)(n+3) - 6(n+3) - 6(n+2)}{12(n+2)(n+3)}$$

M1

$$= \frac{5n^2 + 25n + 30 - 6n - 18 - 6n - 12}{12(n+2)(n+3)}$$

$$= \frac{5n^2 + 13n}{12(n+2)(n+3)}$$

$$= \frac{n(5n+13)}{12(n+2)(n+3)}$$

A1

So $a = 5$, $b = 13$

Alternative:

Proof by Induction:

$n = 1$:

$\frac{1}{8} = \frac{a+b}{12 \times 3 \times 4}$ $\qquad a + b = 18$ ---[1]

$n = 2$:

$\frac{1}{8} + \frac{1}{15} = \frac{2(2a+b)}{12 \times 4 \times 5}$ $\qquad 2a + b = 23$ ---[2]

Solve simultaneously:
[2] – [1] gives $a = 5$, $b = 13$ M1

Assume true for $n = k$:

$$\sum_{r=1}^{k} \frac{1}{(r+1)(r+3)} = \frac{k(5k+13)}{12(k+2)(k+3)}$$

When $n = k + 1$:

$$\sum_{r=1}^{k+1} \frac{1}{(r+1)(r+3)} = \frac{k(5k+13)}{12(k+2)(k+3)} + \frac{1}{(k+2)(k+4)}$$ M1

$$\frac{k(5k+13)}{12(k+2)(k+3)} + \frac{1}{(k+2)(k+4)} = \frac{k(5k+13)(k+4)+12(k+3)}{12(k+2)(k+3)(k+4)}$$ A1

$$= \frac{5k^3+33k^2+52k+12k+36}{12(k+2)(k+3)(k+4)}$$

$$= \frac{5k^3+33k^2+64k+36}{12(k+2)(k+3)(k+4)}$$

$$= \frac{(k+2)(5k^2+23k+18)}{12(k+2)(k+3)(k+4)}$$

$$= \frac{\cancel{(k+2)}(k+1)(5k+18)}{12\cancel{(k+2)}(k+3)(k+4)}$$ M1

$$= \frac{(k+1)(5(k+1)+13)}{12((k+1)+2)((k+1)+3)}$$

$$= \frac{\{k+1\}(5\{k+1\}+13)}{12(\{k+1\}+2)(\{k+1\}+3)}$$

So true for $n = k + 1$

So $$\sum_{r=1}^{n} \frac{1}{(r+1)(r+3)} = \frac{n(5n+13)}{12(n+2)(n+3)}$$ A1

Question 2

Prove by induction that, for all positive integers n,

$$f(n) = 2^{3n+1} + 3(5^{2n+1})$$

is divisible by 17.

(6)

When $n = 1$,

$2^{3n+1} + 3(5^{2n+1}) = 2^4 + 3(5^3) = 16 + 375 = 391$

$391 = 17 \times 23$ so the statement is true for $n = 1$. B1

Assume true for $n = k$,

so $f(k) = 2^{3k+1} + 3(5^{2k+1})$ is divisible by 17. M1

$$f(k+1) - f(k) = 2^{3(k+1)+1} + 3(5^{2(k+1)+1}) - 2^{3k+1} - 3(5^{2k+1})$$

$$= 2^{3k+4} + 3(5^{2k+3}) - 2^{3k+1} - 3(5^{2k+1})$$ M1

$$= 2^{3k+1+3} + 3(5^{2k+1+2}) - 2^{3k+1} - 3(5^{2k+1})$$

$$= 2^{3k+1} \times 2^3 + 3(5^{2k+1} \times 5^2) - 2^{3k+1} - 3(5^{2k+1})$$

$$= 8(2^{3k+1}) - 2^{3k+1} + 75(5^{2k+1}) - 3(5^{2k+1})$$

$$= 7(2^{3k+1}) + 72(5^{2k+1})$$

$$= 7(2^{3k+1}) + 21(5^{2k+1}) + 51(5^{2k+1})$$

$$= 7[(2^{3k+1}) + 3(5^{2k+1})] + 51(5^{2k+1})$$

$$= 7[f(k)] + 17 \times 3(5^{2k+1})$$ A1

$$f(k+1) = 8f(k) + 17 \times 3(5^{2k+1})$$

which must be divisible by 17. A1

If the statement is true for $n = k$ then it has been shown true for $n = k + 1$ and as it is true for $n = 1$, the statement is true for all positive integers n. A1

Question 3

$$f(z) = z^4 + az^3 + 6z^2 + bz + 65,$$

where a and b are real constants.

Given that $z = 3 + 2i$ is a root of the equation $f(z) = 0$, show the roots of $f(z) = 0$ on a single Argand diagram.

(9)

If $3 + 2i$ is a root then the conjugate $3 - 2i$ is also a root. B1

Either:

$(z - (3 + 2i))(z - (3 - 2i))$
$= z^2 - 3z + 2iz - 3z + 9 - 6i - 2iz + 6i + 4$
$= z^2 - 6z + 13$

Or:

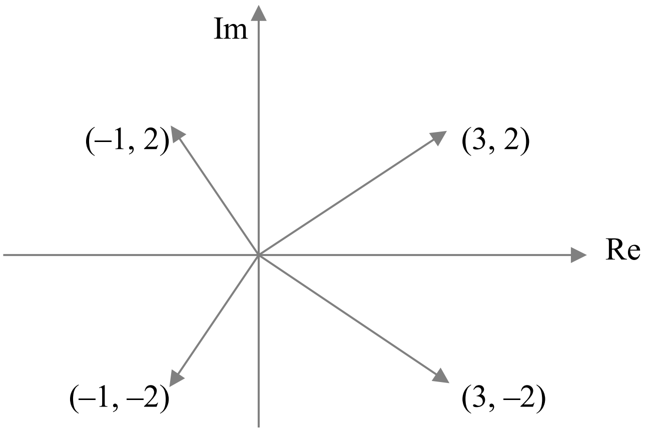
sum of roots = 6
product of roots = 13 M1
So $z^2 - 6z + 13$ A1

$z^4 + az^3 + 6z^2 + bz + 65 = (z^2 - 6z + 13)(z^2 + cz + 5)$

compare z^2 terms:
$6z^2 = 5z^2 - 6cz^2 + 13z^2$
$-12 = -6c$
$c = 2$ M1
$z^2 + 2z + 5 = 0$ A1

Solve by completing the square:
$(z + 1)^2 + 4 = 0$
$z = -1 \pm\sqrt{(-4)}$ M1
$z = -1 \pm 2i$ A1

Argand diagram:



$(3 \pm 2i)$ B1
$(-1 \pm 2i)$ B1

Question 4

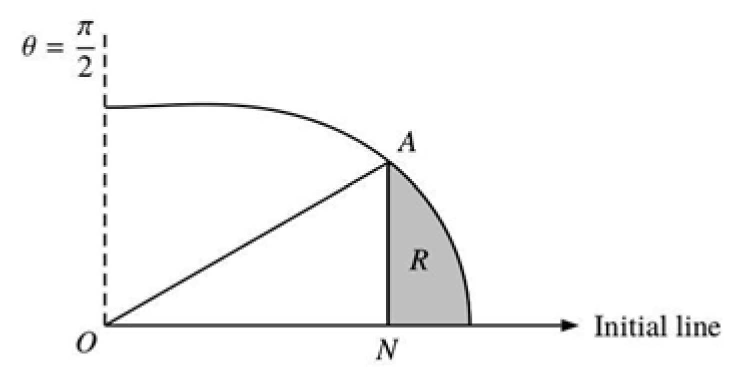


Figure 1

The curve C shown in Figure 1 has polar equation

$$r = 4 + \cos 2\theta \qquad 0 \leq \theta \leq \frac{\pi}{2}$$

At the point A on C, the value of r is $\frac{9}{2}$

The point N lies on the initial line and AN is perpendicular to the initial line.

The finite region R, shown shaded in Figure 1, is bounded by the curve C, the initial line and the line AN.

Find the exact area of the shaded region R, giving your answer in the form $p\pi + q\sqrt{3}$, where p and q are rational numbers to be found.

(9)

At A,

$4 + \cos 2\theta = \frac{9}{2}$

$\cos 2\theta = \frac{1}{2}$

$2\theta = \frac{\pi}{3}$ M1

$\theta = \frac{\pi}{6}$ A1

Formula Book:

Area of a sector $= \frac{1}{2}\int r^2 \, d\theta$

$$\frac{1}{2}\int r^2 \, d\theta = \frac{1}{2}\int \ (4 + \cos 2\theta)^2 \ d\theta$$

$$= \frac{1}{2}\int 16 + 8\cos 2\theta + \cos^2 2\theta \ d\theta$$ M1

$$= \frac{1}{2}\int 16 + 8\cos 2\theta + \frac{1}{2}(1 + \cos 4\theta) \ d\theta$$ M1

$$= \frac{1}{2}\int \frac{33}{2} + 8\cos 2\theta + \frac{1}{2}\cos 4\theta \ d\theta$$

$$= \frac{1}{2}\ [\ \frac{33}{2}\theta + 4\sin 2\theta + \frac{1}{8}\sin 4\theta\ ]$$ A1

Using limits 0 and $\frac{\pi}{6}$:

$$= \frac{1}{2}\left(\frac{33\pi}{12} + 2\sqrt{3} + \frac{\sqrt{3}}{16}\right) - (0)$$ M1

$$= \frac{33\pi}{24} + \frac{33\sqrt{3}}{32}$$

Area of triangle = $\frac{1}{2}(r\cos\theta)(r\sin\theta)$

$$= \frac{1}{2} \times \frac{9}{2} \times \frac{\sqrt{3}}{2} \times \frac{9}{2} \times \frac{1}{2}$$ M1

$$= \frac{81\sqrt{3}}{32}$$

Area of $R = \frac{33\pi}{24} + \frac{33\sqrt{3}}{32} - \frac{81\sqrt{3}}{32}$ M1

$$= \frac{11\pi}{8} - \frac{3\sqrt{3}}{2}$$ A1

$p = \frac{11}{8}, q = \frac{-3}{2}$

Question 5

A pond initially contains 1000 litres of unpolluted water.

The pond is leaking at a constant rate of 20 litres per day.

It is suspected that contaminated water flows into the pond at a constant rate of 25 litres per day and that the contaminated water contains 2 grams of pollutant in every litre of water.

It is assumed that the pollutant instantly dissolves throughout the pond upon entry.

Given that there are x grams of the pollutant in the pond after t days,

(a) show that the situation can be modelled by the differential equation,

$$\frac{dx}{dt} = 50 - \frac{4x}{200 + t}$$

(4)

Rate of increase of water in the pond is 25 – 20 = 5 litres per day.
The pond contains $1000 + 5t$ litres after t days. M1

If x is the amount of pollutant in the pond after t days

Rate of pollutant out $= \dfrac{20}{1000 + 5t} \times x$ g per day M1

$= \dfrac{4x}{200 + t}$

Rate of pollutant in = 25 × 2 g = 50 g per day B1

Overall rate of change = rate in – rate out

$\dfrac{dx}{dt} = 50 - \dfrac{4x}{200 + t}$ A1

(b) Hence find the number of grams of pollutant in the pond after 8 days. **(5)**

$$\frac{dx}{dt} + \frac{4x}{200+t} = 50$$

Integrating factor:

$I = e^{\int \frac{4}{200+t} dt}$

$I = e^{4\ln(200+t)}$

$I = e^{\ln(200+t)^4}$

$I = (200 + t)^4$

Multiply all terms by $(200 + t)^4$:

$$\frac{dx}{dt}(200 + t)^4 + 4(200 + t)^3 x = 50(200 + t)^4$$

[LHS is of the form $u'v + v'u = (uv)'$]

Integrate both sides:

$x(200 + t)^4 = \int 50(200 + t)^4 \, dt$ M1

$x(200 + t)^4 = 10(200 + t)^5 + c$ A1

$x = 0, t = 0,$

$0 = 10(200)^5 + c$

$c = -3.2\times10^{12}$ M1

$t = 8,$

$x(208)^4 = 10(208)^5 - 3.2\times10^{12}$

$x = 2080 - \frac{3.2\times10^{12}}{208^4}$ M1

$x = 370.3916...$

$x = 370$ g A1

(c) Explain how the model could be refined. **(1)**

e.g.
The model should take into account the fact that the pollutant does not dissolve throughout the pond upon entry. B1
The rate of leaking could be made to vary with the volume of water in the pond.

Question 6

$$f(x) = \frac{x+2}{x^2+9}$$

(a) Show that

$$\int f(x)\ \mathrm{d}x = A \ln (x^2+9) + B \arctan\left(\frac{x}{3}\right) + c,$$

where c is an arbitrary constant and A and B are constants to be found.

(4)

$$f(x) = \frac{x}{x^2+9} + \frac{2}{x^2+9}\ .$$ B1

$$\int \frac{x}{x^2+9}\mathrm{d}x = \frac{1}{2}\int \frac{2x}{x^2+9}\mathrm{d}x$$

$$= \frac{1}{2}\ln|x^2+9|$$ M1

Formula Book:

$$\int \frac{1}{x^2+a^2}\mathrm{d}x = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$$

$$\int \frac{2}{x^2+9}\mathrm{d}x = 2\int \frac{1}{x^2+3^2}\mathrm{d}x$$

$$= 2 \times \frac{1}{3}\arctan\left(\frac{x}{3}\right)$$ M1

$$\int f(x)\ \mathrm{d}x = \frac{1}{2}\ln(x^2+9) + \frac{2}{3}\arctan\left(\frac{x}{3}\right) + c$$ A1

(b) Hence show that the mean value of f(x) over the interval [0, 3] is

$$\frac{1}{6}\ \ln 2 + \frac{1}{18}\pi.$$

(3)

Using limits 0 and 3:

$$\int_0^3 \mathrm{f}(x)\ \mathrm{d}x = \frac{1}{2}\ \ln(3^2 + 9) + \frac{2}{3}\ \arctan\left(\frac{3}{3}\right) - \frac{1}{2}\ \ln(0 + 9) - \frac{2}{3}\ \arctan(0)$$

$$= \frac{1}{2}\ \ln 18 - \frac{1}{2}\ \ln 9 + \frac{2}{3}\ \arctan 1$$

$$= \frac{1}{2}\ln\left(\frac{18}{9}\right) + \frac{2}{3}\arctan 1$$ M1

$$= \frac{1}{2}\ln(2) + \frac{2}{3} \times \frac{\pi}{4}$$

$$= \frac{1}{2}\ln 2 + \frac{\pi}{6}$$

Mean value $= \frac{1}{3-0} \times \left(\frac{1}{2}\ln 2 + \frac{\pi}{6}\right)$ M1

$$= \frac{1}{6}\ln 2 + \frac{\pi}{18}$$ A1

(c) Use your answer to part (*b*) to find the mean value, over the interval [0, 3], of

$$f(x) + \ln k,$$

where k is a positive constant, giving your answer in the form $p + \frac{1}{6}\ln q$, where p and q are constants and q is in terms of k.

(2)

Mean value of $f(x) + \ln k$

$= \frac{1}{6}\ln 2 + \frac{\pi}{18} + \ln k$ M1

$= \frac{1}{6}\ln 2 + \frac{\pi}{18} + \frac{1}{6}\ln k^6$

$= \frac{1}{6}\ln 2k^6 + \frac{\pi}{18}$ A1

Question 7

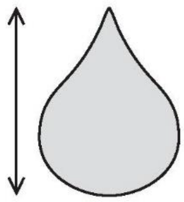


Figure 2

Figure 2 shows the image of a gold pendant which has height 2 cm. The pendant is modelled by a solid of revolution of a curve C about the y-axis. The curve C has parametric equations

$$x = \cos\theta + \frac{1}{2}\sin 2\theta, \qquad y = -(1 + \sin\theta) \qquad 0 \leq \theta \leq 2\pi$$

(a) Show that a Cartesian equation of the curve C is

$$x^2 = -(y^4 + 2y^3)$$

(4)

$x = \cos\theta + \frac{1}{2}\sin 2\theta$

$x = \cos\theta + \sin\theta\cos\theta$

$x = (1 + \sin\theta)\cos\theta$

$x = -y\cos\theta$ M1

$y = -(1 + \sin\theta)$

$\sin\theta = -y - 1$ M1

$\cos^2\theta + \sin^2\theta = 1$

$\left(\frac{-x}{y}\right)^2 + (-y-1)^2 = 1$ M1

$x^2 = y^2[1 - (-y-1)^2]$

$x^2 = y^2[1 - y^2 - 2y - 1]$

$x^2 = -(y^4 + 2y^3)$ A1

(b) Hence, using the model, find, in cm^3, the volume of the pendant.

(4)

$V = \pi \int x^2 \, dy = \pi \int -(y^4 + 2y^3) \, dy$ M1

$= -\pi \left[\frac{1}{5}y^5 + \frac{1}{2}y^4 \right]$ A1

Either:

Putting $x = 0$ in $x^2 = -(y^4 + 2y^3)$

gives $y = 0$ or $y = -2$.

Or:

$\theta = \frac{\pi}{2}$ gives $y = -2$

$\theta = \frac{3\pi}{2}$ gives $y = 0$

The y values 0 and –2 relate to the top and bottom of the pendant.
Therefore using limits 0 and –2,

$V = -\pi \left\{ \left(\frac{1}{5}(0)^5 + \frac{1}{2}(0)^4 \right) - \left(\frac{1}{5}(-2)^5 + \frac{1}{2}(-2)^4 \right) \right\}$ M1

$= 1.6\pi \text{ cm}^3 \quad (5.03 \text{ cm}^3)$ A1

Question 8

The line l_1 has equation $\dfrac{x-2}{4} = \dfrac{y-4}{-2} = \dfrac{z+6}{1}$

The plane Π has equation $x - 2y + z = 6$

The line l_2 is the reflection of the line l_1 in the plane Π.

Find a vector equation of the line l_2

(7)

$$\frac{x-2}{4} = \frac{y-4}{-2} = \frac{z+6}{1} = \lambda$$

The vector form of l_1 is

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$$

Find the coordinates of the intersection of the line and plane:

$x - 2y + z = 6$
$(2 + 4\lambda) - 2(4 - 2\lambda) + (-6 + \lambda) = 6$
$9\lambda - 12 = 6$
$\lambda = 2$ M1

$x = 2 + 4(2) = 10$
$y = 4 - 2(2) = 0$
$z = -6 + 2 = -4$

Intersection is $(10, 0, -4)$ A1

When $\lambda = 0$, $x = 2$, $y = 4$, $z = -6$ so $(2, 4, -6)$ is a point on line l_1

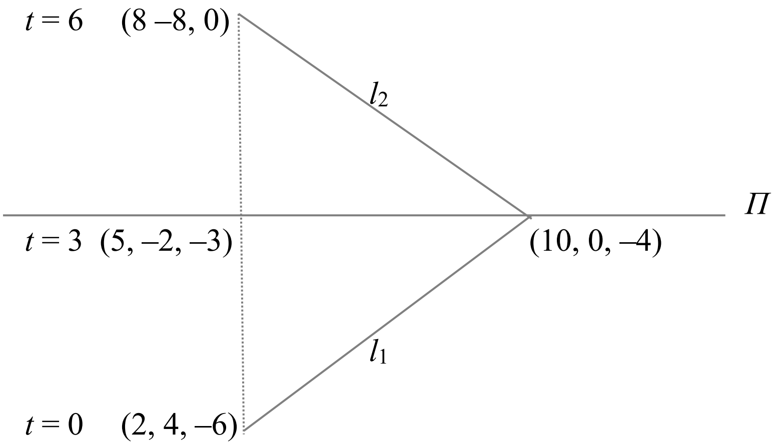
The normal vector to the plane is $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

Vector equation of line perpendicular to plane though $(2, 4, -6)$ is $\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

This line intersects the plane where

$(2 + t) - 2(4 - 2t) + (-6 + t) = 6$
$6t - 12 = 6$
$t = 3$ M1

Since $t = 0$ at the point (2, 4, –6) and $t = 3$ at the plane, then the reflection of this point in the plane is when $t = 6$.
$x = 2 + 6 = 8$
$y = 4 - 2(6) = -8$
$z = -6 + 6 = 0$ M1
So the reflected point is (8, –8, 0). A1



Direction of line l_2 is

$$\begin{pmatrix} 10 \\ 0 \\ -4 \end{pmatrix} - \begin{pmatrix} 8 \\ -8 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ -4 \end{pmatrix}$$ M1

Components can be halved to give $\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$

Equation of line l_2 is

$$\mathbf{r} = \begin{pmatrix} 10 \\ 0 \\ -4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$$ A1

Alternative:
Vector product form is

$$\left[\mathbf{r} - \begin{pmatrix} 10 \\ 0 \\ -4 \end{pmatrix}\right] \times \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = 0$$

Question 9

A company plans to build a new fairground ride. The ride will consist of a capsule that will hold the passengers and the capsule will be attached to a tall tower. The capsule is to be released from rest from a point half way up the tower and then made to oscillate in a vertical line.

The vertical displacement, x metres, of the top of the capsule below its initial position at time t seconds is modelled by the differential equation,

$$m\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + x = 200\cos t, \quad t \geq 0$$

where m is the mass of the capsule including its passengers, in thousands of kilograms. The maximum permissible weight for the capsule, including its passengers, is 30 000 N.

Taking the value of g to be 10 m s^{-2} and assuming the capsule is at its maximum permissible weight,

(a) (i) explain why the value of m is 3

(ii) show that a particular solution to the differential equation is

$$x = 40\sin t - 20\cos t$$

(iii) hence find the general solution of the differential equation.

(8)

(i)
Weight = mass × g $\quad$ $m = 30000/g = 3000$
But mass is in thousands of kg, so $m = 3$ M1

(ii)
$x = 40\sin t - 20\cos t$

$\frac{dx}{dt} = 40\cos t + 20\sin t$

$\frac{d^2x}{dt^2} = -40\sin t + 20\cos t$ M1

LHS of differential equation is

$= 3(-40\sin t + 20\cos t) + 4(40\cos t + 20\sin t) + (40\sin t - 20\cos t)$

$= (60\cos t + 160\cos t - 20\cos t) + (-120\sin t + 80\sin t + 40\sin t)$

$= 200\cos t + 0$ M1

which is RHS of differential equation,
so Particular Integral (PI) is $x = 40\sin t - 20\cos t$ A1

Alternative:

$x = a\cos t + b\sin t$

$\frac{dx}{dt} = -a\sin t + b\cos t$

$\frac{d^2x}{dt^2} = -a\cos t - b\sin t$ M1

$3(-a\cos t - b\sin t) + 4(-a\sin t + b\cos t) + (a\cos t + b\sin t) = 200\cos t$

Compare cos t terms
$-3a + 4b + a = 200$
$2b - a = 100$ ---[1]

Compare sin t terms
$-3b - 4a + b = 0$
$-2b - 4a = 0$ ---[2]

Solve simultaneously, [1] + [2] gives
$-5a = 100$
$a = -20$
$b = 40$ M1

So $x = 40\sin t - 20\cos t$ A1

(iii)
To find the Complimentary Function (CF), solve the auxiliary equation
$3\lambda^2 + 4\lambda + 1 = 0$
$(\lambda + 1)(3\lambda + 1) = 0$
$\lambda = -1$ or $\lambda = -\frac{1}{3}$ M1

Two real values, so solution is of the form
$x = Ae^{-t} + Be^{-(1/3)t}$ A1

General Solution = CF + PI M1

$x = Ae^{-t} + Be^{-(1/3)t} + 40\sin t - 20\cos t$ A1

(b) Using the model, find, to the nearest metre, the vertical distance of the top of the capsule from its initial position, 9 seconds after it is released.

(4)

$t = 0$, $x = 0$,

$0 = Ae^0 + Be^0 + 0 - 20$

$A + B = 20$ ---[1] M1

$\frac{dx}{dt} = -Ae^{-t} - \frac{1}{3}Be^{-(1/3)t} + 40\cos t + 20\sin t$

$\frac{dx}{dt} = 0$ (= v) when $t = 0$ as the capsule is released from rest

$0 = -Ae^0 - \frac{1}{3}Be^0 + 40 + 0$

$A + \frac{1}{3}B = 40$ ---[2] M1

Solve simultaneously, [1] – [2] gives

$\frac{2}{3}B = -20$

$B = -30$

$A = 50$

$x = 50e^{-t} - 30e^{-(1/3)t} + 40\sin t - 20\cos t$ A1

$t = 9$,

$x = 50e^{-9} - 30e^{-3} + 40\sin 9 - 20\cos 9$

$x = 33.2199...$

$x = 33$ m A1

Core Pure Mathematics 2

Question 1

The roots of the equation

$$x^3 - 8x^2 + 28x - 32 = 0$$

are α, β and γ

Without solving the equation, find the value of

(i) $\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}$

(ii) $(\alpha + 2)(\beta + 2)(\gamma + 2)$

(iii) $\alpha^2 + \beta^2 + \gamma^2$

(8)

(i)

$\alpha + \beta + \gamma = -(-8) = 8$

$\alpha\beta + \beta\gamma + \gamma\alpha = 28$

$\alpha\beta\gamma = -(-32) = 32$ B1

$\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma} = \dfrac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$ M1

$= \dfrac{28}{32}$

$= \dfrac{7}{8}$ A1

(ii)

$(\alpha + 2)(\beta + 2)(\gamma + 2) = (\alpha\beta + 2\alpha + 2\beta + 4)(\gamma + 2)$ M1

$= \alpha\beta\gamma + 2\alpha\beta + 2\alpha\gamma + 4\alpha + 2\beta\gamma + 4\beta + 4\gamma + 8$

$= \alpha\beta\gamma + 2(\alpha\beta + \alpha\gamma + \beta\gamma) + 4(\alpha + \beta + \gamma) + 8$ A1

$= 32 + 2(28) + 4(8) + 8$

$= 128$ A1

Alternative:

$(x-2)^3 - 8(x-2)^2 + 28(x-2) - 32 = 0$ M1

$x^3 - 6x^2 + 12x - 8 - 8x^2 + 32x - 32 + 28x - 56 - 32 = 0$

The powers of x are not important here so could write:
$\ldots\ \ldots\ \ldots - 8\ \ldots\ \ldots - 32\ \ldots - 56 - 32 = 0$

constant term:
$-8 - 32 - 56 - 32 = -128$ A1

So product of roots $= (\alpha + 2)(\beta + 2)(\gamma + 2)$
$= -(-128) = 128$ A1

(iii)
$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$ M1
$= 8^2 - 2(28)$
$= 8$ A1

Question 2

The plane Π_1 has vector equation

$$\mathbf{r}.(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 5$$

(*a*) Find the perpendicular distance from the point (6, 2, 12) to the plane Π_1

(3)

Formula book:
The perpendicular distance of (α, β, γ) from $n_1x + n_2y + n_3z + d = 0$ is
$$\frac{|n_1\alpha + n_2\beta + n_3\gamma + d|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$$

Cartesian equation of plane Π_1
$3x - 4y + 2z - 5 = 0$

$$\begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 6 \\ 2 \\ 12 \end{pmatrix} = 18 - 8 + 24$$ M1

Perpendicular distance $= \dfrac{18 - 8 + 24 - 5}{\sqrt{3^2 + (-4)^2 + 2^2}}$ M1

$= \dfrac{29}{\sqrt{29}}$

$= \sqrt{29}$ A1

The plane Π_2 has vector equation

$$\mathbf{r} = \lambda(2\mathbf{i} + \mathbf{j} + 5\mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} - 2\mathbf{k})$$

where λ and μ are scalar parameters.

(b) Show that the vector $-\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is perpendicular to Π_2

(2)

Calculate the scalar product between the given vector and both direction vectors:

$$\begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} = -2 - 3 + 5 = 0 \qquad \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = -1 + 3 - 2 = 0$$ M1

So $-\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is perpendicular to plane Π_2 A1

(c) Show that the acute angle between Π_1 and Π_2 is 52° to the nearest degree.

(3)

Calculate the scalar product between the two normal vectors:

$$\begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} = -3 + 12 + 2 = 11$$ M1

$$\sqrt{(-1)^2 + (-3)^2 + 1^2} = \sqrt{11}$$

$$\sqrt{3^2 + (-4)^2 + 2^2} = \sqrt{29}$$

$\mathbf{a}.\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$

$11 = \sqrt{11}\sqrt{29} \cos\theta$

$$\cos\theta = \sqrt{\frac{11}{29}}$$ M1

$\theta = 51.98398...$

Angle between normal vectors is the same as angle between planes, so
$\theta = 52°$ A1

Question 3

(i)

$$\mathbf{M} = \begin{pmatrix} 2 & a & 4 \\ 1 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix}$$

where a is a constant.

(a) For which values of a does the matrix **M** have an inverse?

(2)

Determinant of **M**:

$|\mathbf{M}| = 2((-1 \times -1) - (-1 \times 2)) - a((1 \times -1) - (-1 \times -1)) + 4((1 \times 2) - (-1 \times -1))$

$= 2(1 + 2) - a(-1 - 1) + 4(2 - 1)$

$= 6 + 2a + 4$

$= 0$ if no inverse

$2a + 10 = 0$

$a = -5$ M1

The matrix **M** has an inverse when $a \neq -5$ A1

Given that **M** is non-singular,

(b) find $\mathbf{M}^{-1}$ in terms of a

(4)

Calculate the Minors:

$$\begin{pmatrix} (-1\times-1)-(-1\times2) & (1\times-1)-(-1\times-1) & (1\times2)-(-1\times-1) \\ (a\times-1)-(2\times4) & (2\times-1)-(4\times-1) & (2\times2)-(a\times-1) \\ (a\times-1)-(4\times-1) & (2\times-1)-(4\times1) & (2\times-1)-(a\times1) \end{pmatrix}$$

Calculate the Cofactors:

As above, changing selected signs according to $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

Minors: Cofactors:

$$\begin{pmatrix} 3 & -2 & 1 \\ -a-8 & 2 & a+4 \\ 4-a & -6 & -2-a \end{pmatrix} \qquad \begin{pmatrix} 3 & 2 & 1 \\ a+8 & 2 & -a-4 \\ 4-a & 6 & -2-a \end{pmatrix}$$

(either) B1

Inverse $\mathbf{M}^{-1} = \dfrac{1}{|\mathbf{M}|}\text{adj}(\mathbf{M})$ M1

$$\mathbf{M}^{-1} = \frac{1}{2a+10}\begin{pmatrix} 3 & a+8 & 4-a \\ 2 & 2 & 6 \\ 1 & -a-4 & -2-a \end{pmatrix}$$

(2 correct rows or columns) A1
(all correct) A1

(ii) Prove by induction that for all positive integers n,

$$\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^n = \begin{pmatrix} 3^n & 0 \\ 3(3^n - 1) & 1 \end{pmatrix}$$

(6)

When $n = 1$,

$$\text{LHS} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$$

$$\text{RHS} = \begin{pmatrix} 3^1 & 0 \\ 3(3^1 - 1) & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$$

So the statement is true for $n = 1$. B1

Assume true for $n = k$, so

$$\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^k = \begin{pmatrix} 3^k & 0 \\ 3(3^k - 1) & 1 \end{pmatrix}$$ M1

$$\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3^k & 0 \\ 3(3^k - 1) & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$$ M1

$$= \begin{pmatrix} 3 \times 3^k & 0 \\ 3 \times 3(3^k - 1) + 6 & 1 \end{pmatrix}$$ A1

$$= \begin{pmatrix} 3^{k+1} & 0 \\ 3(3^{k+1}) - 9 + 6 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3^{k+1} & 0 \\ 3(3^{k+1} - 1) & 1 \end{pmatrix}$$ A1

If the statement is true for $n = k$ then it has been shown true for $n = k + 1$, and as it is true for $n = 1$, the statement is true for all positive integers n. A1

Question 4

A complex number z has modulus 1 and argument θ.

(a) Show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta, \qquad n \in \mathbb{Z}^+$$

(2)

$z^n + z^{-n} = \cos n\theta + \mathrm{i}\sin n\theta + \cos n\theta - \mathrm{i}\sin n\theta$	M1
$= 2\cos n\theta$	A1

(b) Hence, show that

$$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$$

(5)

$n = 1$, $z + z^{-1} = 2\cos\theta$	
$(z + z^{-1})^4 = (2\cos\theta)^4$	
$(z + z^{-1})^4 = 16\cos^4\theta$	B1
$(z + z^{-1})^4 = z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4}$	M1
$(z + z^{-1})^4 = (z^4 + z^{-4}) + 4(z^2 + z^{-2}) + 6$	A1
$16\cos^4\theta = 2\cos 4\theta + 4(2\cos 2\theta) + 6$	M1
$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$	A1

Question 5

$$y = \sin x \sinh x$$

(a) Show that $\frac{d^4y}{dx^4} = -4y$

(4)

$y = \sin x \sinh x$

Product rule: $(uv)' = uv' + u'v$

$\frac{dy}{dx} = \sin x \cosh x + \cos x \sinh x$ M1

$\frac{d^2y}{dx^2} = \sin x \sinh x + \cos x \cosh x + \cos x \cosh x - \sin x \sinh x$ M1

$= 2 \cos x \cosh x$

$\frac{d^3y}{dx^3} = 2 \cos x \sinh x - 2 \sin x \cosh x$ M1

$\frac{d^4y}{dx^4} = 2 \cos x \cosh x - 2 \sin x \sinh x - 2 \sin x \sinh x - 2 \cos x \cosh x$

$= -4 \sin x \sinh x$

$= -4y$ A1

(b) Hence find the first three non-zero terms of the Maclaurin series for y, giving each coefficient in its simplest form. **(4)**

When $x = 0$, $\sin x = 0$, $\sinh x = 0$, $\cos x = 1$ and $\cosh x = 1$.
So the only non zero terms will be where the derivative contains $\cos x$ and $\cosh x$ only, (2nd, 6th, 10th …)

$$\left(\frac{d^2y}{dx^2}\right)_0 = 2\times1\times1 = 2$$

$$\left(\frac{d^6y}{dx^6}\right)_0 = -4\left(\frac{d^2y}{dx^2}\right) = -4\times2 = -8$$

$$\left(\frac{d^{10}y}{dx^{10}}\right)_0 = -4\left(\frac{d^6y}{dx^6}\right) = -4\times-8 = 32$$ B1

Formula Book:
Maclaurin series
$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^r}{r!} f^{(r)}(0) + \ldots$

$$y = \frac{x^2}{2!}(2) + \frac{x^6}{6!}(-8) + \frac{x^{10}}{10!}(32)$$ M1

A1

$$= x^2 - \frac{x^6}{90} + \frac{x^{10}}{113400}$$ A1

(c) Find an expression for the nth non-zero term of the Maclaurin series for y. **(2)**

2, 6, 10, 14 … is an arithmetic sequence so has
$a = 2$, $d = 4$ and general term $4n - 2$

2, –8, 32, –128 … is a geometric sequence so has
$a = 2$, $r = -4$ and general term $2(-4)^{n-1}$ M1

So n^{th} non zero term is $\frac{2(-4)^{n-1} \times x^{4n-2}}{(4n-2)!}$ A1

Question 6

(a) (i) Show on an Argand diagram the locus of points given by the values of z satisfying

$$|z - 4 - 3\mathbf{i}| = 5$$

Taking the initial line as the positive real axis with the pole at the origin and given that

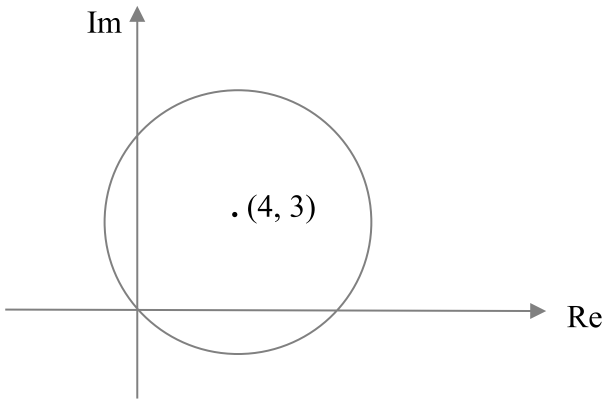
$\theta \in [\alpha, \alpha + \pi]$, where $\alpha = -\arctan\left(\frac{4}{3}\right)$,

(ii) show that this locus of points can be represented by the polar curve with equation

$$r = 8\cos\theta + 6\sin\theta$$

(6)

(i)



(any circle passing through origin) M1
(fully correct) A1

(ii)

$|z - 4 - 3\text{i}| = 5$

$|x + \text{i}y - 4 - 3\text{i}| = 5$

$|x - 4 + \text{i}(y - 3)| = 5$

$(x - 4)^2 + (y - 3)^2 = 5^2$ M1
A1

$(r\cos\theta - 4)^2 + (r\sin\theta - 3)^2 = 25$

$r^2\cos^2\theta - 8r\cos\theta + 16 + r^2\sin^2\theta - 6r\sin\theta + 9 = 25$

$r^2(\cos^2\theta + \sin^2\theta) - 8r\cos\theta - 6r\sin\theta = 0$

$r^2(1) - 8r\cos\theta - 6r\sin\theta = 0$ M1

$r = 8\cos\theta + 6\sin\theta$ A1

The set of points A is defined by

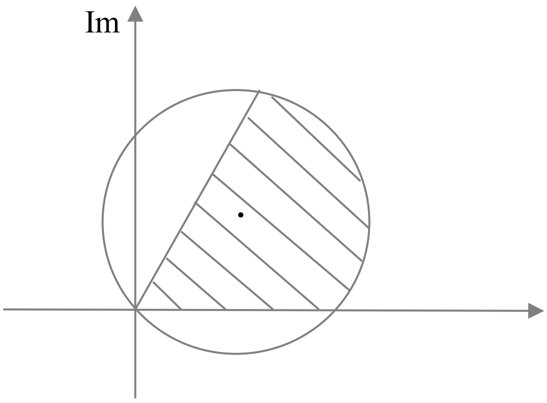
$$A = \{ z : 0 \leq \arg z \leq \frac{\pi}{3} \} \cap \{ z : |z - 4 - 3\mathbf{i}| \leqslant 5 \}$$

(b) (i) Show, by shading on your Argand diagram, the set of points A.

(ii) Find the **exact** area of the region defined by A, giving your answer in simplest form.

(7)

(i)



(rays at $\frac{\pi}{3}$ and 0) B1

(correct shading) B1

(ii)

Formula Book:

Area of a sector $= \frac{1}{2}\int r^2 \, d\theta$

$$\frac{1}{2}\int r^2 \, d\theta = \frac{1}{2}\int (8\cos\theta + 6\sin\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int 64\cos^2\theta + 96\cos\theta\sin\theta + 36\sin^2\theta \, d\theta$$ M1

$$= \frac{1}{2}\int 64\times\frac{1}{2}(1 + \cos 2\theta) + 48\times 2\cos\theta\sin\theta + 36\times\frac{1}{2}(1 - \cos 2\theta) \, d\theta$$ M1

$$= \frac{1}{2}\int 32(1 + \cos 2\theta) + 48\sin 2\theta + 18(1 - \cos 2\theta) \, d\theta$$

$$= \frac{1}{2}\int 14\cos 2\theta + 50 + 48\sin 2\theta \, d\theta$$ A1

$$= \frac{1}{2}[7\sin 2\theta + 50\theta - 24\cos 2\theta]$$

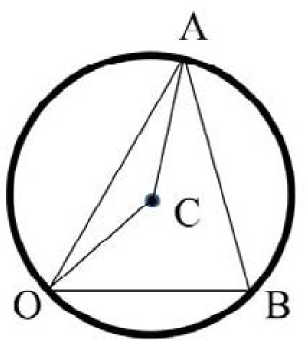
Using limits 0 and $\frac{\pi}{3}$:

$$= \frac{1}{2}\left\{7\left(\frac{\sqrt{3}}{2}\right) + 50\left(\frac{\pi}{3}\right) - 24\left(\frac{-1}{2}\right) - (0 + 0 - 24)\right\}$$ M1

$$= \frac{7\sqrt{3}}{4} + \frac{25\pi}{3} + 18$$ A1

Alternative:

Geometric approach, eg by finding sector + 2 triangles:



Angle $ACB = 2 \times$ Angle AOB

$$= \frac{2\pi}{3}$$

Area of sector $ACB = \frac{1}{2}(5)^2\frac{2\pi}{3}$

$$= \frac{25\pi}{3}$$

B is (8, 0), C is (4, 3)

Area of triangle $OCB = \frac{1}{2} \times 8 \times 3$ M1

$$= 12$$

Sector area ACB + triangle area $OCB = \frac{25\pi}{3} + 12$ A1

cos (angle OCB) = $\frac{5^2 + 5^2 - 8^2}{2\times5\times5}$

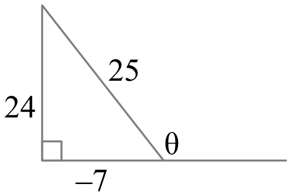
Angle $OCB = \cos^{-1}\left(\frac{-7}{25}\right)$

Angle $ACO = 2\pi - \frac{2\pi}{3} - \cos^{-1}\left(\frac{-7}{25}\right)$

Area of triangle $OAC = \frac{1}{2}(5)^2 \sin\left(\frac{4\pi}{3} - \cos^{-1}\left(\frac{-7}{25}\right)\right)$ M1

Using compound angle formula $\sin(X - Y) = \sin X \cos Y - \cos X \sin Y$:

Area $OAC = \frac{25}{2}\left\{\sin\left(\frac{4\pi}{3}\right)\cos\left(\cos^{-1}\left(\frac{-7}{25}\right)\right) - \cos\left(\frac{4\pi}{3}\right)\sin\left(\cos^{-1}\left(\frac{-7}{25}\right)\right)\right\}$



If $\cos\theta = \frac{-7}{25}$,

$\sqrt{25^2 - 7^2} = 24$

so $\sin\theta = \frac{24}{25}$

Area $OAC = \frac{25}{2}\left\{\left(\frac{-\sqrt{3}}{2}\right)\left(\frac{-7}{25}\right) - \left(\frac{-1}{2}\right)\left(\frac{24}{25}\right)\right\}$

$= \frac{25}{2}\left\{\left(\frac{7\sqrt{3}}{50}\right) + \left(\frac{12}{25}\right)\right\}$

$= \frac{7\sqrt{3}}{4} + 6$

Total area $= \frac{25\pi}{3} + 12 + \frac{7\sqrt{3}}{4} + 6$ M1

$= \frac{7\sqrt{3}}{4} + \frac{25\pi}{3} + 18$ A1

Question 7

At the start of the year 2000, a survey began of the number of foxes and rabbits on an island.

At time t years after the survey began, the number of foxes, f, and the number of rabbits, r, on the island are modelled by the differential equations

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 0.2f + 0.1r$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -0.2f + 0.4r$$

(a) Show that $\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} - 0.6\frac{\mathrm{d}f}{\mathrm{d}t} + 0.1f = 0$

(3)

$$10\frac{\mathrm{d}f}{\mathrm{d}t} = 2f + r$$

$$r = 10\frac{\mathrm{d}f}{\mathrm{d}t} - 2f$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = 10\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} - 2\frac{\mathrm{d}f}{\mathrm{d}t}$$ M1

$$10\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} - 2\frac{\mathrm{d}f}{\mathrm{d}t} = -0.2f + 0.4r$$

$$10\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} - 2\frac{\mathrm{d}f}{\mathrm{d}t} = -0.2f + 0.4(10\frac{\mathrm{d}f}{\mathrm{d}t} - 2f)$$ M1

$$10\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} - 2\frac{\mathrm{d}f}{\mathrm{d}t} = -0.2f + 4\frac{\mathrm{d}f}{\mathrm{d}t} - 0.8f$$

$$10\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} - 6\frac{\mathrm{d}f}{\mathrm{d}t} + f = 0$$

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} - 0.6\frac{\mathrm{d}f}{\mathrm{d}t} + 0.1f = 0$$ A1

(b) Find a general solution for the number of foxes on the island at time t years. **(4)**	

Auxiliary equation:

$m^2 - 0.6m + 0.1 = 0$

$(m - 0.3)^2 - 0.09 + 0.1 = 0$

$(m - 0.3)^2 = -0.01$

$m = 0.3 \pm \sqrt{(-0.01)}$ M1

$m = 0.3 \pm 0.1\mathrm{i}$ A1

Complex solutions $\alpha \pm \beta\mathrm{i}$ give a complimentary function (CF) of the form
$\mathrm{e}^{\alpha t}(A\cos\beta t + B\sin\beta t)$ M1

$f = \mathrm{e}^{0.3t}(A\cos 0.1t + B\sin 0.1t)$ A1

(c) Hence find a general solution for the number of rabbits on the island at time t years. **(3)**	

Product rule:

$\frac{\mathrm{d}f}{\mathrm{d}t} = 0.3\,\mathrm{e}^{0.3t}(A\cos 0.1t + B\sin 0.1t) + \mathrm{e}^{0.3t}(-0.1A\sin 0.1t + 0.1B\cos 0.1t)$ M1

$= \mathrm{e}^{0.3t}\{(0.3A + 0.1B)\cos 0.1t + (0.3B - 0.1A)\sin 0.1t\}$

$r = 10\frac{\mathrm{d}f}{\mathrm{d}t} - 2f$

$r = 10(\mathrm{e}^{0.3t}\{(0.3A + 0.1B)\cos 0.1t + (0.3B - 0.1A)\sin 0.1t\}) - 2\,\mathrm{e}^{0.3t}(A\cos 0.1t + B\sin 0.1t)$

$r = \mathrm{e}^{0.3t}\{(3A + B)\cos 0.1t + (3B - A)\sin 0.1t\} - 2\,\mathrm{e}^{0.3t}(A\cos 0.1t + B\sin 0.1t)$ M1

$r = \mathrm{e}^{0.3t}\{(3A + B)\cos 0.1t + (3B - A)\sin 0.1t - 2A\cos 0.1t - 2B\sin 0.1t\}$

$r = \mathrm{e}^{0.3t}\{(A + B)\cos 0.1t + (B - A)\sin 0.1t\}$ A1

At the start of the year 2000 there were 6 foxes and 20 rabbits on the island.

(d) (i) According to this model, in which year are the rabbits predicted to die out?

(ii) According to this model, how many foxes will be on the island when the rabbits die out?

(iii) Use your answers to parts (i) and (ii) to comment on the model.

(7)

(i)
$t = 0, f = 6,$
$6 = e^0 (A \cos 0 + B \sin 0)$
$A = 6$ M1

$t = 0, r = 20$
$20 = e^0 \{(A + B) \cos 0 + (B - A) \sin 0\}$
$20 = A + B$
$B = 14$ M1

$r = e^{0.3t} \{20 \cos 0.1t + 8 \sin 0.1t\} = 0$ M1

$8 \sin 0.1t = -20 \cos 0.1t$

$\tan 0.1t = -2.5$ A1

$0.1t = (-1.1903 \text{ or}) -1.1903 + \pi$

$0.1t = 1.9513...$

$t = 19.513...$

Therefore during 2019. A1

(ii)
$f = e^{0.3(19.513)} (6 \cos 1.9513 + 14 \sin 1.9513)$

$f = 3754.45...$

$f = 3750$ (3sf) B1

(iii)
e.g. the model predicts a large number of foxes are on the island when the rabbits have died out and this may not be sensible. B1

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