AS and A level Mathematics and Further Mathematics 2017

Getting Ready to Teach Online course
The A level reforms

• All new AS and A levels will be assessed at the same standard as they are currently
• All new AS and A levels will be fully linear
• AS levels will be stand-alone qualifications
• The content of the AS level can be a sub-set of the A level content to allow co-teachability, but marks achieved in the AS will not count towards the A level
The A level reforms - content

A level Mathematics

• 100% core content

• Pure mathematics (broadly same as C1 to C4)

• Mechanics (mainly from M1 and M2)

• Statistics (mainly from S1 and S2)

• AS content shown in bold font
The A level reforms - content

• Requirement for the assessment of problem solving, communication, proof, modelling, application of techniques
• Requirement for a pre-release large data set (mathematics only)
• Requirement that candidates have a calculator with
  • the ability to compute summary statistics and access probabilities from standard statistical distributions
  • an iterative function
  • the ability to perform calculations with matrices up to at least order 3 × 3 (further mathematics only)
Our design principles

• Separate pure and applied papers
• Simple 2:1 ratio of pure to applied (A level Mathematics)
• Single large data set for the lifetime of the qualification – may be amended
• Further mathematics designed to aid parallel delivery with mathematics
• No non-calculator assessment
Today’s agenda

Focus – Mechanics content at AS and A Level

• Information regarding specification changes
• Course planning and pre-requisites
• General themes in planning for mechanics
• Specific topic areas
  – Kinematics with calculus (AS)
  – Projectiles (A Level)
  – Ladder problems (A Level)
• Questions arising and further support available
Poll 1

How much mechanics teaching experience do you have?

- M1 Edexcel
- M2 Edexcel
- M3 Edexcel
- M4 Edexcel
- Other exam boards
- None / never taught mechanics
Overarching theme 1: Mathematical argument, language and proof

A Level Mathematics students must use the mathematical notation set out in the booklet *Mathematical Formulae and Statistical Tables* and be able to recall the mathematical formulae and identities set out in *Appendix 1*.

<table>
<thead>
<tr>
<th>Knowledge/skill</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>OT1.1</strong> Construct and present mathematical arguments through appropriate use of diagrams; sketching graphs; logical deduction; precise statements involving correct use of symbols and connecting language, including: constant, coefficient, expression, equation, function, identity, index, term, variable.</td>
</tr>
<tr>
<td><strong>OT1.2</strong> Understand and use mathematical language and syntax as set out in the content.</td>
</tr>
<tr>
<td><strong>OT1.3</strong> Understand and use language and symbols associated with set theory, as set out in the content. Apply to solutions of inequalities and probability.</td>
</tr>
<tr>
<td><strong>OT1.4</strong> Understand and use the definition of a function; domain and range of functions.</td>
</tr>
<tr>
<td><strong>OT1.5</strong> Comprehend and critique mathematical arguments, proofs and justifications of methods and formulae, including those relating to applications of mathematics.</td>
</tr>
</tbody>
</table>
## Overarching theme 2: Mathematical problem solving

<table>
<thead>
<tr>
<th>Knowledge/skill</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>OT2.1</td>
<td>Recognise the underlying mathematical structure in a situation and simplify and abstract appropriately to enable problems to be solved.</td>
</tr>
<tr>
<td>OT2.2</td>
<td>Construct extended arguments to solve problems presented in an unstructured form, including problems in context.</td>
</tr>
<tr>
<td>OT2.3</td>
<td>Interpret and communicate solutions in the context of the original problem.</td>
</tr>
<tr>
<td>OT2.4</td>
<td>Understand that many mathematical problems cannot be solved analytically, but numerical methods permit solution to a required level of accuracy.</td>
</tr>
<tr>
<td>OT2.5</td>
<td>Evaluate, including by making reasoned estimates, the accuracy or limitations of solutions, including those obtained using numerical methods.</td>
</tr>
<tr>
<td>OT2.6</td>
<td>Understand the concept of a mathematical problem-solving cycle, including specifying the problem, collecting information, processing and representing information and interpreting results, which may identify the need to repeat the cycle.</td>
</tr>
<tr>
<td>OT2.7</td>
<td>Understand, interpret and extract information from diagrams and construct mathematical diagrams to solve problems, including in mechanics.</td>
</tr>
</tbody>
</table>
### Overarching theme 3: Mathematical modelling

<table>
<thead>
<tr>
<th>Knowledge/skill</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>OT3.1</strong> Translate a situation in context into a mathematical model, making simplifying assumptions.</td>
<td></td>
</tr>
<tr>
<td><strong>OT3.2</strong> Use a mathematical model with suitable inputs to engage with and explore situations (for a given model or a model constructed or selected by the student).</td>
<td></td>
</tr>
<tr>
<td><strong>OT3.3</strong> Interpret the outputs of a mathematical model in the context of the original situation (for a given model or a model constructed or selected by the student).</td>
<td></td>
</tr>
<tr>
<td><strong>OT3.4</strong> Understand that a mathematical model can be refined by considering its outputs and simplifying assumptions; evaluate whether the model is appropriate.</td>
<td></td>
</tr>
<tr>
<td><strong>OT3.5</strong> Understand and use modelling assumptions.</td>
<td></td>
</tr>
</tbody>
</table>
# Assessment structure

## AS assessment

<table>
<thead>
<tr>
<th>Paper 1: Pure Mathematics</th>
<th>62.5%</th>
<th>2 hours</th>
<th>100 marks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Paper 2: Mechanics and Statistics</td>
<td>37.5%</td>
<td>1 hour 15 minutes</td>
<td>60 marks</td>
</tr>
</tbody>
</table>

## A level assessment

<table>
<thead>
<tr>
<th>Paper 1: Pure Mathematics</th>
<th>33%</th>
<th>2 hours</th>
<th>100 marks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Paper 2: Pure Mathematics</td>
<td>33%</td>
<td>2 hours</td>
<td>100 marks</td>
</tr>
<tr>
<td>Paper 3: Mechanics and Statistics</td>
<td>33%</td>
<td>2 hours</td>
<td>100 marks</td>
</tr>
</tbody>
</table>

Any pure content can be assessed on either paper.
Applied Content Changes

• Standardised content across all boards
• More emphasis on Statistics at AS
• Statistics relies heavily on the use of calculators for distributions
## Content changes

| 6 | 6.1 | Understand and use fundamental quantities and units in the S.I. system: length, time, mass. Understand and use derived quantities and units: velocity, acceleration, force, weight, moment. | Students may be required to convert one unit into another e.g. km h\(^{-1}\) into m s\(^{-1}\) |
## Content changes

<table>
<thead>
<tr>
<th>7</th>
<th>Kinematics</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.1</td>
<td>Understand and use the language of kinematics: position; displacement; distance travelled; velocity; speed; acceleration.</td>
</tr>
<tr>
<td>7.2</td>
<td>Understand, use and interpret graphs in kinematics for motion in a straight line: displacement against time and interpretation of gradient; velocity against time and interpretation of gradient and area under the graph.</td>
</tr>
</tbody>
</table>
| 7.3 | Understand, use and derive the formulae for constant acceleration for motion in a straight line. Extend to 2 dimensions using vectors. | Derivation may use knowledge of sections 7.2 and/or 7.4
Understand and use suvat formulae for constant acceleration in 2-D,
e.g. \( v = u + at \), \( r = ut + \frac{1}{2}at^2 \) with vectors given in \( \vec{i} - \vec{j} \) or column vector form.
Use vectors to solve problems. |
## Content changes

<table>
<thead>
<tr>
<th>Section</th>
<th>Topic</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.4</td>
<td>Use calculus in kinematics for motion in a straight line:</td>
<td>$v = \frac{dr}{dt}$, $a = \frac{dv}{dt} = \frac{d^2r}{dt^2}$, $r = \int v , dt$, $v = \int a , dt$ Extend to 2 dimensions using vectors.</td>
</tr>
<tr>
<td></td>
<td>The level of calculus required will be consistent with that in Sections 7 and 8 in Paper 1 and Sections 6 and 7 in Paper 2.</td>
<td>Differentiation and integration of a vector with respect to time. E.g. Given $\mathbf{r} = t^2 \mathbf{i} + t^2 \mathbf{j}$, find $\mathbf{i}$ and $\mathbf{j}$ at a given time.</td>
</tr>
<tr>
<td>7.5</td>
<td>Model motion under gravity in a vertical plane using vectors; projectiles.</td>
<td>Derivation of formulae for time of flight, range and greatest height and the derivation of the equation of the path of a projectile may be required.</td>
</tr>
</tbody>
</table>
## Content changes

<table>
<thead>
<tr>
<th>8</th>
<th>8.1 Understand the concept of a force; understand and use Newton’s first law.</th>
<th>Normal reaction, tension, thrust or compression, resistance.</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>8.2 Understand and use Newton’s second law for motion in a straight line (restricted to forces in two perpendicular directions or simple cases of forces given as 2-D vectors); extend to situations where forces need to be resolved (restricted to 2 dimensions).</td>
<td>Problems will involve motion in a straight line with constant acceleration in scalar form, where the forces act either parallel or perpendicular to the motion. Extend to problems where forces need to be resolved, e.g. a particle moving on an inclined plane. Problems may involve motion in a straight line with constant acceleration in vector form, where the forces are given in i – j form or as column vectors.</td>
</tr>
<tr>
<td>8</td>
<td>8.3 Understand and use weight and motion in a straight line under gravity; gravitational acceleration, ( g ), and its value in S.I. units to varying degrees of accuracy.</td>
<td>The default value of ( g ) will be 9.8 m s(^{-2}) but some questions may specify another value, e.g. ( g = 10 ) m s(^{-2}). The inverse square law for gravitation is not required and ( g ) may be assumed to be constant, but students should be aware that ( g ) is not a universal constant but depends on location.</td>
</tr>
</tbody>
</table>
### Content changes

<table>
<thead>
<tr>
<th>8.4</th>
<th>Understand and use Newton’s third law; equilibrium of forces on a particle and motion in a straight line; application to problems involving smooth pulleys and connected particles; resolving forces in 2 dimensions; equilibrium of a particle under coplanar forces.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Problems may be set where forces need to be resolved (restricted to forces in two perpendicular directions or simple cases of forces given as 2-D vectors).</td>
</tr>
<tr>
<td></td>
<td>Connected particle problems could include problems with particles in contact e.g. lift problems.</td>
</tr>
<tr>
<td></td>
<td>Problems may be set where forces need to be resolved, e.g. at least one of the particles is moving on an inclined plane.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>8.5</th>
<th>Understand and use addition of forces; resultant forces; dynamics for motion in a plane.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Students may be required to resolve a vector into two components or use a vector diagram, e.g. problems involving two or more forces, given in magnitude-direction form.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>8</th>
<th><strong>Forces and Newton’s laws continued</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>8.6</td>
<td>Understand and use the $F \leq \mu R$ model for friction; coefficient of friction; motion of a body on a rough surface; limiting friction and statics.</td>
</tr>
<tr>
<td></td>
<td>An understanding of $F = \mu R$ when a particle is moving.</td>
</tr>
<tr>
<td></td>
<td>An understanding of $F \leq \mu R$ in a situation of equilibrium.</td>
</tr>
<tr>
<td></td>
<td>9.1</td>
</tr>
<tr>
<td>---</td>
<td>-----</td>
</tr>
</tbody>
</table>
Content summary AS

- **Kinematics**
  - SUVAT, 1D horizontal and vertical motion
  - Velocity time graphs
  - Calculus in 1D

- **Forces and Newton’s Laws**
  - Statics: equilibrium in 2D **without resolving** components (perpendicular only)
  - Connected particles using smooth pulleys
  - Simple 2D vector problems
Content summary A Level

- **Kinematics**
  - Extending to 2D (motion in a vertical plane)
  - Use of vectors in kinematics
  - Projectile motion

- **Forces and Newton’s Laws**
  - Statics: extended to include resolution of forces
  - Coefficient of friction

- **Moments**
  - Including ladder problems

  *Impulse and momentum is now in Further Mechanics*
<table>
<thead>
<tr>
<th>Topic</th>
<th>Content</th>
<th>2008 Specification</th>
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<tbody>
<tr>
<td>Kinematics</td>
<td>Understand and use the language of kinematics: position; displacement; distance travelled; velocity; speed; acceleration.</td>
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<td>Kinematics</td>
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<td></td>
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<td>Kinematics</td>
<td>Use calculus in kinematics for motion in a straight line. Extend to 2 dimensions using vectors.</td>
<td></td>
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<td>Kinematics</td>
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<td></td>
</tr>
<tr>
<td>Forces and Newton’s laws</td>
<td>Understand the concept of a force; understand and use Newton’s first law.</td>
<td>M1</td>
</tr>
<tr>
<td>--------------------------</td>
<td>-------------------------------------------------------------------</td>
<td>----</td>
</tr>
<tr>
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<td>Understand and use Newton’s second law for motion in a straight line (restricted to forces in two perpendicular directions or simple cases of forces given as 2-D vectors)</td>
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<td>Forces and Newton’s laws</td>
<td>Understand and use weight and motion in a straight line under gravity; gravitational acceleration, ( g ), and its value in S.I. units to varying degrees of accuracy.</td>
<td>M1</td>
</tr>
<tr>
<td>Forces and Newton’s laws</td>
<td>Understand and use Newton’s third law; equilibrium of forces on a particle and motion in a straight line; application to problems involving smooth pulleys and connected particles</td>
<td>M1</td>
</tr>
<tr>
<td>Forces and Newton’s laws</td>
<td></td>
<td>M1</td>
</tr>
</tbody>
</table>
Assessment objectives

<table>
<thead>
<tr>
<th>AO1: Use and apply standard techniques.</th>
<th>50% (A Level)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Learners should be able to:</td>
<td>60% (AS)</td>
</tr>
<tr>
<td>- select and correctly carry out routine procedures</td>
<td></td>
</tr>
<tr>
<td>- accurately recall facts, terminology and definitions</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Strands</th>
<th>Elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. select and correctly carry out routine procedures</td>
<td>1a - select routine procedures</td>
</tr>
<tr>
<td>1b - correctly carry out routine procedures</td>
<td></td>
</tr>
<tr>
<td>2. accurately recall facts, terminology and definitions</td>
<td>This strand is a single element</td>
</tr>
</tbody>
</table>
Assessment objectives

AO2: Reason, interpret and communicate mathematically

Learners should be able to:
- construct rigorous mathematical arguments (including proofs)
- make deductions and inferences
- assess the validity of mathematical arguments
- explain their reasoning
- use mathematical language and notation correctly

<table>
<thead>
<tr>
<th>Strands</th>
<th>Elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. construct rigorous mathematical arguments (including proofs)</td>
<td>This strand is a single element</td>
</tr>
<tr>
<td>2. make deductions and inferences</td>
<td>2a – make deductions</td>
</tr>
<tr>
<td></td>
<td>2b – make inferences</td>
</tr>
<tr>
<td>3. assess the validity of mathematical arguments</td>
<td>This strand is a single element</td>
</tr>
<tr>
<td>4. explain their reasoning</td>
<td>This strand is a single element</td>
</tr>
<tr>
<td>5. use mathematical language and notation correctly</td>
<td>This strand is a single element</td>
</tr>
</tbody>
</table>

“Spot the mistakes” style questions
Specific notation may be required
Assessment objectives

AO3: Solve problems within mathematics and in other contexts

Learners should be able to:
- translate problems in mathematical and non-mathematical contexts into mathematical processes
- interpret solutions to problems in their original context, and, where appropriate, evaluate their accuracy and limitations
- translate situations in context into mathematical models
- use mathematical models
- evaluate the outcomes of modelling in context, recognise the limitations of models and, where appropriate, explain how to refine them

25% (A Level)
20% (AS)
### Assessment objectives

<table>
<thead>
<tr>
<th>Strands</th>
<th>Elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. translate problems in mathematical and non-mathematical contexts into mathematical processes</td>
<td>1a – translate problems in mathematical contexts into mathematical processes</td>
</tr>
<tr>
<td></td>
<td>1b – translate problems in non-mathematical contexts into mathematical processes</td>
</tr>
<tr>
<td>2. interpret solutions to problems in their original context, and, where appropriate evaluate their accuracy and limitations</td>
<td>2a – interpret solutions to problems in their original context</td>
</tr>
<tr>
<td></td>
<td>2b – where appropriate, evaluation the accuracy and limitations of solutions to problems</td>
</tr>
<tr>
<td>3. translate situations in context into mathematical models</td>
<td>This strand is a single element</td>
</tr>
<tr>
<td>4. use mathematical models</td>
<td>This strand is a single element</td>
</tr>
<tr>
<td>5. evaluate the outcomes of modelling in context, recognise the limitations of models and, where appropriate, explain how to refine them</td>
<td>5a – evaluate the outcomes of modelling in context</td>
</tr>
<tr>
<td></td>
<td>5b – recognise the limitations of models</td>
</tr>
<tr>
<td></td>
<td>5c – where appropriate, explain how to refine models</td>
</tr>
</tbody>
</table>
## Assessment Objectives

### Breakdown of Assessment Objectives

<table>
<thead>
<tr>
<th>Paper</th>
<th>AO1 %</th>
<th>AO2 %</th>
<th>AO3 %</th>
<th>Total for all Assessment Objectives</th>
</tr>
</thead>
<tbody>
<tr>
<td>Paper 1: Pure Mathematics 1</td>
<td>16.00–17.33</td>
<td>9.33–10.67</td>
<td>6.00–7.33</td>
<td>33.33%</td>
</tr>
<tr>
<td>Paper 2: Pure Mathematics 2</td>
<td>16.00–17.33</td>
<td>9.33–10.67</td>
<td>6.00–7.33</td>
<td>33.33%</td>
</tr>
<tr>
<td>Paper 3: Statistics and Mechanics</td>
<td>16.00–17.33</td>
<td>5.67–7.00</td>
<td>9.67–11.00</td>
<td>33.33%</td>
</tr>
<tr>
<td><strong>Total for GCE A Level</strong></td>
<td><strong>48-52</strong></td>
<td><strong>23-27</strong></td>
<td><strong>23-27</strong></td>
<td><strong>100%</strong></td>
</tr>
</tbody>
</table>
Students will be given the following formulae, in the formula booklet, in their examinations for both AS and A Level Mathematics.

However, note that they do need to be able to DERIVE these formulae too, according to the specification.

**Mechanics**

**Kinematics**

For motion in a straight line with constant acceleration:

\[ v = u + at \]
\[ s = ut + \frac{1}{2} at^2 \]
\[ s = vt - \frac{1}{2} at^2 \]
\[ v^2 = u^2 + 2as \]
\[ s = \frac{1}{2} (u + v)t \]
Formulae for Mechanics

What they will need to learn:

- **S.I. units** and know how to convert between
- Corresponding vector equations for SUVAT
- **Calculus in kinematics** extending to vectors
- **In forces and Newton’s laws**, $F = ma$
- $F = \mu R$ when a particle is moving and $F \leq \mu R$ in a situation of equilibrium
- Concept of force $x$ (perp.) distance for moments
# Course planning

## Year 1: AS Mathematics applied content

### Section B – Mechanics

<table>
<thead>
<tr>
<th>Unit</th>
<th>Title</th>
<th>Estimated hours</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>Quantities and units in mechanics</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Introduction to mathematical modelling and standard S.I. units of length, time and mass</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Definitions of force, velocity, speed, acceleration and weight and displacement; Vector and scalar quantities</td>
<td>2</td>
</tr>
<tr>
<td>7</td>
<td>Kinematics 1 (constant acceleration)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Graphical representation of velocity, acceleration and displacement</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>Motion in a straight line under constant acceleration; suvat formulae for constant acceleration; Vertical motion under gravity</td>
<td>6</td>
</tr>
<tr>
<td>8</td>
<td>Forces &amp; Newton's laws</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Newton’s first law, force diagrams, equilibrium, introduction to ( \mathbf{i}, \mathbf{j} ) system</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>Newton’s second law, ‘( F = ma )’, connected particles (no resolving forces or use of ( F = \mu R )); Newton’s third law: equilibrium, problems involving smooth pulleys</td>
<td>6</td>
</tr>
<tr>
<td>9</td>
<td>Kinematics 2 (variable acceleration)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Variable force; Calculus to determine rates of change for kinematics</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>Use of integration for kinematics problems i.e. ( r = \int v , dt, v = \int a , dt )</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td><strong>Total</strong></td>
<td><strong>30 hours</strong></td>
</tr>
</tbody>
</table>
### Year 2: Remaining A Level Mathematics applied content

#### Section B – Mechanics

<table>
<thead>
<tr>
<th>Unit</th>
<th>Title</th>
<th>Estimated hours</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>Moments: Forces’ turning effect</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>Forces at any angle</td>
<td></td>
</tr>
<tr>
<td></td>
<td>a Resolving forces</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>b Friction forces (including coefficient of friction $\mu$)</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>Applications of kinematics: Projectiles</td>
<td>5</td>
</tr>
<tr>
<td>7</td>
<td>Applications of forces</td>
<td></td>
</tr>
<tr>
<td></td>
<td>a Equilibrium and statics of a particle (including ladder problems)</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>b Dynamics of a particle</td>
<td>4</td>
</tr>
<tr>
<td>8</td>
<td>Further kinematics</td>
<td></td>
</tr>
<tr>
<td></td>
<td>a Constant acceleration (equations of motion in 2D; the $\mathbf{i}$, $\mathbf{j}$ system)</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>b Variable acceleration (use of calculus and finding vectors $\mathbf{r}$ and $\mathbf{r}$ at a given time)</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>30 hours</td>
</tr>
</tbody>
</table>
Course planning

- SoW document contains full detail of the dependencies for each topic under the prior knowledge section for that topic.

- Use the interactive SoW to see dependencies clearly and quickly, and to ensure that you don’t miss anything when deciding order of teaching.

- Take particular care if teaching Further Mechanics parallel delivery.
General themes

- When planning and teaching Mechanics, there are a few important general themes to consider.
- These should be embedded throughout your teaching of all Mechanics topics.
  - Ideas of modelling and interpretation
  - Use of diagrams to assist in problem solving
  - Reference to practical applications
Introducing the concept of modelling to students

• Explain that a mathematical model is a mathematical description of a real-life situation where simplifying assumptions have been made. It can be used to estimate solutions to real-world problems.

• This allows us to describe the situation using manageable equations or graphs to solve the problem.

• The interpretation of the solution in the context of the problem is one of the most important aspects of modelling.

• It is possible the model may need to be refined and assumptions reconsidered.
Modelling and interpretation

Key points to embed throughout

- Common assumptions used throughout Mechanics, e.g. ignoring air resistance
- Use of SI units
- Giving reasons why your model is or is not realistic
- Use of prior maths knowledge (from pure work or even GCSE) when modelling. Some examples are bearings, unit conversions, trigonometry
- You may be expected to use your own COMMON SENSE about real life circumstances, e.g. knowing that time cannot be negative, or that a person completing a 10k run in 2.8 seconds is not a realistic solution
Use of diagrams

Visualising written information

Which of these does your brain process quicker?

1. The locus of points equidistant from a single given point

2.
Use of diagrams

Visualising written information

Why is this?

It’s not just about the time it takes to read the information

Human brains process visual information up to 60,000 times faster than information conveyed through text
Practical applications

- Mechanics is the area of maths which lends itself best to practical demonstrations and investigations.
- Where possible use this to your advantage in your teaching.
- Numerous pre-existing resources for practical activities or demos that can be used in class.
Specific areas of content

Three chosen topic areas for detailed support

• Kinematics with calculus (AS)
• Projectiles (A Level)
• Ladder problems (A Level)
Poll 2

Use the poll to select your **top two** priorities out of the three areas below

- Kinematics with calculus (AS)
- Projectiles (A Level)
- Ladder problems (A Level)
Projectiles
Projectiles

**SAMs question**

A boy throws a stone with speed $U \text{ m s}^{-1}$ from a point $O$ at the top of a vertical cliff. The point $O$ is 18 m above sea level.

The stone is thrown at an angle $\alpha$ above the horizontal, where $\tan \alpha = \frac{3}{4}$.

The stone hits the sea at the point $S$ which is at a horizontal distance of 36 m from the foot of the cliff, as shown in Figure 2.
The stone is modelled as a particle moving freely under gravity with $g = 10 \text{ m s}^{-2}$

Find

(a) the value of $U$, 

(b) the speed of the stone when it is 10.8 m above sea level, giving your answer to 2 significant figures.

(c) Suggest two improvements that could be made to the model.
Projectiles

SAMs question

Figure 2
Projectiles

Prior knowledge

GCSE (9-1) in Mathematics at Higher Tier

G20  Trigonometry

AS Mathematics – Mechanics content

7.3  suvat formulae (See Unit 7b of the SoW)
8.3  Vertical motion under gravity (See Unit 7b of the SoW)
8.2  i, j (2D) vectors (See Unit 8a of the SoW)

AS Mathematics – Pure Mathematics content

10.1  i, j (2D) vectors (See Unit 5 of the SoW)
10.2  Magnitude and direction of a vector
5  Trigonometry (See Unit 4 of the SoW)

A level Mathematics – Pure Mathematics content

5.5  sec² x = 1 + tan² x identity and solving trigonometric equations (See Unit 6 of the SoW)
Projectiles

Starter ideas – introduce the topic and get them interested...

• Throw something around!

• Discuss applications in sports – make it relevant to the interests of your group

• Look at a simulation of projectile motion

https://phet.colorado.edu/en/simulation/projectile-motion

• Watch a video about projectile motion – which falls faster? Show clip here
PROJECTIVES

By the end of the sub-unit, students should:

- be able to find the time of flight of a projectile;
- be able to find the range and maximum height of a projectile;
- be able to derive formulae to find the greatest height, the time of flight and the horizontal range (for a full trajectory);
- know how to modify projectile equations to take account of the height of release;
- be able to derive and use the equation of the path.
Projectiles

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

• Students often find projectile questions challenging, sometimes confusing the horizontal and vertical aspects of the motion, for example by including the horizontal component of velocity in an equation for the vertical motion.

• Other common mistakes include considering only one component of velocity when finding speeds and making sign errors when producing quadratic equations (to find $t$).
Projectiles

**KEY POINTS**

- Projectiles and parabolas (including symmetry)
- Modelling assumptions
- Displacement, velocity and acceleration are vectors with components in the horizontal and vertical directions
- Treat horizontal and vertical components separately
- Each of the components obey the suvat formulae
- \( t \) will be a common term in horizontal and vertical components
Projectiles

Horizontal projection

For horizontal projections...

\begin{align*}
\rightarrow + \hspace{1cm} \downarrow + \\
\text{s}= & \\
\text{u}= & U \\
\text{v}= & U \\
\text{a}= & 0 \\
\text{t}= & \\
\text{s}= & \\
\text{u}= & 0 \\
\text{v}= & \\
\text{a}= & 9.8 \\
\text{t}= &
\end{align*}
Projectiles

Horizontal projection

Example:

A pebble is projected horizontally from a platform which is 20m above ground level. It travels 28m horizontally before hitting the ground. Find

a) The time in the air

b) The initial speed of the stone
Projectiles

**Horizontal projection**

*Example:*

A pebble is projected horizontally from a platform which is 20m above ground level. It travels 28m horizontally before hitting the ground. Find

a) The time in the air

b) The initial speed of the stone

\[ s = 28 \]

\[ u = U \]

\[ v = U \]

\[ a = 0 \]

\[ t = \]

For the vertical motion:

\[ s = ut + \frac{1}{2} a t^2 \]

\[ 20 = 0 + \frac{1}{2} (9.8) t^2 \]

\[ t^2 = 4.08163 \ldots \]

\[ t = 2.02030 \ldots \]

\[ = 2.0 \text{ s (2 s.f.)} \]
Projectiles

Horizontal projection

Example:

A pebble is projected horizontally from a platform which is 20m above ground level. It travels 28m horizontally before hitting the ground. Find

a) The time in the air
b) The initial speed of the stone

\[
\begin{align*}
\text{For } & \rightarrow + \\
\text{For } & \downarrow + \\
\text{u} & = 0 \\
\text{v} & = U \\
\text{a} & = 9.8 \\
\text{t} & = 2.0 \\
\text{s} & = 20 \\
\text{u} & = U \\
\text{a} & = 0 \\
\text{t} & = 2.0 \\
\text{s} & = 28 \\
\text{u} & = U \\
\text{a} & = 0 \\
\text{t} & = 2.0 \\
\text{s} & = 28 \\
\text{u} & = U \\
\text{a} & = 0 \\
\text{t} & = 2.0 \\
\end{align*}
\]

\[
\begin{align*}
28 & = ut + \frac{1}{2} at^2 \\
28 & = U (2.0) \\
28 & = 14 \text{ms}^{-1} \quad (2 \text{s.f.})
\end{align*}
\]
Projectiles

Projection at any angle, using vectors

In general...

- Particles are projected with a velocity
  \[ u = (u_x \mathbf{i} + u_y \mathbf{j}) \text{ms}^{-1} \]
- \( \mathbf{i} \) and \( \mathbf{j} \) are unit vectors acting in a vertical plane, horizontally and vertically respectively
- Initial speed is the magnitude of the velocity vector, \( u \)
- When an initial velocity is given as a vector, the values of \( u_x \) and \( u_y \) are the horizontal and vertical components of the velocity
Projectiles

Projection at any angle, using vectors

Example:

A particle is projected with initial velocity \( \mathbf{u} = (5i + 8j) \text{ms}^{-1} \). Find the initial speed of the particle and its angle of projection.

\[
\begin{align*}
\text{Speed} &= |\mathbf{u}| = \sqrt{5^2 + 8^2} \\
&= \sqrt{89} \\
&= 9.433981 \ldots
\end{align*}
\]

\[
\tan \alpha = \frac{8}{5}
\]

\[
\alpha = 57.994616 \ldots
\]

Initial speed = \( \sqrt{89} \text{ms}^{-1} \)

Angle of projection = 59° above the horizontal
Projectiles

Projection at any angle

In general...

\[ s = \]
\[ u = U \cos \alpha \]
\[ v = U \cos \alpha \]
\[ a = 0 \]
\[ t = \]

\[ s = \]
\[ u = U \sin \alpha \]
\[ v = \]
\[ a = -9.8 \]
\[ t = \]
Projectiles

Projection at any angle

Video tutorial:

A particle is projected at an angle of elevation of 25 degrees with a speed of 20 m\(\text{s}^{-1}\) from a height of 10 m above the ground.

Find the time of flight, maximum height and range.

\[
\begin{array}{c|c|c}
\rightarrow + & \uparrow + \\
\hline
s = & s = \\
u = U \cos \alpha & u = U \sin \alpha \\
v = U \cos \alpha & v = \\
a = 0 & a = -9.8 \\
t = & t = 
\end{array}
\]

Source: Exam Solutions
A particle is projected at an angle of elevation of $25^\circ$ with a speed of $20\ \text{ms}^{-1}$ from a height of $10\text{m}$ above the ground. Find the time of flight, maximum height, and range.

Time of flight: $T$

Vertical motion:

$s = -10$
$u = 20\sin25^\circ = 8.452...$
$v = ?$
$a = -9.8$
$t = T$

Using $s = ut + \frac{1}{2}at^2$

Maximum height:

$s = h$
$u = 8.452...$
$v = 0$
$a = -9.8$
$t = ?$

Maximum height = $13.6\text{m}$ (1dp)

Range:

$R = (20\cos25^\circ)(2.531...)$

$= 45.87...$

$= 45.9\text{m}$ (1dp)
Projectiles

Projecting from a point above ground level

A boy throws a stone with speed $U \text{ m s}^{-1}$ from a point $O$ at the top of a vertical cliff. The point $O$ is 18 m above sea level.

The stone is thrown at an angle $\alpha$ above the horizontal, where $\tan \alpha = \frac{3}{4}$.

The stone hits the sea at the point $S$ which is at a horizontal distance of 36 m from the foot of the cliff, as shown in Figure 2.

The stone is modelled as a particle moving freely under gravity with $g = 10 \text{ m s}^{-2}$

Find

(a) the value of $U$, 

(b) the speed of the stone when it is 10.8 m above sea level, giving your answer to 2 significant figures. 

(c) Suggest two improvements that could be made to the model.
A boy throws a stone with speed $U \text{ms}^{-1}$ from a point $O$ at the top of a vertical cliff. The point $O$ is $18\text{ m}$ above sea level.

The stone is thrown at an angle $\alpha$ above the horizontal, where $\tan \alpha = \frac{3}{4}$. 

$$\cos \alpha = \frac{4}{5} \quad \sin \alpha = \frac{3}{5}$$
Projectiles

SAMs solution

Find

(a) the value of \( U \),
Projectiles

SAMs solution

Find

(a) the value of $U$,

\[ 45 = ut \quad 1 \quad 25 + 2 - 3ut - 90 = 0 \quad 2 \]

Solving simultaneously...

\[ 25t^2 - 3(45) - 90 = 0 \]
\[ 25t^2 - 225 = 0 \]
\[ t^2 - 9 = 0 \]
\[ t^2 = 9 \]
\[ t = \pm 3 \]

Reject $t = -3$, to get $t = 3 \text{ s}$

Sub into $1$...

\[ 45 = 3u \]
\[ u = 15 \text{ ms}^{-1} \]
(b) the **speed** of the stone when it is 10.8 m above sea level, giving your answer to 2 significant figures.

\[ v = \sqrt{u^2 + 2as} \]

\[ v = \sqrt{9^2 + 2(-10)(-7.2)} \]

\[ v = \pm 15 \]

At this point, the stone will be travelling **down**, so **reject** 15, and so

\[ v = -15 \text{ ms}^{-1} \]
(b) the speed of the stone when it is 10.8 m above sea level, giving your answer to 2 significant figures.

Need the speed, so combine the two components ...

\[ \sqrt{12^2 + 15^2} = \sqrt{144 + 225} = \sqrt{369} = 3\sqrt{41} = 19.20937271 \]

= 19 ms\(^{-1}\)
(c) Suggest two improvements that could be made to the model.

2. Using a more accurate value for \( g \), e.g. \( g = 9.8 \text{ ms}^{-2} \)

2. Include air resistance in the calculations
Projectiles

Deriving formulae

From the objectives:

be able to derive formulae to find the greatest height, the time of flight and the horizontal range (for a full trajectory);

- Ensure students are able to derive the equations
- Discourage them from memorising the equations to use in projectile problems as they are difficult and tedious to memorise correctly. Also they can only be used for a full trajectory.
Projectiles

Extension ideas

- Get students to try these problem solving activities from NRICH
  
  Dambusters 1: [http://nrich.maths.org/5837](http://nrich.maths.org/5837)
  
  Dambusters 2: [http://nrich.maths.org/5838](http://nrich.maths.org/5838)

- Questions on clearing an obstacle

- Practical activities (Mechanics in Action – pages 113-117, worksheets 28-30)
Kinematics
Prior Knowledge
GCSE (9-1) in Mathematics at Higher Tier

A14  Plot and interpret graphs (including reciprocal graphs and exponential graphs) and graphs of non-standard functions in real contexts to find approximate solutions to problems such as simple kinematics problems involving distance, speed and acceleration

A15  Calculate or estimate gradients of graphs and area under graphs (including quadratic and non-linear graphs), and interpret results in cases such as distance-time graphs, velocity-time graphs and graphs in financial contexts

AS Mathematics – Pure Mathematics content
Differentiation and integration of polynomials (including turning points and areas under curves)  [See Units 6 and 7]
Kinematics 2 (AS)  
(Variable Acceleration)

Prior Knowledge  
AS Mechanics: Kinematics 1

Draw and interpret kinematics graphs, knowing the significance (where appropriate) of their gradients and the areas underneath them.

Recognise when it is appropriate to use the ‘suvat’ formulae for constant acceleration.

Be able to solve kinematics problems using constant acceleration formulae.
Kinematics 2 (AS)
(Variable Acceleration)

Objectives

• Be able to use calculus (differentiation & integration) in kinematics to model motion in a straight line for a particle moving with variable acceleration or variable force.

• Understand that gradients of the relevant graphs link to rates of change.

• Know how to find max and min velocities by considering zero gradients and understand how this links with the actual motion (i.e. acceleration = 0)

• Understand that the area under a graph is the integral, which leads to a physical quantity.

• Know how to use initial conditions to calculate the constant of integration and refer back to the problem.
A bird leaves its nest at time \( t = 0 \) for a short flight along a straight line. The bird then returns to its nest.

The bird is modelled as a particle moving in a straight horizontal line.

The distance, \( s \) metres, of the bird from its nest at time \( t \) seconds is given by

\[
s = \frac{1}{10} (t^4 - 20t^3 + 100t^2), \quad \text{where } 0 \leq t \leq 10.
\]

(a) Explain the restriction \( 0 \leq t \leq 10 \).

\( 3 \) points

(b) Find the distance of the bird from the nest when the bird first comes to instantaneous rest.

\( 6 \) points
Starter

Consider the following graph representing the velocity $v$ (ms$^{-1}$) of a particle moving in a straight line under the action of a variable acceleration $v = 2t^2 + 2t + 1$ (for $t > 0$).
Gradient

• Group work to draw tangents ‘by hand’ or ICT investigation to find gradient of tangents drawn at \( t=1s, \) \( t=2s, \) etc.

• The unit of the gradient is \( \text{m/s}^2 \) or \( \text{ms}^{-2} \) (rate of change of velocity) is acceleration, \( a. \)

• Link with ‘\( dy/dx \)’ from Pure 1, to \( dv/dt \) (using axes labels)

• If you asked students to construct tangents, then take an average and compare with the calculus value.

• Hence \( dv/dt = a \)
Kinematics 2 (AS)

\[ a = \frac{dv}{dt} \]

\[ s = f(t) \]

\[ v = \frac{ds}{dt} \]

\[ a = \frac{dv}{dt} \]

\[ v = \text{constant} \]

\[ v = \frac{ds}{dt} \]
Kinematics 2 (AS)

Area

• Group work to find approximate area under the curve between t=0s and t=4s (GCSE (9-1))

• The unit of the area under the curve is ‘m’ is the displacement (distance) covered by the particle in the first 4 seconds.

• Link with Integration from Pure 1 to find the exact area under the curve.

• Hence \( \int v \, dt = s \)
Kinematics 2 (AS)

\[ a = \frac{dv}{dt} = 0 \]

\[ v = \frac{ds}{dt} \]

\[ s = \int v \, dt \]

\[ v = \int a \, dt \]

Note: \[ a = \frac{d}{dt} \left( \frac{ds}{dt} \right) = \frac{d^2s}{dt^2} \]
The acceleration of a particle $P$ moving in a straight line is $(t^2 - 9t + 18) \text{ms}^{-2}$, where $t$ is the time in seconds.

(i) Find the values of $t$ for which the acceleration is zero. [2]

(ii) It is given that when $t = 3$ the velocity of $P$ is $9 \text{ms}^{-1}$. Find the velocity of $P$ when $t = 0$. [4]

(iii) Show that the direction of motion of $P$ changes before $t = 1$. [2]
The acceleration of a particle $P$ moving in a straight line is $(t^2 - 9t + 18)\text{ms}^{-2}$, where $t$ is the time in seconds.

(i) Find the values of $t$ for which the acceleration is zero. [2]

(ii) It is given that when $t = 3$ the velocity of $P$ is $9\text{ms}^{-1}$. Find the velocity of $P$ when $t = 0$. [4]

(iii) Show that the direction of motion of $P$ changes before $t = 1$. [2]

\[
\begin{align*}
(1) \ & a = t^2 - 9t + 18 \\
\text{when } a = 0 & \implies t^2 - 9t + 18 = 0 \\
&(t-3)(t-6) = 0 \\
& t-3=0 \text{ or } t-6=0 \\
& t = 3 \text{ or } t = 6 \\
\text{ii) } v &= \int (t^2 - 9t + 18) \, dt \\
& = \frac{t^3}{3} - \frac{9t^2}{2} + 18t + C \\
& \text{when } t = 3, \quad v = 9 \\
& \text{Sub. in (1)} \\
& 9 = \frac{3^3}{3} - \frac{9(3)^2}{2} + 18(3) + C \\
& \therefore 9 = 22.5 + C \implies C = -13.5 \\
& \therefore v = \frac{t^3}{3} - \frac{9t^2}{2} + 18t - 13.5 \\
& \text{when } t = 0 \\
& \therefore v = -13.5 \text{ms}^{-1} \\
& \text{iii) when } t = 1 \\
& \therefore v = \frac{1}{3} - \frac{9}{2} + 18 - 13.5 \\
& = \frac{1}{3} \text{ms}^{-1} \\
& \text{Direction has changed} \\
& \text{gone from } - \text{ to } +
Summary - Variable Acceleration

The following diagram may help students decide whether to differentiate or integrate to solve a problem. ‘d’ for the down arrow means ‘differentiate’. Hence, down from ‘s’ gives ‘v’ or \( \frac{ds}{dt} = v \).

Integration is the opposite of differentiation so up is integrate, so up from ‘a’ gives ‘v’ or integral of ‘a’ with respect to \( t \) gives ‘v’

\[
\begin{align*}
\downarrow s \\
\text{diff} \quad \downarrow v \uparrow \text{int} \\
\quad a \uparrow
\end{align*}
\]

\[
v = \frac{ds}{dt}, \quad a = \frac{dv}{dt} = \frac{d^2s}{dt^2} \quad \text{and} \quad \text{and} \\
s = \int v \, dt, \quad v = \int a \, dt
\]
A bird leaves its nest at time $t = 0$ for a short flight along a straight line. The bird then returns to its nest.

The bird is modelled as a particle moving in a straight horizontal line.

The distance, $s$ metres, of the bird from its nest at time $t$ seconds is given by

$$s = \frac{1}{10} (t^4 - 20t^3 + 100t^2), \text{ where } 0 \leq t \leq 10.$$ 

(a) Explain the restriction $0 \leq t \leq 10.$

(3)

(b) Find the distance of the bird from the nest when the bird first comes to instantaneous rest.

(6)
Kinematics 2 (AS)
8. A bird leaves its nest at time \( t = 0 \) for a short flight along a straight line. The bird then returns to its nest. The bird is modelled as a particle moving in a straight horizontal line. The distance, \( s \) metres, of the bird from its nest at time \( t \) seconds is given by

\[
s = \frac{1}{10} (t^4 - 20t^3 + 100t^2), \quad \text{where } 0 \leq t \leq 10
\]

(a) Explain the restriction, \( 0 \leq t \leq 10 \)

\[(a) \quad \text{Substitute } t = 0 \text{ into } \frac{8}{10} \text{ equation.}
\[
\frac{s}{10} = \frac{1}{10} (0^4 - 20 \times 0^3 + 100 \times 0^2) = 0 \text{ m}
\]

\[
t = 10 \text{ gives } \frac{s}{10} = \frac{1}{10} (10^4 - 20 \times 10^3 + 100 \times 10^2)
\]

\[
\frac{s}{10} = \frac{1}{10} (10000 - 20000 + 10000)
\]

\[
\frac{s}{10} = 0 \text{ m}
\]

So the bird is at or returns to the nest at \( t = 0 \) and \( t = 10 \) s respectively.

Also, factorising \( s \) gives

\[
s = \frac{1}{10} t^2 (t^2 - 20t + 100)
\]

\[
s = \frac{1}{10} t^2 (t - 10)^2 \quad \text{since all terms are squared.}
\]

\[
\text{Since } \frac{s}{10} > 0 \text{ between } 0 \text{ and } 10 \text{ (Always positive).}
\]

[A] After \( t = 10 \) s, the bird leaves the nest again.]
(b) Find the distance of the bird from the nest when the bird first comes to instantaneous rest.

\[ V = \frac{dy}{dt} = \frac{1}{10} (kt^3 - 60t^2 + 200t) \text{ ms}^{-1} \]

At rest, \( V = 0 \)

\[ \therefore \frac{1}{10} (kt^3 - 60t^2 + 200t) = 0 \]

Factorise \( \frac{1}{10} kt (t - 10) (t - 5) = 0 \)

\[ \frac{2}{5} t (t - 10) (t - 5) = 0 \]

\[ k = 0 \quad t = 10 \quad t = 5 \]

Back to nest

\( t = 0 \) is when the bird was at rest (initially)

\( t = 5 \) was when the bird was first at rest

\[ s = \frac{1}{10} (5^2)(5-10)^2 = 62.5 \text{ m} \]

On sub \( t = 5 \) into original \( s \) equation
<table>
<thead>
<tr>
<th>Qn</th>
<th>Scheme</th>
<th>Marks</th>
</tr>
</thead>
</table>
| 8. (a) | Substitution of both $t = 0$ and $t = 10$
$s = 0$ for both $t = 0$ and $t = 10$
Explanation ($s > 0$ for $0 < t < 10$) since $s = \frac{1}{10}t^2(t - 10)^2$                                                                 | M1    |
|     |                                                                                                                                                                                                       | A1    |
|     |                                                                                                                                                                                                       | A1 (3)|
| (b) | Differentiate displacement $s$ with respect to $t$ to give velocity, $v$
$v = \frac{1}{10}(4t^3 - 60t^2 + 200t)$
Interpretation of ‘rest’ to give $v = \frac{1}{10}(4t^3 - 60t^2 + 200t) = \frac{2}{3}t(t - 5)(t - 10) = 0$
t = 0, 5, 10
Select $t = 5$ and substitute into $s$
Distance = 62.5 m                                                                                           | M1    |
|     |                                                                                                                                                                                                       | A1    |
|     |                                                                                                                                                                                                       | M1    |
|     |                                                                                                                                                                                                       | A1    |
|     |                                                                                                                                                                                                       | M1    |
|     |                                                                                                                                                                                                       | A1 ft (6) |
|     |                                                                                                                                                                                                       | (9 marks) |
Deriving the constant acceleration equations via calculus

Starting with constant acceleration $a$, students can derive the earlier (suvat) equations of uniform motion.

So, $v = \int a \, dt$ gives $v = at + c$, but $v = u$ when $t = 0$. Therefore $v = u + at$

Also, $s = \int v \, dt = \int (u + at) \, dt = ut + \frac{1}{2}at^2 + k$
Stress the constant of integration, $s_0$ (i.e. $s$ when $t = 0$)
giving $s = ut + \frac{1}{2}at^2 + s_0$. 
PRIOR KNOWLEDGE

Basic trigonometry, Pythagoras and vectors
Find the magnitude and direction of vectors

AS Mathematics – Mechanics content
Kinematics 1 and equations of motion (Unit 7b)
Kinematics 2 (variable force) (Unit 9)

AS Mathematics – Pure Mathematics content
2D vectors – \( \mathbf{i}, \mathbf{j} \) system (Unit 5)
Topic 1b: Further Kinematics (AL) (2 Dimensions)

Objectives

- Be able to extend techniques for motion in 1 dimension to 2 dimensions by using calculus and vector versions of equations for variable force/acceleration problems.

- Understand the language and notation of kinematics appropriate to variable motion in 2 dimensions, i.e. knowing the notation \( \dot{r} \) and \( \ddot{r} \) for variable acceleration in terms of time.
Further Kinematics (AL) (2 Dimensions)

This topic links directly to, and is an extension of AS Mathematics – Motions can now be more complicated as the forces in the \( \mathbf{i} \) and \( \mathbf{j} \) directions can differ and be variable (i.e. \( \mathbf{F} = m\mathbf{a} \)). Also the notation for 2D motion replaces the displacement, \( s \), with position vector, \( \mathbf{r} \). Velocity, \( \mathbf{v} \), can be defined as \( \dot{\mathbf{r}} \) and the acceleration vector can be called \( \ddot{\mathbf{r}} \) (rather than \( \mathbf{a} \)).

Introduce this notation to students, explaining how the dot above the \( \mathbf{r} \) denotes how many times the \( \mathbf{r} \) has been differentiated with respect to time. Hence \( \ddot{\mathbf{r}} \) (representing the acceleration) effectively means \( \mathbf{r} \) differentiated twice with respect to time or \( \frac{d^2\mathbf{r}}{dt^2} \).
\[ \mathbf{r} = f(t) \mathbf{i} + g(t) \mathbf{j}, \quad \mathbf{v} = \frac{d\mathbf{r}}{dt}, \quad \mathbf{a} = \frac{d\mathbf{v}}{dt} \]

\[ \mathbf{r} = t^2 \mathbf{i} + \frac{t^3}{3} \mathbf{j} \]

\[ \mathbf{v} = \frac{d\mathbf{r}}{dt} = 2t \mathbf{i} + t^2 \mathbf{j} \]

\[ \mathbf{a} = \frac{d\mathbf{v}}{dt} = 2 \mathbf{i} + 2t \mathbf{j} \]

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Further Kinematics AL
(2 Dimensions)

The other vital point to stress is when we integrate $\dot{r}$ (or $\mathbf{v}$) to obtain the displacement $\mathbf{r}$, we have to introduce a vector constant of integration in the form $c\mathbf{i} + k\mathbf{j}$ or $\mathbf{D}$ (rather than just $+c$).

Any conditions provided in the question (e.g. the particle is initially at the point with position vector $(3\mathbf{i} + 2\mathbf{j})$ m) allow us to substitute into the expression for $\mathbf{r}$ and calculate the constants.
Further Kinematics (AL) (2 Dimensions)

Worked Example
Tutorial Clip 2

If \( \mathbf{r} = f(t)\mathbf{i} + g(t)\mathbf{j} \) then \( \mathbf{v} = \frac{d\mathbf{r}}{dt} \) and \( \mathbf{a} = \frac{d\mathbf{v}}{dt} \)

So \( \mathbf{v} = \int \mathbf{a} \, dt \) and \( \mathbf{r} = \int \mathbf{v} \, dt \)

A particle \( P \) is moving in a plane so that at time \( t \) seconds, its acceleration \( \mathbf{a} = (6t\mathbf{i} + 2\mathbf{j}) \text{ ms}^{-2} \). Find the position vector \( \mathbf{r} \), after \( t = 3 \text{ s} \) given that when \( t = 1 \text{ s} \), \( \mathbf{v} = (4\mathbf{i} - \mathbf{j}) \text{ ms}^{-1} \) and \( \mathbf{r} = (2\mathbf{i} + 3\mathbf{j}) \text{ m} \).
A particle $P$ is moving in a plane so that at time $t$ seconds, its acceleration $a = (6t\mathbf{i} + 2\mathbf{j})$ ms$^{-2}$. Find the position vector $\mathbf{r}$, after $t = 3$ s given that when $t = 1$ s, $\mathbf{v} = (4\mathbf{i} - \mathbf{j})$ ms$^{-1}$ and $\mathbf{r} = (2\mathbf{i} + 3\mathbf{j})$ m.

\[
\mathbf{v} = \int (6t\mathbf{i} + 2\mathbf{j}) dt
\]

\[
= 3t^2\mathbf{i} + 2t\mathbf{j} + c\tag{1}
\]

when $t = 1$, $\mathbf{v} = 4\mathbf{i} - \mathbf{j}$

Sub. in (1)

\[
4\mathbf{i} - \mathbf{j} = 3\mathbf{i} + 2\mathbf{j} + c
\]

\[
\therefore c = \mathbf{i} - 3\mathbf{j}
\]

\[
\mathbf{r} = \int \left[(3t^2 + 1)\mathbf{i} + (2t - 3)\mathbf{j}\right] dt
\]

\[
= (t^3 + t)\mathbf{i} + (t^2 - 3t)\mathbf{j} + d\tag{2}
\]

when $t = 1$, $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j}$
A particle $P$ is moving in a plane so that at time $t$ seconds, its acceleration $a = (6t\mathbf{i} + 2\mathbf{j})$ ms$^{-2}$. Find the position vector $\mathbf{r}$, after $t = 3$ s given that when $t = 1$ s, $\mathbf{v} = (4\mathbf{i} - \mathbf{j})$ ms$^{-1}$ and $\mathbf{r} = (2\mathbf{i} + 3\mathbf{j})$ m.

\[ \mathbf{v} = \int (6t\mathbf{i} + 2\mathbf{j}) dt \]
\[ = 3t^2\mathbf{i} + 2t\mathbf{j} + \mathbf{c} \quad \text{(1)} \]

When $t = 1$, $\mathbf{v} = 4\mathbf{i} - \mathbf{j}$.

Sub. in (1)
\[ 4\mathbf{i} - \mathbf{j} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{c} \]
\[ \therefore \mathbf{c} = \mathbf{i} - 3\mathbf{j} \quad \text{(1)} \]

So
\[ \mathbf{v} = \int [(3t^2+1)\mathbf{i} + (2t-3)\mathbf{j}] dt \]
\[ = (t^3 + t)\mathbf{i} + (t^2 - 3t)\mathbf{j} + \mathbf{D} \quad \text{(2)} \]

When $t = 1$, $\mathbf{v} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{D}$
\[ \therefore \mathbf{D} = 5\mathbf{j} \quad \text{sub. in (2)} \]

So
\[ \mathbf{v} = (t^3 + t)\mathbf{i} + (t^2 - 3t + 5)\mathbf{j} \]

When $t = 3$
\[ \mathbf{r} = (30\mathbf{i} + 5\mathbf{j}) \text{ m} \]
Further Kinematics (AL)

Summary - 2 Dimensions

The following diagram may help students decide whether to differentiate or integrate to solve a problem.

‘d’ for the down arrow means ‘differentiate’. Hence, down from \( r \) gives \( v \) or \( \dot{r} \) or \( \frac{dr}{dt} = v \).

Integration is the opposite of differentiation so up is integrate. Up from \( a(\ddot{r}) \) gives \( v(\dot{r}) \) or integral of \( a(\ddot{r}) \) with respect to \( t \) gives \( v(\dot{r}) \).

\[ v = \frac{dr}{dt} \quad \text{and} \quad a = \frac{dv}{dt} \]

\[ v = \int a \, dt \quad \text{and} \quad r = \int v \, dt \]
Ladder problems
Prior Knowledge

AS Mathematics – Mechanics content

• Basic equilibrium (See Unit 8)
• Types of forces and force diagrams
• Assumptions made throughout this course (e.g. particle, rigid, light)
• S.I. units

A-Level Mathematics – Mechanics content

• Moments (See Unit 4)
• Resolving forces (See Unit 5)
• Frictional forces (see Unit 5)
Objectives

• Be able to solve statics problems for a system of forces which are not concurrent (e.g. ladder problems), thus applying the principle of moments for forces at any angle.

• Using friction forces to model a ladder leaning against a rough ground or wall
A uniform ladder $AB$, of length $2a$ and weight $W$, has its end $A$ on rough horizontal ground. The coefficient of friction between the ladder and the ground is $1/4$. The end $B$ of the ladder is resting against a smooth vertical wall, as shown in Figure 1 (next slide).

A builder of weight $7W$ stands at the top of the ladder. To stop the ladder from slipping, the builder’s assistant applies a horizontal force of magnitude $P$ to the ladder at $A$, towards the wall. The force acts in a direction which is perpendicular to the wall.

The ladder rests in equilibrium in a vertical plane perpendicular to the wall and makes an angle $\alpha$ with the horizontal ground, where $\tan \alpha = 5/2$. The builder is modelled as a particle and the ladder is modelled as a uniform rod.
(a) Show that the reaction of the wall on the ladder at $B$ has magnitude $3W$. 

(b) Find, in terms of $W$, the range of possible values of $P$ for which the ladder remains in equilibrium.

(c) Explain briefly how this helps to stop the ladder from slipping.
Show a bicycle pedal in different positions and discuss which one makes turning easier. (See diagrams below.)

A discussion around this can lead to the understanding that the moment of a force, is a measure of its turning effect and is given by the formula:

moment of a force about a point = force \( (F) \) \( \times \) perpendicular distance from the point to the line of action of the force \( (d) \) (the unit is newton metres, N m)

Ask students questions such as: How do we work out the distance, \( d \), in the second bicycle pedal diagram? What additional information do we need? What if the pedal was at the topmost point, vertically above the axle?
Ladder-type problems which will revise and extend our idea of moments. So far, the bars have been horizontal (or at the point of tilting). We now extend to any angle and the forces will not be concurrent. Extend the moments formula to ‘perpendicular force × distance’ and resolve the force to find its component at right angles to the full distance from the moments point.
Show students how to use the alternative formula ‘force × perpendicular distance’, by measuring the perpendicular distance from the moments point to the line of action of the force.

Also make sure that students are clear about the directions of the frictional force (for examples involving rough surfaces) and the reactions at the wall and ground being labelled differently.
A uniform ladder of length 2l and mass 20 kg rests in equilibrium with its base on a rough horizontal floor and its top against a smooth vertical wall. The ladder is inclined at 60° to the horizontal. If the ladder is on the point of slipping and the coefficient of friction is \( \mu \). Find the value of \( \mu \), the reactions at the floor and wall.
A uniform ladder of length $2l$ and mass 20 kg rests in equilibrium with its base on a rough horizontal floor and its top against a smooth vertical wall. The ladder is inclined at 60° to the horizontal. If the ladder is on the point of slipping and the coefficient of friction is $\mu$. Find the value of $\mu$, the reactions at the floor and wall.
A uniform ladder of length $2l$ and mass $20 \text{ kg}$ rests in equilibrium with its base on a rough horizontal floor and its top against a smooth vertical wall. The ladder is inclined at $60^\circ$ to the horizontal. If the ladder is on the point of slipping and the coefficient of friction is $\mu$. Find the value of $\mu$, the reactions at the floor and wall.

\[ R(\uparrow): R_1 - 20g = 0 \Rightarrow R_1 = 20g \Rightarrow R_1 = 196 \text{ N} \]

\[ R(\rightarrow): R_2 - \mu R_1 = 0 \Rightarrow R_2 = \mu R_1 \Rightarrow R_2 = 196\mu \quad (1) \]

\[ (M(A)): (R_2 \sin 60^\circ)(2l) - (20g \cos 60^\circ)(l) = 0 \]

\[ 2R_2 \tan 60^\circ - 20g = 0 \Rightarrow R_2 = \frac{20g}{2\tan 60^\circ} \]

\[ \therefore R_2 = 56.580... = 56.6 \text{ N} \quad (1dp) \quad (2) \]

\[ \text{from (1) sub in (2)} \quad \therefore 196\mu = 56.580... \]

\[ \therefore \mu = 0.288... \]

\[ \therefore \mu = 0.3 \quad (1dp) \]
A uniform ladder $AB$, of length $2a$ and weight $W$, has its end $A$ on rough horizontal ground. The coefficient of friction between the ladder and the ground is $1/4$. The end $B$ of the ladder is resting against a smooth vertical wall, as shown in Figure 1 (next slide).

A builder of weight $7W$ stands at the top of the ladder. To stop the ladder from slipping, the builder’s assistant applies a horizontal force of magnitude $P$ to the ladder at $A$, towards the wall. The force acts in a direction which is perpendicular to the wall.

The ladder rests in equilibrium in a vertical plane perpendicular to the wall and makes an angle $\alpha$ with the horizontal ground, where $\tan \alpha = 5/2$.

The builder is modelled as a particle and the ladder is modelled as a uniform rod.
Figure 1

\[ \mu = \frac{1}{4} \]

\[ \tan \alpha = \frac{5}{2} \]

(a) Show that the reaction of the wall on the ladder at \( B \) has magnitude \( 3W \).

(b) Find, in terms of \( W \), the range of possible values of \( P \) for which the ladder remains in equilibrium.

Often in practice, the builder’s assistant will simply stand on the bottom of the ladder.

(c) Explain briefly how this helps to stop the ladder from slipping.
A uniform ladder $AB$, of length $2a$ and weight $W$, has its end $A$ on rough horizontal ground. The coefficient of friction between the ladder and the ground is $\frac{1}{4}$. The end $B$ of the ladder is resting against a smooth vertical wall, as shown in Figure 1.

A builder of weight $7W$ stands at the top of the ladder. To stop the ladder from slipping, the builder’s assistant applies a horizontal force of magnitude $P$ to the ladder at $A$, towards the wall. The force acts in a direction which is perpendicular to the wall.

The ladder rests in equilibrium in a vertical plane perpendicular to the wall and makes an angle $\alpha$ with the horizontal ground, where $\tan \alpha = \frac{5}{2}$. The builder is modelled as a particle and the ladder is modelled as a uniform rod.
(a) Show that the reaction of the wall on the ladder at B has magnitude 3W.

\[(a) \quad M(A) :\]

\[a \times W \cos \alpha + 2a \times 7W \cos \alpha = 2a (R_2 \sin \alpha)\]

\[15W \cos \alpha = 2R_2 \sin \alpha\]

\[15W \left(\frac{2}{\sqrt{29}}\right) = 2R_2 \left(\frac{5}{\sqrt{29}}\right)\]

\[15W \times 2 = 2R_2 \times 5 \quad \Rightarrow \quad 30W = 10R_2\]

\[R_2 = 3W \quad \text{QED}\]

Reaction at wall is 3W.
(b) Find, in terms of \( W \), the range of possible values of \( P \) for which the ladder remains in equilibrium.

\[
(6) \quad R(\uparrow) = R_1 - 7W - W = 0 \\
\quad R_1 = 8W
\]

\[
F_{\text{max}} = NR_1 = \frac{1}{4}R_1 \quad \text{(see diagram)}
\]

\[
F_{\text{max}} = \frac{1}{4}(8W) = 2W
\]

If the ladder remains in equilibrium, it could be at the point of sliding forwards for a max value of \( P \) (i.e. the mass and it will move forward) or it could be at the point of sliding backwards for a min value of \( P \) (i.e. just enough to prevent it from moving back).

\[
\therefore \quad P_{\text{min}} = 3W - F_{\text{max}} \quad \text{(Friction opposes potential motion)}
\]

\[
P_{\text{min}} = 3W - 2W = W
\]

\[
F_{\text{max}} = 3W + F_{\text{max}} = 3W + 2W = 5W
\]

\[
\therefore \quad W \leq P \leq 5W \quad \text{(Equilibrium Range)}
\]
Often in practice, the builder’s assistant will simply stand on the bottom of the ladder.

(c) Explain briefly how this helps to stop the ladder from slipping.

\[ R_1 - W - 7w - x = 0 \quad R_1 = 8w + x \]

1. The reaction increases, then \( F_{max} \) increases as \( F_{max} = \frac{1}{4} R_1 \) (i.e. Reaction at A increases)

2. If we take moments about A, the additional force at A has no effect on \( R_2 \) (Reaction at Wall)
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<th>Question</th>
<th>Scheme</th>
<th>Marks</th>
<th>AOs</th>
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<td>9(a)</td>
<td>Take moments about $A$ (or any other complete method to produce an equation in $S$, $W$ and $\alpha$ only)</td>
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<td></td>
<td>$W \cos \alpha + 7W \cos \alpha = S \cdot 2a \sin \alpha$</td>
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<td>Use of $\tan \alpha = \frac{5}{2}$ to obtain $S$</td>
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<td>$S = 3W$ *</td>
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<td>(b)</td>
<td>$R = 8W$</td>
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<td>$F = \frac{1}{4} R (= 2W)$</td>
<td>M1</td>
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<td>$P_{\text{MAX}} = 3W + F$ or $P_{\text{MIN}} = 3W - F$</td>
<td>M1</td>
<td>3.4</td>
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<td>$P_{\text{MAX}} = 5W$ or $P_{\text{MIN}} = W$</td>
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<td>$W \leq P \leq 5W$</td>
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<td>(c)</td>
<td>$M(A)$ shows that the reaction on the ladder at $B$ is unchanged</td>
<td>M1</td>
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<td>also $R$ increases (resolving vertically)</td>
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<td>2.4</td>
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<td>which increases max $F$ available</td>
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<td>2.4</td>
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(13 marks)
COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students are often good at drawing force diagrams, but common errors are omitting arrowheads, incorrectly labelling (e.g. 4 kg rather than 4g) and missing off the normal reaction or friction forces.

Students can sometimes struggle to work out the direction of the frictional force.

Common errors in questions involving moments are ignored the weight of the ladder, sine/cosine confusion and missing a distance in one or more terms.
Another Application

A uniform beam of mass 10kg and length 2.4m rests with one end on the ground and is supported at an angle of 20° to the horizontal by a light rope attached three-quarters of the way up it. The rope is at right-angles to the beam. Find the tension in the rope.
A uniform beam of mass 10kg and length 2.4m rests with one end on the ground and is supported at an angle of 20° to the horizontal by a light rope attached three-quarters of the way up it. The rope is at right-angles to the beam. Find the tension in the rope.

Take moments about O to get an equation without X & Y.

Resolve W parallel & perpendicular to the beam to make this easier.
If the force $W$ acting at the centre of mass (which is where it acts on the beam) is replaced by its 2 components, the moments are simplified as $W\sin\theta$ acts through $O$ and $W\cos\theta$ acts at right angles to $O$. 
Total moment of $W$ about $O = W \cos 20 \times 1.2 + W \sin 20 \times 0$

$= W \cos 20 \times 1.2$
When taking moments about O the force diagram effectively reduces to 

Anticlockwise moment = Clockwise moment 

\[ W \cos 20 \times 1.2 = T \times 1.8 \]

\[ T = 10g \cos 20 \times 1.2 \]

\[ T = \frac{10g \cos 20 \times 1.2}{1.8} \]

\[ T = 61.4N \]
Ladder Problems (AL)

Extension:- Hinged Problem

Consider a uniform rod which has one end freely hinged to a wall and the other end tied to a point above the wall, making the bar horizontal.

Discuss the fact that the reaction at the hinge is not perpendicular to the wall and that the lines of actions of all the forces in the system will all meet at one point for equilibrium.

Representing the reaction at the hinge as two perpendicular forces, XN & YN (like the example we have just seen with the ‘Trap Door’), the ‘resolving and taking moments’ solution would be fairly straightforward.
Free support

- **Plan:**
  schemes of work and course planners to help you deliver the qualifications in the best way for your centre
  content mapping documents
  Getting started guide

- **Teach:**
  topic-based resources to use in the classroom, particularly for the new and unfamiliar topics
  content exemplification

- **Track and Assess:**
  specimen papers
  secure mock papers
  practice papers
  assessment guide
  exemplar solutions
  Exam Wizard & Results Plus

- **Develop:**
  a full programme of launch and training events – live and online
  pre-recorded getting ready to teach sessions
  our collaborative network events
  the famous Mathematics Emporium, led by Graham Cumming
Thank you for your attention