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GCE in Further Mathematics (8FM0)
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About this booklet

This booklet has been produced to support mathematics teachers delivering the new Pearson Edexcel Level 3 Advanced Subsidiary and Advanced GCE in Further Mathematics (8FM0 & 9FM0) (first assessment summer 2018/2019).

The booklet provides additional information on all the content from the specification accredited by Ofqual in 2017. It details the content references and provides sample questions for each content topic taken from the Sample Assessment Materials.
1. Proof

Structure and calculation

1.1 Construct proofs using mathematical induction.
Contexts include sums of series, divisibility and powers of matrices.

Sample questions

AS SAMs Paper 1 Q6 (1.1, 4.3)
(a) Prove by induction that for all positive integers \( n \),

\[
\sum_{r=1}^{n} r^2 = \frac{1}{6} n(n + 1)(2n + 1)
\]

(b) Use the standard results for \( \sum_{r=1}^{n} r^3 \) and \( \sum_{r=1}^{n} r \) to show that for all positive integers \( n \),

\[
\sum_{r=1}^{n} r^2 = r(r + 6)(r - 6) = \frac{1}{4} n(n + 1)(n - 8)(n + 9)
\]

(c) Hence find the value of \( n \) that satisfies

\[
\sum_{r=1}^{n} r(r + 6)(r - 6) = 17 \sum_{r=1}^{n} r^2
\]

(Total for Question 6 is 15 marks)

A level SAMs Paper 1 Q2 (1.1)
Prove by induction that, for all positive integers \( n \),

\[ f(n) = 2^{3n} + 1 + 3(5^{2n} + 1) \]

is divisible by 17.

(Total for Question 2 is 6 marks)
1. Proof

A level SAMs Paper 2 Q3 (1.1, 3.1, 3.5, 3.6)

(i) \[
M = \begin{pmatrix}
2 & a & 4 \\
1 & -1 & -1 \\
-1 & 2 & -1 \\
\end{pmatrix}
\]

where \(a\) is a constant.

(a) For which values of \(a\) does the matrix \(M\) have an inverse? \(2\) marks

Given that \(M\) is non-singular,

(b) find \(M^{-1}\) in terms of \(a\) \(4\) marks

(ii) Prove by induction that for all positive integers \(n\),

\[
\begin{pmatrix}
3 & 0 \\
6 & 1 \\
\end{pmatrix}^n = \begin{pmatrix}
3^n & 0 \\
3(3^n) & 1 \\
\end{pmatrix}
\]

\(6\) marks

(Total for Question 3 is 12 marks)
2. Complex numbers

2. Complex numbers

Structure and calculation

2.1 Solve any quadratic equation with real coefficients.
   Solve cubic or quartic equations with real coefficients.

2.2 Add, subtract, multiply and divide complex numbers in the form \(x + iy\) with \(x\) and \(y\) real.
   Understand and use the terms ‘real part’ and ‘imaginary part’.

2.3 Understand and use the complex conjugate.
   Know that non-real roots of polynomial equations with real coefficients occur in conjugate pairs.

2.4 Use and interpret Argand diagrams.

2.5 Convert between the Cartesian form and the modulus-argument form of a complex number.

2.6 Multiply and divide complex numbers in modulus argument form.

2.7 Construct and interpret simple loci in the argand diagram such as \(z - a > r\) and \(\arg (z - a) = \theta\)

2.8 Understand de Moivre’s theorem and use it to find multiple angle formulae and sums of series.

2.9 Know and use the definition \(e^{i\theta} = \cos \theta + i \sin \theta\) and the form \(z = re^{i\theta}\)

2.10 Find the \(n\) distinct \(n\)th roots of \(re^{i\theta}\) for \(r \neq 0\) and know that they form the vertices of a regular \(n\)-gon in the Argand diagram.

2.11 Use complex roots of unity to solve geometric problems.

Sample questions

AS SAMs Paper 1 Q1 (2.1, 2.2, 2.3, 4.1)

\[
f(z) = z^3 + p z^2 + q z - 15,
\]

where \(p\) and \(q\) are real constants.

Given that the equation \(f(z) = 0\) has roots

\[
\alpha, \frac{5}{\alpha} \quad \text{and} \quad \left(\alpha + \frac{5}{\alpha} - 1\right),
\]

(a) solve completely the equation \(f(z) = 0\).  
\(\text{ (5)}\)

(b) Hence find the value of \(p\).  
\(\text{ (2)}\)

(Total for Question 1 is 7 marks)
2. Complex numbers

AS SAMs Paper 1 Q8 (2.4, 2.7)

(a) Shade on an Argand diagram the set of points

\[ \{ z \in \mathbb{C} : |z - 4i| \leq 3 \} \cap \{ -\frac{\pi}{2} < \arg (z + 3 - 4i) \leq \frac{\pi}{4} \}. \]

The complex number \( w \) satisfies \( |w - 4i| = 3 \).

(b) Find the maximum value of \( \arg w \) in the interval \((-\pi, \pi] \).

Give your answer in radians correct to 2 decimal places.

(Total for Question 8 is 8 marks)

A level SAMs Paper 1 Q3 (2.1, 2.3, 2.4)

\[ f(z) = z^4 + az^3 + 6z^2 + bz + 65, \]

where \( a \) and \( b \) are real constants.

Given that \( z = 3 + 2i \) is a root of the equation \( f(z) = 0 \), show the roots of \( f(z) = 0 \) on a single Argand diagram.

(Total for Question 3 is 9 marks)

A level SAMs Paper 2 Q4 (2.2, 2.8)

A complex number \( z \) has modulus 1 and argument \( \theta \).

(a) Show that

\[ z^n + \frac{1}{z^n} = 2 \cos n, \quad n \in \mathbb{Z}^+ \]

(b) Hence, show that

\[ \cos^4 \theta = \frac{1}{8} (\cos 4\theta + 4 \cos 2\theta + 3) \]

(Total for Question 4 is 7 marks)
2. Complex numbers

A level SAMs Paper 2 Q6 (2.4, 2.7, 7.1, 7.3, A level Mathematics Pure Mathematics 5.3, 5.6)

(a) (i) Show on an Argand diagram the locus of points given by the values of $z$ satisfying

$$|z - 4 - 3i| = 5$$

Taking the initial line as the positive real axis with the pole at the origin and given that

$$\theta \in [\alpha, \alpha + \pi], \text{ where } \alpha = - \arctan \frac{4}{3},$$

(ii) show that this locus of points can be represented by the polar curve with equation

$$r = 8 \cos \theta + 6 \sin \theta$$

(6)

The set of points $A$ is defined by

$$A = z : \left\{ \begin{array}{l} z : \arg z = \frac{\pi}{3} \\
\{ z : |z - 4 - 3i| \leq 5 \} \end{array} \right\}$$

(b) (i) Show, by shading on your Argand diagram, the set of points $A$.

(ii) Find the exact area of the region defined by $A$, giving your answer in simplest form.

(7)

(Total for Question 6 is 13 marks)
3. Matrices

Structure and calculation

3.1 Add, subtract and multiply conformable matrices.
   Multiply a matrix by a scalar.
3.2 Understand and use zero and identity matrices.
3.3 Use matrices to represent linear transformations in 2-D.
   Successive transformations.
   Single transformations in 3-D.
3.4 Find invariant points and lines for a linear transformation.
3.5 Calculate determinants of: 2 × 2 and 3 × 3 matrices and interpret as scale factors, including
   the effect on orientation.
3.6 Understand and use singular and non-singular matrices.
   Properties of inverse matrices.
   Calculate and use the inverse of non-singular 2 × 2 matrices and 3 × 3 matrices.
3.7 Solve three linear simultaneous equations in three variables by use of the inverse matrix.
3.8 Interpret geometrically the solution and failure of solution of three simultaneous linear
   equations.

Sample questions

AS SAMs Paper 1 Q3 (3.1, 3.6, 3.7)

Tyler invested a total of £5000 across three different accounts; a savings account, a property bond
account and a share dealing account.

Tyler invested £400 more in the property bond account than in the savings account.

After one year

- the savings account had increased in value by 1.5%,
- the property bond account had increased in value by 3.5%,
- the share dealing account had decreased in value by 2.5%,
- the total value across Tyler’s three accounts had increased by £79.

Form and solve a matrix equation to find out how much money was invested by Tyler in each
account.

(Total for Question 3 is 7 marks)
3. Matrices

AS SAMs Paper 1 Q5 (3.3, 3.5, 3.6)

$$M = \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$$

(a) Show that $M$ is non-singular. (2)

The hexagon $R$ is transformed to the hexagon $S$ by the transformation represented by the matrix $M$.

Given that the area of hexagon $R$ is 5 square units,

(b) find the area of hexagon $S$. (1)

The matrix $M$ represents an enlargement, with centre $(0, 0)$ and scale factor $k$, where $k > 0$, followed by a rotation anti-clockwise through an angle $\theta$ about $(0, 0)$.

(c) Find the value of $k$. (2)

(d) Find the value of $\theta$. (2)

(Total for Question 5 is 7 marks)

A level SAMs Paper 2 Q3 (1.1, 3.1, 3.5, 3.6)

(i)

$$M = \begin{pmatrix} 2 & a & 4 \\ 1 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix}$$

where $a$ is a constant.

(a) For which values of $a$ does the matrix $M$ have an inverse? (2)

Given that $M$ is non-singular,

(b) find $M^{-1}$ in terms of $a$ (4)

(ii) Prove by induction that for all positive integers $n$,

$$\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^n = \begin{pmatrix} 3^n & 0 \\ 3(3^n) & 1 \end{pmatrix}$$

(6)

(Total for Question 3 is 12 marks)
4. Further algebra and functions

Structure and calculation

<table>
<thead>
<tr>
<th>4.1</th>
<th>Understand and use the relationship between roots and coefficients of polynomial equations up to quartic equations.</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.2</td>
<td>Form a polynomial equation whose roots are a linear transformation of the roots of a given polynomial equation (of at least cubic degree).</td>
</tr>
<tr>
<td>4.3</td>
<td>Understand and use formulae for the sums of integers, squares and cubes and use these to sum other series.</td>
</tr>
<tr>
<td>4.4</td>
<td>Understand and use the method of differences for summation of series including use of partial fractions.</td>
</tr>
<tr>
<td>4.5</td>
<td>Find the Maclaurin series of a function including the general term.</td>
</tr>
<tr>
<td>4.6</td>
<td>Recognise and use the Maclaurin series for $e^x$, $\ln(1 + x)$, $\sin x$, $\cos x$ and $(1 + x)^n$, and be aware of the range of values of $x$ for which they are valid (proof not required).</td>
</tr>
</tbody>
</table>

Sample questions

AS SAMs Paper 1 Q4 (4.1 or 4.2 depending on which method is used for solution)

The cubic equation

$$x^3 + 3x^2 - 8x + 6 = 0$$

has roots $a$, $b$ and $c$.

Without solving the equation, find the cubic equation whose roots are $(a - 1)$, $(b - 1)$ and $(c - 1)$, giving your answer in the form $w^3 + pw^2 + qw + r = 0$, where $p$, $q$ and $r$ are integers to be found.

(Total for Question 4 is 5 marks)
4. Further algebra and functions

AS SAMs Paper 1 Q6 (1.1, 4.3)

(a) Prove by induction that for all positive integers \( n \),
\[
\sum_{r=1}^{n} r^2 = \frac{1}{6} n(n + 1)(2n + 1)
\]

(b) Use the standard results for \( \sum_{r=1}^{n} r^3 \) and \( \sum_{r=1}^{n} r \) to show that for all positive integers \( n \),
\[
\sum_{r=1}^{n} r^2 = r(r + 6)(r - 6) = \frac{1}{4} n(n + 1)(n - 8)(n + 9)
\]

(c) Hence find the value of \( n \) that satisfies
\[
\sum_{r=1}^{n} r(r + 6)(r - 6) = 17 \sum_{r=1}^{n} r^2
\]

(Total for Question 6 is 15 marks)

A level SAMs Paper 1 Q1 (4.4)

Prove that
\[
\sum_{r=1}^{n} \frac{1}{(r + 1)(r + 3)} = \frac{n(an + b)}{12(n + 2)(n + 3)},
\]
where \( a \) and \( b \) are constants to be found.

(Total for Question 1 is 5 marks)
The roots of the equation \( x^3 - 8x^2 + 28x - 32 = 0 \) are \( \alpha, \beta \) and \( \gamma \).

Without solving the equation, find the value of

(i) \( \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \)  

(ii) \((\alpha + 2)(\beta + 2)(\gamma + 2)\)  

(iii) \(\alpha^2 + \beta^2 + \gamma^2\)  

(Total for Question 1 is 8 marks)

\[ A \text{ level SAMs Paper 2 Q5 (4.5, 8.2)} \]

\( y = \sin x \sinh x \)

(a) Show that \( \frac{d^3 y}{dx^3} = -4y \)  

(b) Hence find the first three non-zero terms of the Maclaurin series for \( y \), giving each coefficient in its simplest form.  

(c) Find an expression for the \( n \)th non-zero term of the Maclaurin series for \( y \).  

(Total for Question 5 is 10 marks)
5. Further calculus

Structure and calculation

5.1 Derive formulae for and calculate volumes of revolution.
5.2 Evaluate improper integrals where either the integrand is undefined at a value in the range of integration or the range of integration extends to infinity.
5.3 Understand and evaluate the mean value of a function.
5.4 Integrate using partial fractions.
5.5 Differentiate inverse trigonometric functions.
5.6 Integrate functions of the form \( (a^2 - x^2)^{\frac{1}{2}} \) and \( (a^2 - x^2)^{-1} \) and be able to choose trigonometric substitutions to integrate associated functions.

Sample questions

AS SAMs Paper 1 Q7 (5.1)

Diagrams not drawn to scale

Figure 1

Figure 2

Figure 1 shows the central cross-section \( AOBCD \) of a circular birdbath, which is made of concrete. Measurements of the height and diameter of the birdbath, and the depth of the bowl of the birdbath have been taken in order to estimate the amount of concrete that was required to make this birdbath.

Using these measurements, the cross-sectional curve \( CD \), shown in Figure 2, is modelled as a curve with equation

\[
y = 1 + kx^2, \quad -0.2 \leq x \leq 0.2,
\]

where \( k \) is a constant and where \( O \) is the fixed origin.

The height of the birdbath measured 1.16 m and the diameter, \( AB \), of the base of the birdbath measured 0.40 m, as shown in Figure 1.
5. Further calculus

(a) Suggest the maximum depth of the birdbath.  

(b) Find the value of $k$.  

(c) Hence find the volume of concrete that was required to make the birdbath according to this model. Give your answer, in m$^3$, correct to 3 significant figures.  

(d) State a limitation of the model.

It was later discovered that the volume of concrete used to make the birdbath was 0.127 m$^3$ correct to 3 significant figures.

(e) Using this information and the answer to part (c), evaluate the model, explaining your reasoning.

(Total for Question 7 is 12 marks)

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A level SAMs Paper 1 Q6 (5.3, 5.6, A level Mathematics Pure Mathematics 6.4)

$$ f(x) = \frac{x + 2}{x^2 + 9} $$

(a) Show that

$$ \int f(x) \, dx = A \ln (x^2 + 9) + B \arctan \left( \frac{x}{3} \right) + c, $$

where $c$ is an arbitrary constant and $A$ and $B$ are constants to be found.  

(b) Hence show that the mean value of $f(x)$ over the interval $[0, 3]$ is

$$ \frac{1}{6} \ln 2 + \frac{1}{18} \pi. $$

(c) Use your answer to part (b) to find the mean value, over the interval $[0, 3]$, of

$$ f(x) + \ln k, $$

where $k$ is a positive constant, giving your answer in the form $p + \frac{1}{6} \ln q$, where $p$ and $q$ are constants and $q$ is in terms of $k$.  

(Total for Question 6 is 9 marks)
Figure 2 shows the image of a gold pendant which has height 2 cm. The pendant is modelled by a solid of revolution of a curve $C$ about the y-axis. The curve $C$ has parametric equations

$$
x = \cos \theta + \frac{1}{2} \sin 2\theta, \quad y = -(1 + \sin \theta) \quad 0 \leq \theta \leq 2\pi
$$

(a) Show that a Cartesian equation of the curve $C$ is

$$x^3 = -(y^4 + 2y^3)$$  \hspace{1cm} (4)

(b) Hence, using the model, find, in cm$^3$, the volume of the pendant.  \hspace{1cm} (4)

(Total for Question 7 is 8 marks)
6. Further vectors

Structure and calculation

6.1 Understand and use the vector and Cartesian forms of an equation of a straight line in 3-D.
6.2 Understand and use the vector and Cartesian forms of the equation of a plane.
6.3 Calculate the scalar product and use it to express the equation of a plane, and to calculate the angle between two lines, the angle between two planes and the angle between a line and a plane.
6.4 Check whether vectors are perpendicular by using the scalar product.
6.5 Find the intersection of a line and a plane.

Sample questions

AS SAMs Paper 1 Q2 (6.1, 6.2, 6.3, 6.5)

The plane $\Pi$ passes through the point $A$ and is perpendicular to the vector $n$.

Given that

$$\overrightarrow{OA} = \begin{pmatrix} 5 \\ -4 \\ -4 \end{pmatrix} \text{ and } n = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix},$$

where $O$ is the origin,

(a) find a Cartesian equation of $\Pi$.

With respect to the fixed origin $O$, the line $l$ is given by the equation

$$r = \begin{pmatrix} 7 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -5 \\ 3 \end{pmatrix}. \quad (2)$$

The line $l$ intersects the plane $\Pi$ at the point $X$.

(b) Show that the acute angle between the plane $\Pi$ and the line $l$ is $21.2^\circ$, correct to one decimal place. \quad (4)

(c) Find the coordinates of the point $X$. \quad (4)

(Total for Question 2 is 10 marks)
AS SAMs Paper 1 Q9 (6.1, 6.3, 6.4, 6.5)

An octopus is able to catch any fish that swim within a distance of 2 m from the octopus’s position.

A fish $F$ swims from a point $A$ to a point $B$.

The octopus is modelled as a fixed particle at the origin $O$.

Fish $F$ is modelled as a particle moving in a straight line from $A$ to $B$.

Relative to $O$, the coordinates of $A$ are ($-3$, $1$, $-7$) and the coordinates of $B$ are ($9$, $4$, $11$), where the unit of distance is metres.

(a) Use the model to determine whether or not the octopus is able to catch fish $F$.

(b) Criticise the model in relation to fish $F$.

(c) Criticise the model in relation to the octopus.

(Total for Question 9 is 9 marks)

A level SAMs Paper 1 Q8 (6.1, 6.2, 6.3)

The line $l_1$ has equation $\frac{x - 2}{4} = \frac{y - 4}{-2} = \frac{z + 6}{1}$

The plane $\Pi$ has equation $x - 2y + z = 6$

The line $l_2$ is the reflection of the line $l_1$ in the plane $\Pi$.

Find a vector equation of the line $l_2$.

(Total for Question 8 is 7 marks)

A level SAMs Paper 2 Q2 (6.2, 6.3, 6.4, 6.5)

The plane $\Pi_1$ has vector equation $\mathbf{r}.(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 5$

(a) Find the perpendicular distance from the point (6, 2, 12) to the plane $\Pi_1$

The plane $\Pi_2$ has vector equation $\mathbf{r} = \lambda(2\mathbf{i} + \mathbf{j} + 5\mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} - 2\mathbf{k})$

where $\lambda$ and $\mu$ are scalar parameters.

(b) Show that the vector $-\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is perpendicular to $\Pi_2$

(c) Show that the acute angle between $\Pi_1$ and $\Pi_2$ is $52^\circ$ to the nearest degree.

(Total for Question 2 is 8 marks)
7. Polar coordinates

Structure and calculation

7.1 Understand and use polar coordinates and be able to convert between polar and Cartesian coordinates.
7.2 Sketch curves with \( r \) given as a function of \( \theta \), including use of trigonometric functions.
7.3 Find the area enclosed by a polar curve.

Sample questions

A level SAMs Paper 1 Q4 (7.1, 7.3)

The curve \( C \) shown in Figure 1 has polar equation

\[
r = 4 + \cos 2\theta \quad 0 \leq \theta \leq \frac{\pi}{2}
\]

At the point \( A \) on \( C \), the value of \( r \) is \( \frac{9}{2} \)

The point \( N \) lies on the initial line and \( AN \) is perpendicular to the initial line.

The finite region \( R \), shown shaded in Figure 1, is bounded by the curve \( C \), the initial line and the line \( AN \).

Find the exact area of the shaded region \( R \), giving your answer in the form \( p\pi + q\sqrt{3} \), where \( p \) and \( q \) are rational numbers to be found.

(Total for Question 4 is 9 marks)
A level SAMs Paper 2 Q6 (2.4, 2.7, 7.1, 7.3, A level Mathematics Pure Mathematics 5.3, 5.6)

(a) (i) Show on an Argand diagram the locus of points given by the values of \( z \) satisfying
\[
|z - 4 - 3i| = 5
\]
Taking the initial line as the positive real axis with the pole at the origin and given that
\[
\theta \in [\alpha, \alpha + \pi], \quad \text{where} \quad \alpha = \arctan \left(\frac{4}{3}\right),
\]
(ii) show that this locus of points can be represented by the polar curve with equation
\[
r = 8 \cos \theta + 6 \sin \theta
\]
(6)

The set of points \( A \) is defined by
\[
A = z : \begin{cases} z : 0 \leq \arg z < \frac{\pi}{3} \\ z : |z - 4 - 3i| \leq 5 \end{cases}
\]
(b) (i) Show, by shading on your Argand diagram, the set of points \( A \).
(ii) Find the exact area of the region defined by \( A \), giving your answer in simplest form.
(7)

(Total for Question 6 is 13 marks)
8. Hyperbolic functions

Structure and calculation

8.1 Understand the definitions of hyperbolic functions sinh \( x \), cosh \( x \) and tanh \( x \), including their domains and ranges, and be able to sketch their graphs.

8.2 Differentiate and integrate hyperbolic functions.

8.3 Understand and be able to use the definitions of the inverse hyperbolic functions and their domains and ranges.

8.4 Derive and use the logarithmic forms of the inverse hyperbolic functions.

8.5 Integrate functions of the form \( (a^2 + x^2)^{\frac{1}{2}} \) and \( (a^2 - x^2)^{\frac{1}{2}} \) and be able to choose substitutions to integrate associated functions.

Sample questions

A level SAMs Paper 2 Q5 (4.5, 8.2)

\( y = \sin x \sinh x \)

(a) Show that \( \frac{d^4y}{dx^4} = -4y \) \hspace{1cm} (4)

(b) Hence find the first three non-zero terms of the Maclaurin series for \( y \), giving each coefficient in its simplest form. \hspace{1cm} (4)

(c) Find an expression for the \( n \)th non-zero term of the Maclaurin series for \( y \). \hspace{1cm} (2)

(Total for Question 5 is 10 marks)
9. Differential equations

Structure and calculation

9.1 Find and use an integrating factor to solve differential equations of form $\frac{dy}{dx} + P(x)y = Q(x)$ and recognise when it is appropriate to do so.

9.2 Find both general and particular solutions to differential equations.

9.3 Use differential equations in modelling in kinematics and in other contexts.

9.4 Solve differential equations of form $y'' + ay' + by = 0$ where $a$ and $b$ are constants by using the auxiliary equation.

9.5 Solve differential equations of form $y'' + ay' + by = f(x)$ where $a$ and $b$ are constants by solving the homogeneous case and adding a particular integral to the complementary function (in cases where $f(x)$ is a polynomial, exponential or trigonometric function).

9.6 Understand and use the relationship between the cases when the discriminant of the auxiliary equation is positive, zero and negative and the form of solution of the differential equation.

9.7 Solve the equation for simple harmonic motion $\ddot{x}$ and relate the solution to the motion.

9.8 Model damped oscillations using second order differential equations and interpret their solutions.

9.9 Analyse and interpret models of situations with one independent variable and two dependent variables as a pair of coupled first order simultaneous equations and be able to solve them, for example predator-prey models.

Sample questions

A level Mathematics SAMs Paper 1 Q9 (9.2, 9.3, 9.5, 9.6)

A company plans to build a new fairground ride. The ride will consist of a capsule that will hold the passengers and the capsule will be attached to a tall tower. The capsule is to be released from rest from a point half way up the tower and then made to oscillate in a vertical line.

The vertical displacement, $x$ metres, of the top of the capsule below its initial position at time $t$ seconds is modelled by the differential equation,

$$m \frac{d^2x}{dt^2} + 4 \frac{dx}{dt} + x = 200 \cos t, \quad t \geq 0$$

where $m$ is the mass of the capsule including its passengers, in thousands of kilograms. The maximum permissible weight for the capsule, including its passengers, is 30 000 N.

Taking the value of $g$ to be 10 m s$^{-2}$ and assuming the capsule is at its maximum permissible weight,

(a) (i) explain why the value of $m$ is 3

(ii) show that a particular solution to the differential equation is $x = 40 \sin t - 20 \cos t$

(iii) hence find the general solution of the differential equation.

(b) Using the model, find, to the nearest metre, the vertical distance of the top of the capsule from its initial position, 9 seconds after it is released.

(Total for Question 9 is 12 marks)
At the start of the year 2000, a survey began of the number of foxes and rabbits on an island.

At time \(t\) years after the survey began, the number of foxes, \(f\), and the number of rabbits, \(r\), on the island are modelled by the differential equations

\[
\frac{df}{dt} = 0.2f + 0.1r \\
\frac{dr}{dt} = 0.2f + 0.4r
\]

(a) Show that \(\frac{d^2f}{dt^2} - 0.6\frac{df}{dt} + 0.1f = 0\)  

(b) Find a general solution for the number of foxes on the island at time \(t\) years.  

(c) Hence find a general solution for the number of rabbits on the island at time \(t\) years.  

At the start of the year 2000 there were 6 foxes and 20 rabbits on the island.

(d) (i) According to this model, in which year are the rabbits predicted to die out?  

(ii) According to this model, how many foxes will be on the island when the rabbits die out?  

(iii) Use your answers to parts (i) and (ii) to comment on the model.  

(Total for Question 7 is 17 marks)
1. Further trigonometry

Structure and calculation

1.1 The \( t \)-formulae.

Applications of \( t \)-formulae to trigonometric identities.

Applications of \( t \)-formulae to solve trigonometric equations.

Sample questions

AS Further Mathematics SAMs Paper 2A Q1 (1.1, 1.2)

(a) Use the substitution \( t = \tan \frac{x}{2} \) to show that

\[
\sec x - \tan x = \frac{1 - t}{1 + t}, \quad x \neq (2n + 1) \frac{\pi}{2}, \quad n \in \mathbb{Z}
\]

(b) Use the substitution \( t = \tan \frac{x}{2} \) and the answer to part (a) to prove that

\[
\frac{1 - \sin x}{1 + \sin x} = (\sec x - \tan x)^2, \quad x \neq (2n + 1) \frac{\pi}{2}, \quad n \in \mathbb{Z}.
\]

(Total for Question 1 is 6 marks)
A level Further Mathematics SAMs Paper 3A Q8 (1.1, 1.2, A level Mathematics Pure Mathematics 2.9, 5.6, 7.3)

Figure 1 shows the graph of the function \( h(x) \) with equation

\[
h(x) = 45 + 15 \sin x + 21 \sin \left( \frac{x}{2} \right) + 25 \cos \left( \frac{x}{2} \right) \quad x \in [0, 40]
\]

(a) Show that

\[
\frac{dh}{dx} = \frac{(t^2 - 6t - 17)(9t^2 + 4t - 3)}{2(1 + t^2)^2} \quad \text{where} \quad t = \tan \left( \frac{x}{4} \right)
\]

Figure 2 shows a graph of predicted tide heights, in metres, for Portland harbour from 08:00 on the 3rd January 2017 to the end of the 4th January 2017.

The graph of \( k \ h(x) \), where \( k \) is a constant and \( x \) is the number of hours after 08:00 on 3rd of January, can be used to model the predicted tide heights, in metres, for this period of time.

(b) (i) Suggest a value of \( k \) that could be used for the graph of \( k \ h(x) \) to form a suitable model.

(ii) Why may such a model be suitable to predict the times when the tide heights are at their peaks, but not to predict the heights of these peaks?

(c) Use Figure 2 and the result of part (a) to estimate, to the nearest minute, the time of the highest tide height on the 4th January 2017.

(Total for Question 8 is 15 marks)
2. Further calculus

Structure and calculation

2.1 Derivation and use of Taylor series.
2.2 Use of series expansions to find limits.
2.3 Leibnitz’s theorem.
2.4 L’Hospital’s Rule.
2.5 The Weierstrass substitution for integration.

Sample questions

A level Further Mathematics SAMs Paper 3A Q2 (2.3)

Given $k$ is a constant and that

$$y = x^3 e^{kx}$$

use Leibnitz theorem to show that

$$\frac{d^n y}{dx^n} = k^n x^{n-3} e^{kx} (k^3 x^3 + 3nk^2 x^2 + 3n(n-1)k x + n(n-1)(n-2))$$

(Total for Question 2 is 4 marks)
3. Further differential equations

Structure and calculation

<table>
<thead>
<tr>
<th>3.1</th>
<th>Use of Taylor series method for series solution of differential equations.</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.2</td>
<td>Differential equations reducible by means of a given substitution.</td>
</tr>
</tbody>
</table>

Sample questions

A level Further Mathematics SAMs Paper 3A Q3 (3.2)

A vibrating spring, fixed at one end, has an external force acting on it such that the centre of the spring moves in a straight line. At time \( t \) seconds, \( t \geq 0 \), the displacement of the centre \( C \) of the spring from a fixed point \( O \) is \( x \) micrometres.

The displacement of \( C \) from \( O \) is modelled by the differential equation

\[
t^2 \frac{d^2 x}{dt^2} - 2t \frac{dx}{dt} + (2 + t^2)x = t^4
\]

(I)

(a) Show that the transformation \( x = t \, v \) transforms equation (I) into the equation

\[
\frac{d^2 v}{dt^2} + v = t
\]

(II)

(b) Hence find the general equation for the displacement of \( C \) from \( O \) at time \( t \) seconds.

(c) (i) State what happens to the displacement of \( C \) from \( O \) as \( t \) becomes large.

(ii) Comment on the model with reference to this long term behaviour.

(Total for Question 3 is 14 marks)

A level Further Mathematics SAMs Paper 3A Q4 (3.1)

\[
\frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + y = 0
\]

(I)

(a) Show that

\[
\frac{d^5 y}{dx^5} = a x \frac{d^4 y}{dx^4} + b \frac{d^3 y}{dx^3}
\]

where \( a \) and \( b \) are integers to be found.

(b) Hence find a series solution, in ascending powers of \( x \), as far as the term in \( x^5 \),

of the differential equation (I) where \( y = 0 \) and \( \frac{dy}{dx} = 1 \) at \( x = 0 \)

(Total for Question 4 is 9 marks)
3. Further differential equations
4. Coordinate systems

4. Coordinate systems

Structure and calculation

| 4.1 | Cartesian equations for the parabola and rectangular hyperbola, ellipse and hyperbola. |
| 4.2 | Parametric equations for the parabola and rectangular hyperbola, ellipse and hyperbola. |
| 4.3 | The focus-directrix property of the parabola, ellipse and hyperbola, including the eccentricity. |
| 4.4 | Tangents and normals to these curves. |
| 4.5 | Simple loci problems. |

Sample questions

AS Further Mathematics SAMs Paper 2A Q5 (4.1, 4.2, 4.3, 4.4)

[You may quote without proof that for the general parabola $y^2 = 4ax$, $\frac{dy}{dx} = \frac{2a}{y}$]

The parabola $C$ has equation $y^2 = 16x$.

(a) Deduce that the point $P(4p^2, 8p)$ is a general point on $C$.  

(1)

The line $l$ is the tangent to $C$ at the point $P$.

(b) Show that an equation for $l$ is $py = x + 4p^2$  

(3)
4. Coordinate systems

The finite region \( R \), shown shaded in Figure 2, is bounded by the line \( l \), the \( x \)-axis and the parabola \( C \).

The line \( l \) intersects the directrix of \( C \) at the point \( B \), where the \( y \) coordinate of \( B \) is \( \frac{10}{3} \).

Given that \( p > 0 \)

(c) show that the area of \( R \) is 36

(Total for Question 5 is 12 marks)

A level Further Mathematics SAMs Paper 3A Q5 (4.1, 4.3)

The normal to the parabola \( y^2 = 4ax \) at the point \( P(ap^2, 2ap) \) passes through the parabola again at the point \( Q(aq^2, 2aq) \).

The line \( OP \) is perpendicular to the line \( OQ \), where \( O \) is the origin.

Prove that \( p^2 = 2 \)

(Total for Question 5 is 9 marks)

A level Further Mathematics SAMs Paper 3A Q7 (4.1, 4.3, 4.4)

\( P \) and \( Q \) are two distinct points on the ellipse described by the equation \( x^2 + 4y^2 = 4 \)

The line \( l \) passes through the point \( P \) and the point \( Q \).

The tangent to the ellipse at \( P \) and the tangent to the ellipse at \( Q \) intersect at the point \( (r, s) \).

Show that an equation of the line \( l \) is

\[ 4sy + rx = 4 \]

(Total for Question 7 is 8 marks)
5. Further vectors

Structure and calculation

<table>
<thead>
<tr>
<th>5.1</th>
<th>The vector product ( \mathbf{a} \times \mathbf{b} ) of two vectors. Applications of the vector product.</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.2</td>
<td>The scalar triple product ( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) )</td>
</tr>
<tr>
<td>5.3</td>
<td>Applications of vectors to three dimensional geometry involving points, lines and planes.</td>
</tr>
</tbody>
</table>

Sample questions

AS Further Mathematics SAMs Paper 2A Q4 (5.1, 5.2, 5.3)

Figure 1 shows a sketch of a solid sculpture made of glass and concrete. The sculpture is modelled as a parallelepiped.

The sculpture is made up of a concrete solid in the shape of a tetrahedron, shown shaded in Figure 1, whose vertices are \( O(0, 0, 0), A(2, 0, 0), B(0, 3, 1) \) and \( C(1, 1, 2) \), where the units are in metres. The rest of the solid parallelepiped is made of glass which is glued to the concrete tetrahedron.

(a) Find the surface area of the glued face of the tetrahedron. 

(b) Find the volume of glass contained in this parallelepiped.

(c) Give a reason why the volume of concrete predicted by this model may not be an accurate value for the volume of concrete that was used to make the sculpture.

(Total for Question 4 is 10 marks)
A level Further Mathematics SAMs Paper 3A Q6 (5.1, 5.2, 5.3)

A tetrahedron has vertices A(1, 2, 1), B(0, 1, 0), C(2, 1, 3) and D(10, 5, 5).

Find

(a) a Cartesian equation of the plane \(ABC\).

(b) the volume of the tetrahedron \(ABCD\).

The plane \(П\) has equation \(2x - 3y + 3 = 0\)

The point \(E\) lies on the line \(AC\) and the point \(F\) lies on the line \(AD\).

Given that \(П\) contains the point \(B\), the point \(E\) and the point \(F\),

(c) find the value of \(k\) such that \(\vec{AE} = k\vec{AC}\)

Given that \(\vec{AF} = \frac{1}{9}\vec{AD}\)

(d) show that the volume of the tetrahedron \(ABCD\) is 45 times the volume of the tetrahedron \(ABEF\).

(Total for Question 6 is 11 marks)
6. Numerical Methods

6. Numerical Methods

Structure and calculation

<table>
<thead>
<tr>
<th>6.1</th>
<th>Numerical solution of first order and second order differential equations.</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.2</td>
<td>Simpson’s rule.</td>
</tr>
</tbody>
</table>

Sample questions

AS Further Mathematics SAMs Paper 2A Q2 (6.1)

The value, $V$ hundred pounds, of a particular stock $t$ hours after the opening of trading on a given day is modelled by the differential equation

$$\frac{dV}{dt} = \frac{V^2 - 1}{t^2 + tV}, \quad 0 < t < 8.5.$$

A trader purchases £300 of the stock one hour after the opening of trading.

Use two iterations of the approximation formula $\left( \frac{dy}{dx} \right)_0 \approx \frac{y_1 - y_0}{h}$ to estimate, to the nearest £, the value of the trader’s stock half an hour after it was purchased.

(Total for Question 2 is 6 marks)

A level Further Mathematics SAMs Paper 3A Q1 (6.2)

Use Simpson’s Rule with 6 intervals to estimate

$$\int_1^4 \sqrt{1 + x^3} \, dx$$

(Total for Question 1 is 5 marks)
7. Inequalities

Structure and calculation

7.1 The manipulation and solution of algebraic inequalities and inequations, including those involving the modulus sign.

Sample questions

AS Further Mathematics SAMs Paper 2A Q3 (7.1)

Use algebra to find the set of values of $x$ for which

$$\frac{1}{x} < \frac{x}{x+2}$$

(Total for Question 3 is 6 marks)
1. Groups

Structure and calculation

1.1 The Axioms of a group.
1.2 Examples of groups. Cayley tables. Cyclic groups.
1.3 The order of a group and the order of an element. Subgroups.
1.4 Lagrange's theorem.
1.5 Isomorphism.

Sample questions

AS Further Mathematics SAMs Paper 2A Q9 (1.1, 1.2, 1.3, 5.3)

The operation * is defined on the set \( S = \{0, 2, 3, 4, 5, 6\} \) by \( x*y = x + y = xy \mod 7 \).

\[
\begin{array}{cccccc}
* & 0 & 2 & 3 & 4 & 5 & 6 \\
0 & 0 & 2 & 3 & 4 & 5 & 6 \\
2 & 2 & 0 & 5 & 1 & 6 & 3 \\
3 & 3 & 5 & 0 & 2 & 1 & 4 \\
4 & 4 & 1 & 2 & 0 & 3 & 5 \\
5 & 5 & 6 & 1 & 3 & 0 & 2 \\
6 & 6 & 3 & 4 & 5 & 2 & 0 \\
\end{array}
\]

(a) (i) Copy and complete the Cayley table shown above

(ii) Show that \( S \) is a group under the operation *

(You may assume the associative law is satisfied.)

(b) Show that the element 4 has order 3.

(c) Find an element which generates the group and express each of the elements in terms of this generator.

(Total for Question 9 is 11 marks)
A level Further Mathematics SAMs Paper 4A Q4 (1.2, 1.3, 1.5, A level Further Mathematics Core Pure Mathematics 3.3)

(i) A group $G$ contains distinct elements $a$, $b$ and $e$ where $e$ is the identity element and the group operation is multiplication.

Given $a^2b = ba$, prove $ab \neq ba$  

(ii) The set $H = \{1, 2, 4, 7, 8, 11, 13, 14\}$ forms a group under the operation of multiplication modulo 15.

(a) Find the order of each element of $H$.  

(b) Find three subgroups of $H$ each of order 4, and describe each of these subgroups.  

(c) The elements of another group $J$ are the matrices

\[
\begin{pmatrix}
\cos\left(\frac{k}{4}\right) & \sin\left(\frac{k}{4}\right) \\
\sin\left(\frac{k}{4}\right) & \cos\left(\frac{k}{4}\right)
\end{pmatrix}
\]

where $k = 1, 2, 3, 4, 5, 6, 7, 8$ and the group operation is matrix multiplication.

(c) Determine whether $H$ and $J$ are isomorphic, giving a reason for your answer.

(Total for Question 4 is 13 marks)
2. Further calculus

Structure and calculation

2.1 Further Integration — Reduction formulae.
2.2 The calculation of arc length and the area of a surface of revolution.

Sample questions

A level Further Mathematics SAMs Paper 4A Q5 (2.2)

An engineering student makes a miniature arch as part of the design for a piece of coursework. The cross-section of this arch is modelled by the curve with equation

\[ y = A \left( \frac{1}{2} \cosh 2x \right), \quad -\ln a \leq x \leq \ln a \]

where \( a > 1 \) and \( A \) is a positive constant. The curve begins and ends on the \( x \)-axis, as shown in Figure 1.

(a) Show that the length of this curve is \( k \left( a^2 - \frac{1}{a^2} \right) \), stating the value of the constant \( k \).  

The length of the curved cross-section of the miniature arch is required to be 2 m long.

(b) Find the height of the arch, according to this model, giving your answer to 2 significant figures.  

(c) Find also the width of the base of the arch giving your answer to 2 significant figures.  

(d) Give the equation of another curve that could be used as a suitable model for the cross-section of an arch, with approximately the same height and width as you found using the first model. (You do not need to consider the arc length of your curve)

(Total for Question 5 is 12 marks)
2. Further calculus

A level Further Mathematics SAMs Paper 4A Q7 (2.1)

\[ I_n = \int_{0}^{\frac{\pi}{2}} \sin^n x \, dx, \quad n \geq 0 \]

(a) Prove that, for \( n \geq 2 \),

\[ nI_n = (n - 1)I_{n-2} \quad \text{(4)} \]

(b) 

y

O

x

Figure 2

A designer is asked to produce a poster to completely cover the curved surface area of a solid cylinder which has diameter 1 m and height 0.7 m.

He uses a large sheet of paper with height 0.7 m and width of \( \pi \) m.

Figure 2 shows the first stage of the design, where the poster is divided into two sections by a curve.

The curve is given by the equation

\[ y = \sin^2(4x) - \sin^{10}(4x) \]

relative to axes taken along the bottom and left hand edge of the paper.

The region of the poster below the curve is shaded and the region above the curve remains unshaded, as shown in Figure 2.

Find the exact area of the poster which is shaded.

(Total for Question 7 is 9 marks)
3. Further matrix algebra

3. Further matrix algebra

Structure and calculation

| 3.1 | Eigenvalues and eigenvectors of $2 \times 2$ matrices. |
| 3.2 | Reduction of matrices to diagonal form. |
| 3.3 | The use of the Cayley-Hamilton theorem. |

Sample questions

AS Further Mathematics SAMs Paper 2A Q6 (3.1, 3.3)

Given that

$$A = \begin{pmatrix} 3 & 1 \\ 6 & 4 \end{pmatrix},$$

(a) find the characteristic equation of the matrix $A$. 

(b) Hence show that $A^3 = 43A - 42I$. 

(Total for Question 6 is 5 marks)

A level Further Mathematics SAMs Paper 4A Q3 (3.1, 3.2)

The matrix $M$ is given by

$$M = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 4 \end{pmatrix}$$

(a) Show that 4 is an eigenvalue of $M$, and find the other two eigenvalues. 

(b) For each of the eigenvalues find a corresponding eigenvector. 

(c) Find a matrix $P$ such that $P^{-1}MP$ is a diagonal matrix. 

(Total for Question 3 is 10 marks)
4. Further complex numbers

Structure and calculation

4.1 Further loci and regions in the Argand diagram.
4.2 Elementary transformations from the z-plane to the w-plane.

Sample questions

AS Further Mathematics SAMs Paper 2A Q8 (4.1)

A curve $C$ is described by the equation

$$|z - 9 + 12i| = 2|z|.$$

(a) Show that $C$ is a circle, and find its centre and radius. (4)

(b) Sketch $C$ on an Argand diagram. (2)

Given that $w$ lies on $C$,

(c) find the largest value of $a$ and the smallest value of $b$ that must satisfy $a \leq \text{Re}(w) \leq b$. (2)

(Total for Question 8 is 8 marks)

A level Further Mathematics SAMs Paper 4A Q2 (4.2)

A transformation from the $z$-plane to the $w$-plane is given by

$$w = z^2$$

(a) Show that the line with equation $\text{Im}(z) = 1$ in the $z$-plane is mapped to a parabola in the $w$-plane, giving an equation for this parabola. (4)

(b) Sketch the parabola on an Argand diagram. (2)

(Total for Question 2 is 6 marks)
A level Further Mathematics SAMs Paper 4A Q6 (4.1)

A curve has equation

$$|z + 6| = 2 |z - 6|, \quad z \in \mathbb{C}$$

(a) Show that the curve is a circle with equation $x^2 + y^2 - 20x + 36 = 0$  \hfill (2)

(b) Sketch the curve on an Argand diagram. \hfill (2)

The line $l$ has equation $az^* + a^*z = 0$, where $a \in \mathbb{C}$ and $z \in \mathbb{C}$

Given that the line $l$ is a tangent to the curve and that $\arg a = \theta$

(c) Find the possible values of $\tan \theta$ \hfill (5)

(Total for Question 6 is 9 marks)
5. Number theory

Structure and calculation

| 5.1 | An understanding of the division theorem and its application to the Euclidean Algorithm and congruences. |
| 5.2 | Bezout’s identity. |
| 5.3 | Modular arithmetic. Understanding what is meant by two integers \( a \) and \( b \) to be congruent modulo an integer \( n \). Properties of congruences. |
| 5.4 | Fermat’s Little Theorem. |
| 5.5 | Divisibility Tests. |
| 5.6 | Solution of congruence equations. |
| 5.7 | Combinatorics: counting problems, permutations and combinations. |

Sample questions

AS Further Mathematics SAMs Paper 2A Q9 (1.1, 1.2, 1.3, 5.3)

The operation \( * \) is defined on the set \( S = \{0, 2, 3, 4, 5, 6\} \) by \( x* y = x + y = xy \) (mod 7).

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<thead>
<tr>
<th>(*)</th>
<th>0</th>
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</table>

(a)  
(i) Copy and complete the Cayley table shown above  

(ii) Show that \( S \) is a group under the operation \( * \)  

(You may assume the associative law is satisfied.)  

(b) Show that the element 4 has order 3.  

(c) Find an element which generates the group and express each of the elements in terms of this generator.  

(Total for Question 9 is 11 marks)
5. Number theory

AS Further Mathematics SAMs Paper 2A Q7 (5.1, 5.2, 5.5)

(i) Without performing any division, explain why 8184 is divisible by 6. \( \text{(2 marks)} \)

(ii) Use the Euclidean algorithm to find integers \( a \) and \( b \) such that \( 27a + 31b = 1 \). \( \text{(4 marks)} \)

(Total for Question 7 is 6 marks)

---

A level Further Mathematics SAMs Paper 4A Q1 (5.1, 5.7)

(i) Use the Euclidean algorithm to find the highest common factor of 602 and 161.

Show each step of the algorithm. \( \text{(3 marks)} \)

(ii) The digits which can be used in a security code are the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9.

Originally the code used consisted of two distinct odd digits, followed by three distinct even digits.

To enable more codes to be generated, a new system is devised. This uses two distinct even digits, followed by any three other distinct digits. No digits are repeated.

Find the increase in the number of possible codes which results from using the new system. \( \text{(4 marks)} \)

(Total for Question 1 is 7 marks)
Further sequences and series

Structure and calculation

6.1 First and second order recurrence relations.
6.2 The solution of recurrence relations to obtain closed forms.
6.3 Proof by induction of closed forms.

Sample questions

AS Further Mathematics SAMs Paper 2A Q10 (6.1, 6.2, 6.3)

A population of deer on a large estate is assumed to increase by 10% during each year due to natural causes.

The population is controlled by removing a constant number, \( Q \), of the deer from the estate at the end of each year.

At the start of the first year there are 5000 deer on the estate.

Let \( P_n \) be the population of deer at the end of year \( n \).

(a) Explain, in the context of the problem, the reason that the deer population is modelled by the recurrence relation

\[
P_n = 1.1 \ P_{n-1} - Q, \quad P_0 = 5000, \quad n \in \mathbb{Z}^+.
\]

(b) Prove by induction that

\[
P_n = (1.1)^n (5000 - 10Q) + 10Q, \quad n \geq 0.
\]

(c) Explain how the long term behaviour of this population varies for different values of \( Q \).

(Total for Question 10 is 10 marks)
A level Further Mathematics SAMs Paper 4A Q8 (6.1, 6.2)

A staircase has \(n\) steps. A tourist moves from the bottom (step zero) to the top (step \(n\)). At each move up the staircase she can go up either one step or two steps, and her overall climb up the staircase is a combination of such moves.

If \(u_n\) is the number of ways that the tourist can climb up a staircase with \(n\) steps,

(a) explain why \(u_n\) satisfies the recurrence relation

\[
u_n = u_{n-1} + u_{n-2}, \text{ with } u_1 = 1 \text{ and } u_2 = 2
\]

(3)

(b) Find the number of ways in which she can climb up a staircase when there are eight steps.

(1)

(c) Show that the number of ways in which she could climb up to the top of this staircase is given by

\[
\frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{401} - \left( \frac{1 - \sqrt{5}}{2} \right)^{401} \right]
\]

(5)

(Total for Question 8 is 9 marks)
Further Statistics 1

1. Discrete probability distribution

Structure and calculation

1.1 Mean and variance of discrete probability distributions.
Extension of expected value function to include \( E(g(X)) \).

Sample questions

AS Further Mathematics SAMs Paper 2G Q2 (1.1, AS Further Mathematics Core Pure 3.7)

The discrete random variable \( X \) has probability distribution given by

<table>
<thead>
<tr>
<th>( x )</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(X = x) )</td>
<td>( c )</td>
<td>( a )</td>
<td>( a )</td>
<td>( b )</td>
<td>( c )</td>
</tr>
</tbody>
</table>

The random variable \( Y = 2 - 5X \).

Given that \( E(Y) = -4 \) and \( P(Y \geq -3) = 0.45 \).

(a) find the probability distribution of \( X \).

(7)

Given also that \( E(Y^2) = 75 \),

(b) find the exact value of \( \text{Var}(X) \).

(2)

(c) Find \( P(Y > X) \).

(2)

(Total for Question 2 is 11 marks)
The discrete random variable $X$ follows a Poisson distribution with mean 1.4

(a) Write down the value of

(i) $P(X = 1)$,

(ii) $P(X \leq 4)$.

(2)

The manager of a bank recorded the number of mortgages approved each week over a 40 week period.

<table>
<thead>
<tr>
<th>Number of mortgages approved</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>10</td>
<td>16</td>
<td>7</td>
<td>4</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

(b) Show that the mean number of mortgages approved over the 40 week period is 1.4.

(1)

The bank manager believes that the Poisson distribution may be a good model for the number of mortgages approved each week.

She uses a Poisson distribution with a mean of 1.4 to calculate expected frequencies as follows.

<table>
<thead>
<tr>
<th>Number of mortgages approved</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5 or more</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected frequency</td>
<td>9.86</td>
<td>$r$</td>
<td>9.67</td>
<td>4.51</td>
<td>1.58</td>
<td>$s$</td>
</tr>
</tbody>
</table>

(c) Find the value of $r$ and the value of $s$, giving your answers to 2 decimal places.

(2)

The bank manager will test, at the 5% level of significance, whether or not the data can be modelled by a Poisson distribution.

(d) Calculate the test statistic and state the conclusion for this test. State clearly the degrees of freedom and the hypotheses used in the test.

(6)

(Total for Question 4 is 11 marks)
A level Further Mathematics SAMs Paper 3B/4B Q6 (1.1, 7.1, 7.2)

The probability generating function of the discrete random variable $X$ is given by

$$G_X(t) = k(3 + t + 2t^2)^2$$

(a) Show that $k = \frac{1}{36}$

(b) Find $P(X = 3)$

(c) Show that $\text{Var}(X) = \frac{29}{18}$

(d) Find the probability generating function of $2X + 1$

(Total for Question 6 is 14 marks)
2. Poisson & binomial distributions

Structure and calculation

2.1 The Poisson distribution.
   - The additive property of Poisson distributions.
2.2 The mean and variance of the binomial and the Poisson distributions.
2.3 The use of Poisson distribution as an approximation to the binomial distribution.

Sample questions

AS Further Mathematics SAMs Paper 2G Q3 (2.1, 2.2, 2.3)

Two car hire companies hire cars independently of each other.

Car Hire A hires cars at a rate of 2.6 cars per hour.

Car Hire B hires cars at a rate of 1.2 cars per hour.

(a) In a one-hour period, find the probability that each company hires exactly 2 cars. (2)

(b) In a one-hour period, find the probability that the total number of cars hired by the two companies is 3. (2)

(c) In a 2 hour period, find the probability that the total number of cars hired by the two companies is less than 9. (2)

On average, 1 in 250 new cars produced at a factory has a defect.

In a random sample of 600 new cars produced at the factory,

(d) (i) find the mean of the number of cars with a defect, (2)

   (ii) find the variance of the number of cars with a defect. (2)

(e) (i) Use a Poisson approximation to find the probability that no more than 4 of the cars in the sample have a defect. (2)

   (ii) Give a reason to support the use of a Poisson approximation. (2)

(Total for Question 3 is 10 marks)
2. Poisson & binomial distributions

AS Further Mathematics SAMs Paper 2G Q4 (1.1, 2.1, 6.1)

The discrete random variable $X$ follows a Poisson distribution with mean 1.4

(a) Write down the value of

(ii) $P(X = 1)$,

(ii) $P(X \leq 4)$.

The manager of a bank recorded the number of mortgages approved each week over a 40 week period.

<table>
<thead>
<tr>
<th>Number of mortgages approved</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>10</td>
<td>16</td>
<td>7</td>
<td>4</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

(b) Show that the mean number of mortgages approved over the 40 week period is 1.4.

The bank manager believes that the Poisson distribution may be a good model for the number of mortgages approved each week.

She uses a Poisson distribution with a mean of 1.4 to calculate expected frequencies as follows.

<table>
<thead>
<tr>
<th>Number of mortgages approved</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5 or more</th>
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</thead>
<tbody>
<tr>
<td>Expected frequency</td>
<td>9.86</td>
<td>$r$</td>
<td>9.67</td>
<td>4.51</td>
<td>1.58</td>
<td>$s$</td>
</tr>
</tbody>
</table>

(c) Find the value of $r$ and the value of $s$, giving your answers to 2 decimal places.

The bank manager will test, at the 5% level of significance, whether or not the data can be modelled by a Poisson distribution.

(d) Calculate the test statistic and state the conclusion for this test. State clearly the degrees of freedom and the hypotheses used in the test.

(Total for Question 4 is 11 marks)
A level Further Mathematics SAMs Paper 3B/4B Q2 (2.1, 2.2, 2.3)

A call centre routes incoming telephone calls to agents who have specialist knowledge to deal with the call. The probability of a caller, chosen at random, being connected to the wrong agent is \( p \).

The probability of at least 1 call in 5 consecutive calls being connected to the wrong agent is 0.049.

The call centre receives 1000 calls each day.

(a) Find the mean and variance of the number of wrongly connected calls a day. \( \text{(7 marks)} \)

(b) Use a Poisson approximation to find, to 3 decimal places, the probability that more than 6 calls each day are connected to the wrong agent. \( \text{(2 marks)} \)

(c) Explain why the approximation used in part (b) is valid. \( \text{(2 marks)} \)

The probability that more than 6 calls each day are connected to the wrong agent using the binomial distribution is 0.8711 to 4 decimal places.

(d) Comment on the accuracy of your answer in part (b). \( \text{(1 mark)} \)

(Total for Question 2 is 12 marks)

A level Further Mathematics SAMs Paper 3B/4B Q4 (2.2, 5.1)

A random sample of 100 observations is taken from a Poisson distribution with mean 2.3.

Estimate the probability that the mean of the sample is greater than 2.5.

(Total for Question 4 is 4 marks)
3. Geometric and negative binomial distributions

Structure and calculation

| 3.1  | Geometric and negative binomial distributions. |
| 3.2  | Mean and variance of a geometric distribution with parameter $p$. |
| 3.3  | Mean and variance of negative binomial distribution with $\text{P}(X = x) = \binom{x-1}{r-1} p^r (1-p)^{(x-r)}$ |

Sample questions

A level Further Mathematics SAMs Paper 3B/4B Q5 (3.1, 3.3)

The probability of Richard winning a prize in a game at the fair is 0.15. Richard plays a number of games.

(a) Find the probability of Richard winning his second prize on his 8th game, (2)

(b) State two assumptions that have to be made, for the model used in part (a) to be valid. (2)

Mary plays the same game, but has a different probability of winning a prize. She plays until she has won $r$ prizes. The random variable $G$ represents the total number of games Mary plays.

(c) Given that the mean and standard deviation of $G$ are 18 and 6 respectively, determine whether Richard or Mary has the greater probability of winning a prize in a game. (4)

(Total for Question 5 is 8 marks)

A level Further Mathematics SAMs Paper 3B/4B Q7 (3.1, 4.2, 8.1)

Sam and Tessa are testing a spinner to see if the probability, $p$, of it landing on red is less than $\frac{1}{5}$. They both use a 10% significance level.

Sam decides to spin the spinner 20 times and record the number of times it lands on red.

(a) Find the critical region for Sam’s test. (2)

(b) Write down the size of Sam’s test. (1)

Tessa decides to spin the spinner until it lands on red and she records the number of spins.

(c) Find the critical region for Tessa’s test. (6)

(d) Find the size of Tessa’s test. (1)

(e) (i) Show that the power function for Sam’s test is given by $(1-p)^9 (1 + 19p)$ (1)

(ii) Find the power function for Tessa’s test. (4)

(f) With reference to parts (b), (d) and (e), state, giving your reasons, whether you would recommend Sam’s test or Tessa’s test when $p = 0.15$ (4)
3. Geometric and negative binomial distributions

(Total for Question 7 is 18 marks)
4. Hypothesis Testing

Structure and calculation

4.1 Extend ideas of hypothesis tests to test for the mean of a Poisson distribution.
4.2 Extend hypothesis testing to test for the parameter $p$ of a geometric distribution.

Sample questions

A level Further Mathematics SAMs Paper 3B/4B Q1 (4.1)

Bacteria are randomly distributed in a river at a rate of 5 per litre of water. A new factory opens and a scientist claims it is polluting the river with bacteria. He takes a sample of 0.5 litres of water from the river near the factory and finds that it contains 7 bacteria.

Stating your hypotheses clearly test, at the 5% level of significance, whether there is evidence that the level of pollution has increased.

(Total for Question 1 is 5 marks)

A level Further Mathematics SAMs Paper 3B/4B Q7 (3.1, 4.2, 8.1)

Sam and Tessa are testing a spinner to see if the probability, $p$, of it landing on red is less than $\frac{1}{5}$.

They both use a 10% significance level.

Sam decides to spin the spinner 20 times and record the number of times it lands on red.

(a) Find the critical region for Sam’s test.

(b) Write down the size of Sam’s test.

Tessa decides to spin the spinner until it lands on red and she records the number of spins.

(c) Find the critical region for Tessa’s test.

(d) Find the size of Tessa’s test.

(e) (i) Show that the power function for Sam’s test is given by

$$ (1 - p)^{19} (1 + 19p) $$

(ii) Find the power function for Tessa’s test.

(f) With reference to parts (b), (d) and (e), state, giving your reasons, whether you would recommend Sam’s test or Tessa’s test when $p = 0.15$

(Total for Question 7 is 18 marks)
5. Central Limit Theorem

Structure and calculation

5.1 Applications of the Central Limit Theorem to other distributions.

Sample questions

A level Further Mathematics SAMs Paper 3B/4B Q4 (2.2, 5.1)

A random sample of 100 observations is taken from a Poisson distribution with mean 2.3. Estimate the probability that the mean of the sample is greater than 2.5.

(Total for Question 4 is 4 marks)
6. Chi Squared Tests

6. Chi Squared Tests

Structure and calculation

### 6.1 Goodness of fit tests and Contingency tables.

**The null and alternative hypotheses.**

The use of \( \sum \frac{(O_i - E_i)^2}{E_i} \) as approximate \( \chi^2 \) statistic.

**Degrees of freedom.**

#### Sample questions

**AS Further Mathematics SAMs Paper 2G Q1 (6.1)**

A university foreign language department carried out a survey of prospective students to find out which of three languages they were most interested in studying.

A random sample of 150 prospective students gave the following results.

<table>
<thead>
<tr>
<th>Gender</th>
<th>French</th>
<th>Spanish</th>
<th>Mandarin</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>23</td>
<td>22</td>
<td>20</td>
</tr>
<tr>
<td>Female</td>
<td>38</td>
<td>32</td>
<td>15</td>
</tr>
</tbody>
</table>

A test is carried out at the 1% level of significance to determine whether or not there is an association between gender and choice of language.

(a) State the null hypothesis for this test.  

(b) Show that the expected frequency for females choosing Spanish is 30.6.  

(c) Calculate the test statistic for this test, stating the expected frequencies you have used.  

(d) State whether or not the null hypothesis is rejected. Justify your answer.  

(e) Explain whether or not the null hypothesis would be rejected if the test was carried out at the 10% level of significance.  

(Total for Question 1 is 8 marks)
The discrete random variable $X$ follows a Poisson distribution with mean 1.4

(a) Write down the value of

(i) $P(X = 1)$,

(ii) $P(X \leq 4)$.

The manager of a bank recorded the number of mortgages approved each week over a 40 week period.

<table>
<thead>
<tr>
<th>Number of mortgages approved</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>10</td>
<td>16</td>
<td>7</td>
<td>4</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

(b) Show that the mean number of mortgages approved over the 40 week period is 1.4.

The bank manager believes that the Poisson distribution may be a good model for the number of mortgages approved each week.

She uses a Poisson distribution with a mean of 1.4 to calculate expected frequencies as follows.

<table>
<thead>
<tr>
<th>Number of mortgages approved</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5 or more</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected frequency</td>
<td>9.86</td>
<td>$r$</td>
<td>9.67</td>
<td>4.51</td>
<td>1.58</td>
<td>$s$</td>
</tr>
</tbody>
</table>

(c) Find the value of $r$ and the value of $s$, giving your answers to 2 decimal places.

The bank manager will test, at the 5% level of significance, whether or not the data can be modelled by a Poisson distribution.

(d) Calculate the test statistic and state the conclusion for this test. State clearly the degrees of freedom and the hypotheses used in the test.

(Total for Question 4 is 11 marks)
Bags of £1 coins are paid into a bank. Each bag contains 20 coins.

The bank manager believes that 5% of the £1 coins paid into the bank are fakes.
He decides to use the distribution $X \sim B(20, 0.05)$ to model the random variable $X$, the number of fake £1 coins in each bag.

The bank manager checks a random sample of 150 bags of £1 coins and records the number of fake coins found in each bag. His results are summarised in Table 1. He then calculates some of the expected frequencies, correct to 1 decimal place.

<table>
<thead>
<tr>
<th>Number of fake coins in each bag</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4 or more</th>
</tr>
</thead>
<tbody>
<tr>
<td>Observed frequency</td>
<td>43</td>
<td>62</td>
<td>26</td>
<td>13</td>
<td>6</td>
</tr>
<tr>
<td>Expected frequency</td>
<td>53.8</td>
<td>56.6</td>
<td>8.9</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table 1**

(a) Carry out a hypothesis test, at the 5% significance level, to see if the data supports the bank manager’s statistical model. State your hypotheses clearly.

The assistant manager thinks that a binomial distribution is a good model but suggests that the proportion of fake coins is higher than 5%. She calculates the actual proportion of fake coins in the sample and uses this value to carry out a new hypothesis test on the data. Her expected frequencies are shown in Table 2.

<table>
<thead>
<tr>
<th>Number of fake coins in each bag</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4 or more</th>
</tr>
</thead>
<tbody>
<tr>
<td>Observed frequency</td>
<td>43</td>
<td>62</td>
<td>26</td>
<td>13</td>
<td>6</td>
</tr>
<tr>
<td>Expected frequency</td>
<td>44.5</td>
<td>55.7</td>
<td>33.2</td>
<td>12.5</td>
<td>4.1</td>
</tr>
</tbody>
</table>

**Table 2**

(b) Explain why there are 2 degrees of freedom in this case.

(c) Given that she obtains a $\chi^2$ test statistic of 2.67, test the assistant manager’s hypothesis that the binomial distribution is a good model for the number of fake coins in each bag. Use a 5% level of significance and state your hypotheses clearly.

(Total for Question 3 is 14 marks)
## 7. Probability generating functions

### Structure and calculation

| 7.1 | Definitions, derivations and applications. Use of the probability generating function for the negative binomial, geometric, binomial and Poisson distributions. |
| 7.2 | Use to find the mean and variance. |
| 7.3 | Probability generating function of the sum of independent random variables. |

### Sample questions

**A level Further Mathematics SAMs Paper 3B/4B Q6 (1.1, 7.1, 7.2)**

The probability generating function of the discrete random variable $X$ is given by

$$G_X(t) = k \left(3 + t + 2t^2\right)^2$$

(a) Show that $k = \frac{1}{36}$  

(b) Find $P(X = 3)$  

(c) Show that $\text{Var}(X) = \frac{29}{18}$  

(d) Find the probability generating function of $2X + 1$

(Total for Question 6 is 14 marks)
8. Quality of tests

Structure and calculation

8.1 Type I and Type II errors. Size and Power of Test. The power function.

Sample questions

A level Further Mathematics SAMs Paper 3B/4B Q7 (3.1, 4.2, 8.1)

Sam and Tessa are testing a spinner to see if the probability, \( p \), of it landing on red is less than \( \frac{1}{5} \). They both use a 10% significance level.

Sam decides to spin the spinner 20 times and record the number of times it lands on red.

(a) Find the critical region for Sam’s test.

(b) Write down the size of Sam’s test.

Tessa decides to spin the spinner until it lands on red and she records the number of spins.

(c) Find the critical region for Tessa’s test.

(d) Find the size of Tessa’s test.

(e) (i) Show that the power function for Sam’s test is given by

\[
(1 - p)^{19} (1 + 19p)
\]

(ii) Find the power function for Tessa’s test.

(f) With reference to parts (b), (d) and (e), state, giving your reasons, whether you would recommend Sam’s test or Tessa’s test when \( p = 0.15 \)

(Total for Question 7 is 18 marks)
1. Linear regression

Structure and calculation

### 1.1 Least squares linear regression. The concept of residuals and minimising the sum of squares of residuals.

### 1.2 Residuals. The residual sum of squares (RSS).

#### Sample questions

**AS Further Mathematics SAMs Paper 2G Q7 (1.1, 1.2, 3.1)**

A scientist wants to develop a model to describe the relationship between the average daily temperature, $x \, ^\circ C$, and a household’s daily energy consumption, $y \, kWh$, in winter.

A random sample of the average temperature and energy consumption are taken from 10 winter days and are summarised below.

\[
\begin{align*}
\sum x &= 12 & \sum x^2 &= 24.76 & \sum y &= 251 & \sum y^2 &= 6341 & \sum xy &= 284.8 \\
S_{xx} &= 10.36 & S_{xy} &= 40.9
\end{align*}
\]

(a) Find the product moment correlation coefficient between $y$ and $x$.  

(b) Find the equation of the regression line of $y$ on $x$ in the form $y = a + bx$. 

(c) Use your equation to estimate the daily energy consumption when the average daily temperature is $2 \, ^\circ C$. 

(d) Calculate the residual sum of squares (RSS). 

The table shows the residual for each value of $x$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>–0.4</th>
<th>–0.2</th>
<th>0.3</th>
<th>0.8</th>
<th>1.1</th>
<th>1.4</th>
<th>1.8</th>
<th>2.1</th>
<th>2.5</th>
<th>2.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Residual</td>
<td>–0.63</td>
<td>–0.32</td>
<td>–0.52</td>
<td>–0.73</td>
<td>0.74</td>
<td>2.22</td>
<td>1.84</td>
<td>0.32</td>
<td>$f$</td>
<td>–1.88</td>
</tr>
</tbody>
</table>

(e) Find the value of $f$.  

(f) By considering the signs of the residuals, explain whether or not the linear regression model is a suitable model for these data.  

(Total for Question 7 is 11 marks)
A level Further Mathematics SAMs Paper 4E Q6 (1.1, 1.2)

A random sample of 10 female pigs was taken. The number of piglets, $x$, born to each female pig and their average weight at birth, $m$ kg, was recorded. The results were as follows:

<table>
<thead>
<tr>
<th>Number of piglets, $x$</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average weight at birth, $m$ kg</td>
<td>1.50</td>
<td>1.20</td>
<td>1.40</td>
<td>1.40</td>
<td>1.23</td>
<td>1.30</td>
<td>1.20</td>
<td>1.15</td>
<td>1.25</td>
<td>1.15</td>
</tr>
</tbody>
</table>

(You may use $S_{xx} = 82.5$ and $S_{mm} = 0.12756$ and $S_{xm} = -2.29$)

(a) Find the equation of the regression line of $m$ on $x$ in the form $m = a + bx$ as a model for these results.

(b) Show that the residual sum of squares (RSS) is 0.064 to 3 decimal places.

(c) Calculate the residual values.

(d) Write down the outlier.

(e) (i) Comment on the validity of ignoring this outlier.

   (ii) Ignoring the outlier, produce another model.

   (iii) Use this model to estimate the average weight at birth if $x = 15$

   (iv) Comment, giving a reason, on the reliability of your estimate.

(Total for Question 6 is 12 marks)
2. Continuous probability distributions

Structure and calculation

2.1 The concept of a continuous random variable. The probability density function and the cumulative distribution function for a continuous random variable.

2.2 Relationship between probability density and cumulative distribution functions.

2.3 Mean and variance of continuous random variables. Extension of expected value function to include \( E(g(X)) \). Mode, median and percentiles of continuous random variables. Idea of skewness.

2.4 The continuous uniform (rectangular) distribution.

Sample questions

AS Further Mathematics SAMs Paper 2G Q6 (2.1, 2.3)

The continuous random variable \( X \) has probability density function

\[
f(x) = \begin{cases} 
\frac{1}{18} (11 - 2x) & 1 \leq x \leq 4 \\
0 & \text{otherwise.}
\end{cases}
\]

(a) Find \( P(X < 3) \) \hspace{2cm} (2)

(b) State, giving a reason, whether the upper quartile of \( X \) is greater than 3, less than 3 or equal to 3. \hspace{2cm} (1)

Given that \( E(X) = \frac{9}{4} \),

(c) use algebraic integration to find \( \text{Var}(X) \). \hspace{2cm} (3)

The cumulative distribution function of \( X \) is given by

\[
F(x) = \begin{cases} 
0 & x < 1 \\
\frac{1}{18} (11 - 2x) & 1 \leq x \leq 4 \\
1 & \text{otherwise.}
\end{cases}
\]

(d) Show that \( c = -10 \). \hspace{2cm} (2)

(e) Find the median of \( X \), giving your answer to 3 significant figures. \hspace{2cm} (3)

(Total for Question 6 is 11 marks)
2. Continuous probability distributions

AS Further Mathematics SAMs Paper 2G Q8 (2.3, 2.4)

The continuous random variable $X$ is uniformly distributed over the interval $[-3, 5]$.

(a) Sketch the probability density function $f(x)$ of $X$.  

(b) Find the value of $k$ such that $P(X < 2[k - X]) = 0.25$.  

(c) Use algebraic integration to show that $E(X^3) = 17$.  

(Total for Question 8 is 8 marks)

A level Further Mathematics SAMs Paper 4E Q1 (2.1, 2.2, 2.3, 2.4)

The three independent random variables $A$, $B$ and $C$ each have a continuous uniform distribution over the interval $[0, 5]$.

(a) Find the probability that $A$, $B$ and $C$ are all greater than 3.  

The random variable $Y$ represents the maximum value of $A$, $B$ and $C$.  

The cumulative distribution function of $Y$ is

$$F(y) = \begin{cases} 
0 & y < 0 \\
\frac{y^3}{125} & 0 \leq y \leq 5 \\
1 & y > 5 
\end{cases}$$

(b) Using algebraic integration, show that $\text{Var}(Y) = 0.9375$  

(c) Find the mode of $Y$, giving a reason for your answer.  

(d) Describe the skewness of the distribution of $Y$. Give a reason for your answer.  

(e) Find the value of $k$ such that $P(k < Y < 2k) = 0.189$  

(Total for Question 1 is 13 marks)
3. Correlation

3. Correlation

Structure and calculation

<table>
<thead>
<tr>
<th>3.1</th>
<th>Use of formulae to calculate the product moment correlation.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Knowledge of conditions for the use of the product moment correlation.</td>
</tr>
<tr>
<td></td>
<td>A knowledge of effects of coding.</td>
</tr>
</tbody>
</table>

| 3.2 | Spearman’s rank correlation coefficient, its use and interpretation. |
| 3.3 | Testing the hypothesis that a correlation is zero using either Spearman’s rank correlation or the product moment correlation coefficient. |

Sample questions

AS Further Mathematics SAMs Paper 2G Q7 (1.1, 1.2, 3.1)

A scientist wants to develop a model to describe the relationship between the average daily temperature, \(x\) °C, and a household’s daily energy consumption, \(y\) kWh, in winter.

A random sample of the average temperature and energy consumption are taken from 10 winter days and are summarised below.

\[
\begin{align*}
\sum x &= 12 & \sum x^2 &= 24.76 & \sum y &= 251 & \sum y^2 &= 6341 & \sum xy &= 284.8 \\
S_{xx} &= 10.36 & S_{xy} &= 40.9
\end{align*}
\]

(a) Find the product moment correlation coefficient between \(y\) and \(x\).  

(b) Find the equation of the regression line of \(y\) on \(x\) in the form \(y = a + bx\). 

(c) Use your equation to estimate the daily energy consumption when the average daily temperature is 2 °C. 

(d) Calculate the residual sum of squares (RSS). 

The table shows the residual for each value of \(x\).

<table>
<thead>
<tr>
<th>(x)</th>
<th>-0.4</th>
<th>-0.2</th>
<th>0.3</th>
<th>0.8</th>
<th>1.1</th>
<th>1.4</th>
<th>1.8</th>
<th>2.1</th>
<th>2.5</th>
<th>2.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Residual</td>
<td>-0.63</td>
<td>-0.32</td>
<td>-0.52</td>
<td>-0.73</td>
<td>0.74</td>
<td>2.22</td>
<td>1.84</td>
<td>0.32</td>
<td>(f)</td>
<td>-1.88</td>
</tr>
</tbody>
</table>

(e) Find the value of \(f\). 

(f) By considering the signs of the residuals, explain whether or not the linear regression model is a suitable model for these data. 

(Total for Question 7 is 11 marks)
3. Correlation

A level Further Mathematics SAMs Paper 4E Q2 (3.2, 3.3)

A researcher claims that, at a river bend, the water gradually gets deeper as the distance from the inner bank increases. He measures the distance from the inner bank, \( b \) cm, and the depth of a river, \( s \) cm, at 7 positions. The results are shown in the table below.

<table>
<thead>
<tr>
<th>Position</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance from inner bank ( b ) cm</td>
<td>100</td>
<td>200</td>
<td>300</td>
<td>400</td>
<td>500</td>
<td>600</td>
<td>700</td>
</tr>
<tr>
<td>Depth ( s ) cm</td>
<td>60</td>
<td>75</td>
<td>85</td>
<td>76</td>
<td>110</td>
<td>120</td>
<td>104</td>
</tr>
</tbody>
</table>

The Spearman’s rank correlation coefficient between \( b \) and \( s \) is \( \frac{6}{7} \).

(a) Stating your hypotheses clearly, test whether or not the data provides support for the researcher’s claim. Use a 1% level of significance.

(b) Without re-calculating the correlation coefficient, explain how the Spearman’s rank correlation coefficient would change if

(i) the depth for G is 109 instead of 104

(ii) an extra value H with distance from the inner bank of 800 cm and depth 130 cm is included.

The researcher decided to collect extra data and found that there were now many tied ranks.

(c) Describe how you would find the correlation with many tied ranks.

(Total for Question 2 is 9 marks)
3. Correlation

A level Further Mathematics SAMs Paper 4E Q7 (3.1)

Over a period of time, researchers took 10 blood samples from one patient with a blood disease. For each sample, they measured the levels of serum magnesium, \( s \) mg/dl, in the blood and the corresponding level of the disease protein, \( d \) mg/dl. One of the researchers coded the data for each sample using \( x = 10s \) and \( y = 10(d - 9) \) but spilt ink over his work.

The following summary statistics and unfinished scatter diagram are the only remaining information.

\[
d^2 = 1081.74 \quad S_{ds} = 59.524
\]

and

\[
y = 64 \quad S_{xx} = 2658.9
\]

(a) Use the formula for \( S_{xx} \) to show that \( S_{ds} = 26.589 \)

(b) Find the value of the product moment correlation coefficient between \( s \) and \( d \).

(c) With reference to the unfinished scatter diagram, comment on your result in part (b).

(Total for Question 7 is 8 marks)
4. Combinations of random variables

Structure and calculation

4.1 Distribution of linear combinations of independent Normal random variables.

Sample questions

A Level Further Mathematics SAMs Paper 4E Q5 (4.1)

Scaffolding poles come in two sizes, long and short. The length $L$ of a long pole has the normal distribution $\text{N}(19.6, 0.6^2)$. The length $S$ of a short pole has the normal distribution $\text{N}(4.8, 0.3^2)$. The random variables $L$ and $S$ are independent.

A long pole and a short pole are selected at random.

(a) Find the probability that the length of the long pole is more than 4 times the length of the short pole. Show your working clearly.

(b) Find the distribution of $T$.

(c) Find $P(|L - T| < 0.2)$

(Total for Question 5 is 13 marks)
5. Estimation, confidence intervals and tests using a normal distribution

Structure and calculation

<table>
<thead>
<tr>
<th>Section</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1</td>
<td>Concepts of standard error, estimator, bias. Quality of estimators.</td>
</tr>
<tr>
<td>5.2</td>
<td>Concept of a confidence interval and its interpretation.</td>
</tr>
<tr>
<td>5.3</td>
<td>Confidence limits for a Normal mean, with variance known.</td>
</tr>
<tr>
<td>5.4</td>
<td>Hypothesis test for the difference between the means of two Normal distributions with variances known.</td>
</tr>
<tr>
<td>5.5</td>
<td>Use of large sample results to extend to the case in which the population variances are unknown.</td>
</tr>
</tbody>
</table>

Sample questions

A level Further Mathematics SAMs Paper 4E Q3 (5.2, 6.1, 7.1)

A nutritionist studied the levels of cholesterol, $X$ mg/cm$^3$, of male students at a large college. She assumed that $X$ was distributed $N(\mu, \sigma^2)$ and examined a random sample of 25 male students. Using this sample she obtained unbiased estimates of $\mu$ and $\sigma^2$ as $\hat{\mu}$ and $\hat{\sigma}^2$.

A 95% confidence interval for $\mu$ was found to be (1.128, 2.232)

(a) Show that $\hat{\sigma}^2 = 1.79$ (correct to 3 significant figures)  
(b) Obtain a 95% confidence interval for $\sigma^2$  

(Total for Question 3 is 7 marks)

A level Further Mathematics SAMs Paper 4E Q4 (5.2, 6.2, 7.3)

The times, $x$ seconds, taken by the competitors in the 100 m freestyle events at a school swimming gala are recorded. The following statistics are obtained from the data.

<table>
<thead>
<tr>
<th>No of competitors</th>
<th>Sample mean $\bar{x}$</th>
<th>$x^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Girls</td>
<td>8</td>
<td>83.1</td>
</tr>
<tr>
<td>Boys</td>
<td>7</td>
<td>88.9</td>
</tr>
</tbody>
</table>

Following the gala, a mother claims that girls are faster swimmers than boys. Assuming that the times taken by the competitors are two independent random samples from normal distributions,

(a) test, at the 10% level of significance, whether or not the variances of the two distributions are the same. State your hypotheses clearly.  
(b) Stating your hypotheses clearly, test the mother’s claim. Use a 5% level of significance.  

(Total for Question 4 is 13 marks)
6. Other Hypothesis Tests and confidence intervals

Structure and calculation

<table>
<thead>
<tr>
<th>6.1</th>
<th>Hypothesis test and confidence interval for the variance of a Normal distribution.</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.2</td>
<td>Hypothesis test that two independent random samples are from Normal populations with equal variances.</td>
</tr>
</tbody>
</table>

Sample questions

A level Further Mathematics SAMs Paper 4E Q3 (5.2, 6.1, 7.1)

A nutritionist studied the levels of cholesterol, X mg/cm³, of male students at a large college. She assumed that X was distributed N(μ, σ²) and examined a random sample of 25 male students. Using this sample she obtained unbiased estimates of μ and σ² as \( \hat{\mu} \) and \( \hat{\sigma}^2 \).

A 95% confidence interval for μ was found to be (1.128, 2.232)

(a) Show that \( \hat{\sigma}^2 = 1.79 \) (correct to 3 significant figures) (4)

(b) Obtain a 95% confidence interval for \( \sigma^2 \) (3)

(Total for Question 3 is 7 marks)

A level Further Mathematics SAMs Paper 4E Q4 (5.2, 6.2, 7.3)

The times, x seconds, taken by the competitors in the 100 m freestyle events at a school swimming gala are recorded. The following statistics are obtained from the data.

<table>
<thead>
<tr>
<th>No of competitors</th>
<th>Sample mean ( \bar{x} )</th>
<th>( x^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Girls</td>
<td>8</td>
<td>83.1</td>
</tr>
<tr>
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</tr>
</tbody>
</table>

Following the gala, a mother claims that girls are faster swimmers than boys. Assuming that the times taken by the competitors are two independent random samples from normal distributions,

(a) test, at the 10% level of significance, whether or not the variances of the two distributions are the same. State your hypotheses clearly. (7)

(b) Stating your hypotheses clearly, test the mother’s claim. Use a 5% level of significance. (6)

(Total for Question 4 is 13 marks)
7. Confidence intervals and tests using the $t$–distribution

Structure and calculation

<table>
<thead>
<tr>
<th>7.1</th>
<th>Hypothesis test and confidence interval for the mean of a Normal distribution with unknown variance.</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.2</td>
<td>Paired $t$-test.</td>
</tr>
<tr>
<td>7.3</td>
<td>Hypothesis test and confidence interval for the difference between two means from independent Normal distributions when the variances are equal but unknown. Use of the pooled estimate of variance.</td>
</tr>
</tbody>
</table>

Sample questions

A level Further Mathematics SAMs Paper 4E Q3 (5.2, 6.1, 7.1)

A nutritionist studied the levels of cholesterol, $X$ mg/cm$^3$, of male students at a large college. She assumed that $X$ was distributed $N(\mu, \sigma^2)$ and examined a random sample of 25 male students. Using this sample she obtained unbiased estimates of $\mu$ and $\sigma^2$ as $\bar{\mu}$ and $\bar{\sigma}^2$.

A 95% confidence interval for $\mu$ was found to be (1.128, 2.232)

(a) Show that $\bar{\sigma}^2 = 1.79$ (correct to 3 significant figures)  

(b) Obtain a 95% confidence interval for $\sigma^2$  

(Total for Question 3 is 7 marks)

A level Further Mathematics SAMs Paper 4E Q4 (5.2, 6.2, 7.3)

The times, $x$ seconds, taken by the competitors in the 100 m freestyle events at a school swimming gala are recorded. The following statistics are obtained from the data.

<table>
<thead>
<tr>
<th>No of competitors</th>
<th>Sample mean $\bar{x}$</th>
<th>$x^2$</th>
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Following the gala, a mother claims that girls are faster swimmers than boys. Assuming that the times taken by the competitors are two independent random samples from normal distributions,

(a) test, at the 10% level of significance, whether or not the variances of the two distributions are the same. State your hypotheses clearly.

(b) Stating your hypotheses clearly, test the mother’s claim. Use a 5% level of significance.  

(Total for Question 4 is 13 marks)
1. Momentum and impulse

Further Mechanics 1

1. Momentum and impulse

Structure and calculation

| 1.1 | Momentum and impulse. The impulse-momentum principle. The principle of conservation of momentum applied to two spheres colliding directly. |
| 1.2 | Momentum as a vector. The impulse-momentum principle in vector form. |

Sample questions

AS Further Mathematics SAMs Paper 2J Q4 (1.1, 4.1, 4.2, AS Pure Mathematics 2.4)

A particle $P$ of mass $3m$ is moving in a straight line on a smooth horizontal table.

A particle $Q$ of mass $m$ is moving in the opposite direction to $P$ along the same straight line. The particles collide directly. Immediately before the collision the speed of $P$ is $u$ and the speed of $Q$ is $2u$. The velocities of $P$ and $Q$ immediately after the collision, measured in the direction of motion of $P$ before the collision, are $v$ and $w$ respectively. The coefficient of restitution between $P$ and $Q$ is $e$.

(a) Find an expression for $v$ in terms of $u$ and $e$. \(6\)

Given that the direction of motion of $P$ is changed by the collision,

(a) find the range of possible values of $e$. \(2\)

(c) Show that $w = \frac{u}{4} (1 + 9e)$. \(2\)

Following the collision with $P$, the particle $Q$ then collides with and rebounds from a fixed vertical wall which is perpendicular to the direction of motion of $Q$. The coefficient of restitution between $Q$ and the wall is $f$.

Given that $e = \frac{1}{3}$, and that $P$ and $Q$ collide again in the subsequent motion,

(d) find the range of possible values of $f$. \(6\)

(Total for Question 4 is 16 marks)
1. Momentum and impulse

A level Further Mathematics SAMs Paper 3C/4C Q1 (1.2, 2.1)

A particle $P$ of mass 0.5 kg is moving with velocity $(4i + j) \text{ m s}^{-1}$ when it receives an impulse $(2i - j) \text{ N s}$.

Show that the kinetic energy gained by $P$ as a result of the impulse is 12 J.

(Total for Question 1 is 6 marks)

A level Further Mathematics SAMs Paper 3C/4C Q3 (1.1, 4.1, 4.2)

A particle of mass $m$ kg lies on a smooth horizontal surface.
Initially the particle is at rest at a point $O$ between two fixed parallel vertical walls.
The point $O$ is equidistant from the two walls and the walls are 4 m apart.
At time $t = 0$ the particle is projected from $O$ with speed $u \text{ m s}^{-1}$ in a direction perpendicular to the walls.
The coefficient of restitution between the particle and each wall is $\frac{3}{4}$.
The magnitude of the impulse on the particle due to the first impact with a wall is $\lambda mu \text{ N s}$.

(a) Find the value of $\lambda$.

The particle returns to $O$, having bounced off each wall once, at time $t = 7$ seconds.
(b) Find the value of $u$.

(Total for Question 3 is 8 marks)
A level Further Mathematics SAMs Paper 3C/4C Q6 (1.2, 5.1, A level Further Mathematics Core pure Mathematics 6.3)

[In this question \( \mathbf{i} \) and \( \mathbf{j} \) are perpendicular unit vectors in a horizontal plane.]

A smooth uniform sphere \( A \) has mass \( 2m \) kg and another smooth uniform sphere \( B \), with the same radius as \( A \), has mass \( 3m \) kg.

The spheres are moving on a smooth horizontal plane when they collide obliquely.

Immediately before the collision the velocity of \( A \) is \((3 \mathbf{i} + 3 \mathbf{j}) \) m s\(^{-1}\) and the velocity of \( B \) is \((-5 \mathbf{i} + 2 \mathbf{j}) \) m s\(^{-1}\).

At the instant of collision, the line joining the centres of the spheres is parallel to \( \mathbf{i} \).

The coefficient of restitution between the spheres is \( \frac{1}{4} \).

(a) Find the velocity of \( B \) immediately after the collision.

(b) Find, to the nearest degree, the size of the angle through which the direction of motion of \( B \) is deflected as a result of the collision.

(Total for Question 6 is 9 marks)

A level Further Mathematics SAMs Paper 3C/4C Q8 (1.1, 2.1, 4.1)

A particle \( P \) of mass \( 2m \) and a particle \( Q \) of mass \( 5m \) are moving along the same straight line on a smooth horizontal plane.

They are moving in opposite directions towards each other and collide directly.

Immediately before the collision the speed of \( P \) is \( 2u \) and the speed of \( Q \) is \( u \).

The direction of motion of \( Q \) is reversed by the collision.

The coefficient of restitution between \( P \) and \( Q \) is \( e \).

(a) Find the range of possible values of \( e \).

Given that \( e = \frac{1}{3} \)

(b) show that the kinetic energy lost in the collision is \( \frac{40mu^2}{7} \).

(c) Without doing any further calculation, state how the amount of kinetic energy lost in the collision would change if \( e > \frac{1}{3} \).

(Total for Question 8 is 14 marks)
2. Work, energy and power

Structure and calculation

2.1 Kinetic and potential energy, work and power. The work-energy principle. The principle of conservation of mechanical energy.

Sample questions

AS Further Mathematics SAMs Paper 2J Q2 (2.1)

A small stone of mass 0.5 kg is thrown vertically upwards from a point A with an initial speed of 25 m s\(^{-1}\). The stone first comes to instantaneous rest at the point B which is 20 m vertically above the point A. As the stone moves it is subject to air resistance. The stone is modelled as a particle.

(a) Find the energy lost due to air resistance by the stone, as it moves from A to B.

The air resistance is modelled as a constant force of magnitude \(R\) newtons.

(b) Find the value of \(R\).

(c) State how the model for air resistance could be refined to make it more realistic.

(Total for Question 2 is 6 marks)

AS Further Mathematics SAMs Paper 2J Q3 (2.1)

[In this question use \(g = 10\) m s\(^{-2}\)]

A jogger of mass 60 kg runs along a straight horizontal road at a constant speed of 4 m s\(^{-1}\). The total resistance to the motion of the jogger is modelled as a constant force of magnitude 30 N.

(a) Find the rate at which the jogger is working.

The jogger now comes to a hill which is inclined to the horizontal at an angle \(\alpha\), where \(\sin \alpha = \frac{1}{\sqrt{3}}\). Because of the hill, the jogger reduces her speed to 3 m s\(^{-1}\) and maintains this constant speed as she runs up the hill. The total resistance to the motion of the jogger from non-gravitational forces continues to be modelled as a constant force of magnitude 30 N.

(b) Find the rate at which she has to work in order to run up the hill at 3 m s\(^{-1}\).

(Total for Question 3 is 8 marks)
A level Further Mathematics SAMs Paper 3C/4C Q1 (1.2, 2.1)

A particle $P$ of mass 0.5 kg is moving with velocity $(4\mathbf{i} + \mathbf{j})$ m s$^{-1}$ when it receives an impulse $(2\mathbf{i} - \mathbf{j})$ N s.

Show that the kinetic energy gained by $P$ as a result of the impulse is 12 J.

(Total for Question 1 is 6 marks)

A level Further Mathematics SAMs Paper 3C/4C Q2 (2.1)

A parcel of mass 5 kg is projected with speed 8 m s$^{-1}$ up a line of greatest slope of a fixed rough inclined ramp.

The ramp is inclined at angle $\alpha$ to the horizontal, where $\sin \alpha = \frac{1}{7}$

The parcel is projected from the point $A$ on the ramp and comes to instantaneous rest at the point $B$ on the ramp, where $AB = 14$ m.

The coefficient of friction between the parcel and the ramp is $\mu$.

In a model of the parcel’s motion, the parcel is treated as a particle.

(a) Use the work-energy principle to find the value of $\mu$.

(b) Suggest one way in which the model could be refined to make it more realistic.

(Total for Question 2 is 6 marks)
A level Further Mathematics SAMs Paper 3C/4C Q4 (2.1, 5.2)

Figure 1

Figure 1 represents the plan view of part of a horizontal floor, where \(AB\) and \(BC\) are perpendicular vertical walls. The floor and the walls are modelled as smooth.

A ball is projected along the floor towards \(AB\) with speed \(u\) m \(s^{-1}\) on a path at an angle of 60° to \(AB\). The ball hits \(AB\) and then hits \(BC\).

The ball is modelled as a particle.

The coefficient of restitution between the ball and wall \(AB\) is \(\frac{1}{\sqrt{3}}\)

The coefficient of restitution between the ball and wall \(BC\) is \(\frac{2}{\sqrt{5}}\)

(a) Show that, using this model, the final kinetic energy of the ball is 35% of the initial kinetic energy of the ball.

(b) In reality the floor and the walls may not be smooth. What effect will the model have had on the calculation of the percentage of kinetic energy remaining?

(Total for Question 4 is 9 marks)
A level Further Mathematics SAMs Paper 3C/4C Q5 (2.1)

A car of mass 600 kg is moving along a straight horizontal road.

At the instant when the speed of the car is \( v \, \text{m s}^{-1} \), the resistance to the motion of the car is modelled as a force of magnitude \((200 + 2v)\) N.

The engine of the car is working at a constant rate of 12 kW.

(a) Find the acceleration of the car at the instant when \( v = 20 \) (4)

Later on the car is moving up a straight road inclined at an angle \( \theta \) to the horizontal, where \( \sin \theta = \frac{1}{14} \)

At the instant when the speed of the car is \( v \, \text{m s}^{-1} \), the resistance to the motion of the car from non-gravitational forces is modelled as a force of magnitude \((200 + 2v)\) N.

The engine is again working at a constant rate of 12 kW.

At the instant when the car has speed \( w \, \text{m s}^{-1} \), the car is decelerating at 0.05 m s\(^{-2}\).

(b) Find the value of \( w \). (5)

(Total for Question 5 is 9 marks)

A level Further Mathematics SAMs Paper 3C/4C Q7 (2.1, 3.1, 3.2)

A particle \( P \) of mass \( m \) is attached to one end of a light elastic string of natural length \( a \) and modulus of elasticity \( 3mg \).

The other end of the string is attached to a fixed point \( O \) on a ceiling.

The particle hangs freely in equilibrium at a distance \( d \) vertically below \( O \).

(a) Show that \( d = \frac{4}{3}a \). (3)

The point \( A \) is vertically below \( O \) such that \( OA = 2a \).

The particle is held at rest at \( A \), then released and first comes to instantaneous rest at the point \( B \).

(b) Find, in terms of \( g \), the acceleration of \( P \) immediately after it is released from rest. (3)

(c) Find, in terms of \( g \) and \( a \), the maximum speed attained by \( P \) as it moves from \( A \) to \( B \). (5)

(d) Find, in terms of \( a \), the distance \( OB \). (3)

(Total for Question 7 is 14 marks)
A level Further Mathematics SAMs Paper 3C/4C Q8 (1.1, 2.1, 4.1)

A particle $P$ of mass $2m$ and a particle $Q$ of mass $5m$ are moving along the same straight line on a smooth horizontal plane.

They are moving in opposite directions towards each other and collide directly.

Immediately before the collision the speed of $P$ is $2u$ and the speed of $Q$ is $u$.

The direction of motion of $Q$ is reversed by the collision.

The coefficient of restitution between $P$ and $Q$ is $e$.

(a) Find the range of possible values of $e$. \hfill (8)

Given that $e = \frac{1}{3}$

(b) show that the kinetic energy lost in the collision is $\frac{40mu^2}{7}$. \hfill (5)

(c) Without doing any further calculation, state how the amount of kinetic energy lost in the collision would change if $e > \frac{1}{3}$ \hfill (1)

(Total for Question 8 is 14 marks)
3. Elastic strings and springs and elastic energy

Structure and calculation

3.1 Elastic strings and springs. Hooke’s law.
3.2 Energy stored in an elastic string or spring.

Sample questions

A level Further Mathematics SAMs Paper 3C/4C Q7 (2.1, 3.1, 3.2)

A particle $P$ of mass $m$ is attached to one end of a light elastic string of natural length $a$ and modulus of elasticity $3mg$.

The other end of the string is attached to a fixed point $O$ on a ceiling.

The particle hangs freely in equilibrium at a distance $d$ vertically below $O$.

(a) Show that $d = \frac{4}{3}a$.

The point $A$ is vertically below $O$ such that $OA = 2a$.

The particle is held at rest at $A$, then released and first comes to instantaneous rest at the point $B$.

(b) Find, in terms of $g$, the acceleration of $P$ immediately after it is released from rest.

(c) Find, in terms of $g$ and $a$, the maximum speed attained by $P$ as it moves from $A$ to $B$.

(d) Find, in terms of $a$, the distance $OB$.

(Total for Question 7 is 14 marks)
4. Elastic collisions in one dimension

Structure and calculation

4.2 Successive direct impacts of spheres and/or a sphere with a smooth plane surface.

Sample questions

AS Further Mathematics SAMs Paper 2J Q1 (4.1, 4.2)

A small ball of mass 0.1 kg is dropped from a point which is 2.4 m above a horizontal floor. The ball falls freely under gravity, strikes the floor and bounces to a height of 0.6 m above the floor. The ball is modelled as a particle.

(a) Show that the coefficient of restitution between the ball and the floor is 0.5. (6)
(b) Find the height reached by the ball above the floor after it bounces on the floor for the second time. (3)
(c) By considering your answer to part (b), describe the subsequent motion of the ball. (1)

(Total for Question 1 is 10 marks)
4. Elastic collisions in one dimension

AS Further Mathematics SAMs Paper 2J Q4 (1.1, 4.1, 4.2, AS Pure Mathematics 2.4)

A particle P of mass $3m$ is moving in a straight line on a smooth horizontal table.

A particle Q of mass $m$ is moving in the opposite direction to P along the same straight line. The particles collide directly. Immediately before the collision the speed of P is $u$ and the speed of Q is $2u$. The velocities of P and Q immediately after the collision, measured in the direction of motion of P before the collision, are $v$ and $w$ respectively. The coefficient of restitution between P and Q is $e$.

(a) Find an expression for $v$ in terms of $u$ and $e$.  

Given that the direction of motion of P is changed by the collision,

(b) find the range of possible values of $e$.  

(c) Show that $w = \frac{u}{4} (1 + 9e)$.  

Following the collision with P, the particle Q then collides with and rebounds from a fixed vertical wall which is perpendicular to the direction of motion of Q. The coefficient of restitution between Q and the wall is $f$.

Given that $e = \frac{5}{7}$, and that P and Q collide again in the subsequent motion,

(d) find the range of possible values of $f$.  

(Total for Question 4 is 16 marks)

A level Further Mathematics SAMs Paper 3C/4C Q3 (1.1, 4.1, 4.2)

A particle of mass $m$ kg lies on a smooth horizontal surface.

Initially the particle is at rest at a point O between two fixed parallel vertical walls.

The point O is equidistant from the two walls and the walls are 4 m apart.

At time $t = 0$ the particle is projected from O with speed $u$ m s$^{-1}$ in a direction perpendicular to the walls.

The coefficient of restitution between the particle and each wall is $\frac{3}{4}$.

The magnitude of the impulse on the particle due to the first impact with a wall is $\lambda mu$ N s.

(a) Find the value of $\lambda$.  

The particle returns to O, having bounced off each wall once, at time $t = 7$ seconds.

(b) Find the value of $u$.  

(Total for Question 3 is 8 marks)
4. Elastic collisions in one dimension

A level Further Mathematics SAMs Paper 3C/4C Q8 (1.1, 2.1, 4.1)

A particle $P$ of mass $2m$ and a particle $Q$ of mass $5m$ are moving along the same straight line on a smooth horizontal plane.

They are moving in opposite directions towards each other and collide directly. Immediately before the collision the speed of $P$ is $2u$ and the speed of $Q$ is $u$. The direction of motion of $Q$ is reversed by the collision. The coefficient of restitution between $P$ and $Q$ is $e$.

(a) Find the range of possible values of $e$. 

Given that $e = \frac{1}{3}$ 

(b) Show that the kinetic energy lost in the collision is $\frac{40mu^2}{7}$.

(c) Without doing any further calculation, state how the amount of kinetic energy lost in the collision would change if $e > \frac{1}{3}$.

(Total for Question 8 is 14 marks)
5. Elastic collisions in two dimensions

Structure and calculation

5.1 Applications of the Central Limit Theorem to other distributions.

Sample questions

A level Further Mathematics SAMs Paper 3C/4C Q4 (2.1, 5.2)

Figure 1

Figure 1 represents the plan view of part of a horizontal floor, where $AB$ and $BC$ are perpendicular vertical walls. The floor and the walls are modelled as smooth.

A ball is projected along the floor towards $AB$ with speed $u \text{ m s}^{-1}$ on a path at an angle of $60^\circ$ to $AB$. The ball hits $AB$ and then hits $BC$.

The ball is modelled as a particle.

The coefficient of restitution between the ball and wall $AB$ is $\frac{1}{\sqrt{3}}$.

The coefficient of restitution between the ball and wall $BC$ is $\frac{2}{\sqrt{5}}$.

(a) Show that, using this model, the final kinetic energy of the ball is 35% of the initial kinetic energy of the ball.

(b) In reality the floor and the walls may not be smooth. What effect will the model have had on the calculation of the percentage of kinetic energy remaining?

(Total for Question 4 is 9 marks)

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A level Further Mathematics SAMs Paper 3C/4C Q6 (1.2, 5.1, A level Further Mathematics Core pure Mathematics 6.3)

[In this question \( \mathbf{i} \) and \( \mathbf{j} \) are perpendicular unit vectors in a horizontal plane.]

A smooth uniform sphere \( A \) has mass \( 2m \) kg and another smooth uniform sphere \( B \), with the same radius as \( A \), has mass \( 3m \) kg.

The spheres are moving on a smooth horizontal plane when they collide obliquely.

Immediately before the collision the velocity of \( A \) is \((3\mathbf{i} + 3\mathbf{j}) \text{ m s}^{-1}\) and the velocity of \( B \) is \((-5\mathbf{i} + 2\mathbf{j}) \text{ m s}^{-1}\).

At the instant of collision, the line joining the centres of the spheres is parallel to \( \mathbf{i} \).

The coefficient of restitution between the spheres is \( \frac{1}{4} \)

(a) Find the velocity of \( B \) immediately after the collision.  

(b) Find, to the nearest degree, the size of the angle through which the direction of motion of \( B \) is deflected as a result of the collision.

(Total for Question 6 is 9 marks)
1. Motion in a circle

**Further Mechanics 2**

1. Motion in a circle

**Structure and calculation**

<table>
<thead>
<tr>
<th>1.1</th>
<th>Angular speed $v = r\omega$</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Uniform motion of a particle moving in a horizontal circle.</td>
</tr>
<tr>
<td></td>
<td>Radial acceleration in circular motion.</td>
</tr>
<tr>
<td></td>
<td>The forms $r\omega^2$ and $\frac{v^2}{r}$ are required.</td>
</tr>
</tbody>
</table>

| 1.2 | Motion of a particle in a vertical circle. Radial and tangential acceleration in circular motion.  |
|     | Kinetic and potential energy and the conservation of energy principle applied to motion in a vertical circle.  |

**Sample questions**

**AS Further Mathematics SAMs Paper 2J Q6 (1.1)**

A light inextensible string has length $7a$. One end of the string is attached to a fixed point $A$ and the other end of the string is attached to a fixed point $B$, with $A$ vertically above $B$ and $AB = 5a$.

A particle of mass $m$ is attached to a point $P$ on the string where $AP = 4a$. The particle moves in a horizontal circle with constant angular speed $\omega$, with both $AP$ and $BP$ taut.

(a) Show that

(i) the tension in $AP$ is $\frac{4m}{25} (9a\omega^2 + 5g)$,

(ii) the tension in $BP$ is $\frac{3m}{25} (16a\omega^2 - 5g)$.

(b) Show that

$$3\pi \sqrt{\frac{a}{5g}} < S < 8\pi \sqrt{\frac{a}{5g}}$$

(c) State how in your calculations you have used the assumption that the string is light.

(Total for Question 6 is 16 marks)
A level Further Mathematics SAMs Paper 4F Q2 (1.1)

A hollow right circular cone, of base diameter $4a$ and height $4a$ is fixed with its axis vertical and vertex $V$ downwards, as shown in Figure 1.

A particle of mass $m$ moves in a horizontal circle with centre $C$ on the rough inner surface of the cone with constant angular speed $\omega$.

The height of $C$ above $V$ is $3a$.

The coefficient of friction between the particle and the inner surface of the cone is $\frac{1}{4}$.

Find, in terms of $a$ and $g$, the greatest possible value of $\omega$.

(Total for Question 2 is 8 marks)
A small bead $B$ of mass $m$ is threaded on a circular hoop.
The hoop has centre $O$ and radius $a$ and is fixed in a vertical plane.
The bead is projected with speed $\sqrt{7/2}ga$ from the lowest point of the hoop.
The hoop is modelled as being smooth.
When the angle between $OB$ and the downward vertical is $\theta$, the speed of $B$ is $v$.

(a) Show that $v^2 = ga\left(\frac{3}{2} + 2\cos\theta\right)$

(b) Find the size of $\theta$ at the instant when the contact force between $B$ and the hoop is first zero.

(c) Give a reason why your answer to part (b) is not likely to be the actual value of $\theta$.

(d) Find the magnitude and direction of the acceleration of $B$ at the instant when $B$ is first at instantaneous rest.

(Total for Question 6 is 14 marks)
Sample questions

AS Further Mathematics SAMs Paper 2J Q7 (2.1, 2.2)

Figure 1

Figure 1 shows the shape and dimensions of a template $OPQRSTUV$ made from thin uniform metal.

$OP = 5$ m, $PQ = 2$ m, $QR = 1$ m, $RS = 1$ m, $TU = 2$ m, $UV = 1$ m, $VO = 3$ m.

Figure 1 also shows a coordinate system with $O$ as origin and the $x$-axis and $y$-axis along $OP$ and $OV$ respectively. The unit of length on both axes is the metre. The centre of mass of the template has coordinates $(\bar{x}, \bar{y})$.

(a) (i) Show that $\bar{y} = 1$,

(ii) Find the value of $\bar{x}$.

(7)

A new design requires the template to have its centre of mass at the point $(2.5, 1)$. In order to achieve this, two circular discs, each of radius $r$ metres, are removed from the template which is shown in Figure 1, to form a new template $L$. The centre of the first disc is $(0.5, 0.5)$ and the centre of the second disc is $(0.5, a)$ where $a$ is a constant.

(b) Find the value of $r$.

(4)

(c) (i) Explain how symmetry can be used to find the value of $a$.

(ii) Find the value of $a$.

(2)
2. Centres of mass of plane figures

The template $L$ is now freely suspended from the point $U$ and hangs in equilibrium.

(d) Find the size of the angle between the line $TU$ and the horizontal. \hspace{1cm} (3)

(Total for Question 7 is 16 marks)

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A level Further Mathematics SAMs Paper 4F Q5 (2.1, 2.2, 3.2)

A shop sign is modelled as a uniform rectangular lamina $ABCD$ with a semicircular lamina removed.

The semicircle has radius $a$, $BC = 4a$ and $CD = 2a$.

The centre of the semicircle is at the point $E$ on $AD$ such that $AE = d$, as shown in Figure 3.

(a) Show that the centre of mass of the sign is $\frac{44a}{3(16 - \pi)}$ from $AD$ \hspace{1cm} (4)

The sign is suspended using vertical ropes attached to the sign at $A$ and at $B$ and hangs in equilibrium with $AB$ horizontal.

The weight of the sign is $W$ and the ropes are modelled as light inextensible strings.

(b) Find, in terms of $W$ and $\pi$, the tension in the rope attached at $B$. \hspace{1cm} (2)

The rope attached at $B$ breaks and the sign hangs freely in equilibrium suspended from $A$, with $AD$ at an angle $\alpha$ to the downward vertical.

Given that $\tan \alpha = \frac{11}{18}$

(c) find $d$ in terms of $a$ and $\pi$. \hspace{1cm} (6)

(Total for Question 5 is 12 marks)
3. Further centres of mass

Structure and calculation

3.1 Centre of mass of uniform and non-uniform rigid bodies and composite bodies.
3.2 Equilibrium of rigid bodies under the action of coplanar forces.
3.3 Toppling and sliding of a rigid body on a rough plane.

Sample questions

A level Further Mathematics SAMs Paper 4F Q1 (3.1)

A flag pole is 15 m long.
The flag pole is non-uniform so that, at a distance $x$ metres from its base, the mass per
unit length of the flag pole, $m$ kg m$^{-1}$ is given by the formula $m = 10 \left(1 - \frac{x}{25}\right)$.
The flag pole is modelled as a rod.
(a) Show that the mass of the flag pole is 105 kg. (3)
(b) Find the distance of the centre of mass of the flag pole from its base. (4)

(Total for Question 1 is 7 marks)
A uniform solid cylinder has radius $2a$ and height $h$ ($h > a$).

A solid hemisphere of radius $a$ is removed from the cylinder to form the vessel $V$.

The plane face of the hemisphere coincides with the upper plane face of the cylinder.

The centre $O$ of the hemisphere is also the centre of the upper plane face of the cylinder, as shown in Figure 2.

(a) Show that the centre of mass of $V$ is \( \frac{3(8h^2 - a^2)}{8(6h - a)} \) from $O$.

The vessel $V$ is placed on a rough plane which is inclined at an angle $\phi$ to the horizontal.

The lower plane circular face of $V$ is in contact with the inclined plane.

Given that $h = 5a$, the plane is sufficiently rough to prevent $V$ from slipping and $V$ is on the point of toppling.

(b) find, to three significant figures, the size of the angle $\phi$.

(Total for Question 3 is 9 marks)
3. Further centres of mass

A level Further Mathematics SAMs Paper 4F Q5 (2.1, 2.2, 3.2)

A shop sign is modelled as a uniform rectangular lamina $ABCD$ with a semicircular lamina removed.

The semicircle has radius $a$, $BC = 4a$ and $CD = 2a$.

The centre of the semicircle is at the point $E$ on $AD$ such that $AE = d$, as shown in Figure 3.

(a) Show that the centre of mass of the sign is $\frac{44a}{3(16 - \pi)}$ from $AD$.

The sign is suspended using vertical ropes attached to the sign at $A$ and at $B$ and hangs in equilibrium with $AB$ horizontal.

The weight of the sign is $W$ and the ropes are modelled as light inextensible strings.

(b) Find, in terms of $W$ and $\pi$, the tension in the rope attached at $B$.

The rope attached at $B$ breaks and the sign hangs freely in equilibrium suspended from $A$, with $AD$ at an angle $\alpha$ to the downward vertical.

Given that $\tan \alpha = \frac{11}{18}$

(c) find $d$ in terms of $a$ and $\pi$.

(Total for Question 5 is 12 marks)
4. Further dynamics

Structure and calculation

| 4.1 | Newton’s laws of motion, for a particle moving in one dimension, when the applied force is variable. |
| 4.2 | Simple harmonic motion. Oscillations of a particle attached to the end of elastic string(s) or spring(s). Kinetic, potential and elastic energy in the context of SHM. |

Sample questions

A level Further Mathematics SAMs Paper 4F Q4 (4.1, 5.1, A level Mathematics Pure Mathematics 8.2, 8.6, 8.7)

A car of mass 500 kg moves along a straight horizontal road.

The engine of the car produces a constant driving force of 1800 N.

The car accelerates from rest from the fixed point $O$ at time $t = 0$ and at time $t$ seconds the car is $x$ metres from $O$, moving with speed $v$ m s$^{-1}$.

When the speed of the car is $v$ m s$^{-1}$, the resistance to the motion of the car has magnitude $2v^2$ N.

At time $T$ seconds, the car is at the point $A$, moving with speed 10 m s$^{-1}$.

(a) Show that $T = \frac{25}{6} \ln 2$  

(b) Show that the distance from $O$ to $A$ is $125 \ln \frac{9}{8}$ m.  

(Total for Question 4 is 11 marks)
A level Further Mathematics SAMs Paper 4F Q7 (4.2)

Two points $A$ and $B$ are 6 m apart on a smooth horizontal surface.

A light elastic string of natural length 2 m and modulus of elasticity 20 N, has one end attached to the point $A$.

A second light elastic string of natural length 2 m and modulus of elasticity 50 N, has one end attached to the point $B$.

A particle $P$ of mass 3.5 kg is attached to the free end of each string.

The particle $P$ is held at the point on $AB$ which is 2 m from $B$ and then released from rest.

In the subsequent motion both strings remain taut.

(a) Show that $P$ moves with simple harmonic motion about its equilibrium position. \hspace{1cm} (7)

(b) Find the maximum speed of $P$. \hspace{1cm} (2)

(c) Find the length of time within each oscillation for which $P$ is closer to $A$ than to $B$. \hspace{1cm} (5)

(Total for Question 7 is 14 marks)
5. Further kinematics

Structure and calculation

5.1 Kinematics of a particle moving in a straight line when the acceleration is a function of the time \((t)\) or velocity \((v)\).

Sample questions

AS Further Mathematics SAMs Paper 2J Q5 (5.1)

A particle \(P\) moves on the \(x\)-axis. At time \(t\) seconds the velocity of \(P\) is \(v\) m s\(^{-1}\) in the direction of \(x\) increasing, where

\[
v = (t - 2)(3t - 10), \quad t \geq 0.
\]

When \(t = 0\), \(P\) is at the origin \(O\).

(a) Find the acceleration of \(P\) at time \(t\) seconds.

(b) Find the total distance travelled by \(P\) in the first 2 seconds of its motion.

(c) Show that \(P\) never returns to \(O\), explaining your reasoning.

(Total for Question 5 is 8 marks)

A level Further Mathematics SAMs Paper 4F Q4 (4.1, 5.1, A level Mathematics Pure Mathematics 8.2, 8.6, 8.7)

A car of mass 500 kg moves along a straight horizontal road.

The engine of the car produces a constant driving force of 1800 N.

The car accelerates from rest from the fixed point \(O\) at time \(t = 0\) and at time \(t\) seconds the car is \(x\) metres from \(O\), moving with speed \(v\) m s\(^{-1}\).

When the speed of the car is \(v\) m s\(^{-1}\), the resistance to the motion of the car has magnitude \(2v^2\) N.

At time \(T\) seconds, the car is at the point \(A\), moving with speed 10 m s\(^{-1}\).

(a) Show that \(T = \frac{25}{6} \ln 2\)

(b) Show that the distance from \(O\) to \(A\) is \(125\ln \frac{9}{8}\) m.

(Total for Question 4 is 11 marks)
1. Algorithms and graph theory

Structure and calculation

1.1 The general ideas of algorithms and the implementation of an algorithm given by flow chart or text.
1.2 Bin packing, bubble sort and quick sort.
1.3 Use of the order of the nodes to determine whether a graph is Eulerian, semi-Eulerian or neither.
1.4 The planarity algorithm for planar graphs.

Sample questions

AS Further Mathematics SAMs Paper 2K Q4 (1.3)

(a) Explain why it is not possible to draw a graph with exactly 5 nodes with orders 1, 3, 4, 4 and 5. (1)

A connected graph has exactly 5 nodes and contains 18 arcs. The orders of the 5 nodes are $2^{2x-1}$, $2^2$, $x+1$, $2^{x+1}-3$ and $11-x$.

(b) (i) Calculate $x$.
(ii) State whether the graph is Eulerian, semi-Eulerian or neither. You must justify your answer. (6)

(c) Draw a graph which satisfies all of the following conditions:
- The graph has exactly 5 nodes.
- The nodes have orders 2, 2, 4, 4 and 4.
- The graph is not Eulerian. (2)

(Total for Question 4 is 9 marks)

A level Further Mathematics SAMs Paper 3D/4D Q1 (1.1, 1.2)

A list of $n$ numbers needs to be sorted into descending order starting at the left-hand end of the list.

(a) Describe how to carry out the first pass of a bubble sort on the numbers in the list. (2)

Bubble sort is a quadratic order algorithm.

A computer takes approximately 0.021 seconds to apply a bubble sort to a list of 2000 numbers.

(b) Estimate the time it would take the computer to apply a bubble sort to a list of 50 000 numbers. Make your method clear. (2)

(Total for Question 1 is 4 marks)
(a) Define what is meant by a **planar** graph.

(b) Starting at A, find a Hamiltonian cycle for the graph in Figure 1.

Arc AG is added to Figure 1 to create the graph shown in Figure 2.

(c) Use the planarity algorithm to determine whether the graph shown in Figure 2 is planar. You must make your working clear and justify your answer.

(Total for Question 2 is 7 marks)
Sample questions

AS Further Mathematics SAMs Paper 2K Q1 (2.2, 3.1)

Figure 1

[The total weight of the network is 189]

Figure 1 represents a network of pipes in a building. The number on each arc is the length, in metres, of the corresponding pipe.

(a) Use Dijkstra’s algorithm to find the shortest path from A to F. State the path and its length.  
(5)

On a particular day, Gabriel needs to check each pipe. A route of minimum length, which traverses each pipe at least once and which starts and finishes at A, needs to be found.

(b) Use an appropriate algorithm to find the pipes that will need to be traversed twice. You must make your method and working clear.  
(4)

(c) State the minimum length of Gabriel’s route.  
(1)

A new pipe, BG, is added to the network. A route of minimum length that traverses each pipe, including BG, needs to be found. The route must start and finish at A. Gabriel works out that the addition of the new pipe increases the length of the route by twice the length of BG.

(d) Calculate the length of BG. You must show your working.  
(2)

(Total for Question 1 is 12 marks)
A level Further Mathematics SAMs Paper 3D/4D Q3 (2.1, 3.2)

(a) Explain clearly the difference between the classical travelling salesperson problem and the practical travelling salesperson problem.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
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<tbody>
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</tbody>
</table>

The table shows the least distances, in km, by road between seven towns, A, B, C, D, E, F and G. The least distance between B and G is $x$ km, where $x > 25$.

Preety needs to visit each town at least once, starting and finishing at A. She wishes to minimise the total distance she travels.

(b) Starting by deleting B and all of its arcs, find a lower bound for Preety’s route.

Preety found the nearest neighbour routes from each of A and C. Given that the sum of the lengths of these routes is 331 km,

(c) find $x$, making your method clear.

(d) Write down the smallest interval that you can be confident contains the optimal length of Preety’s route. Give your answer as an inequality.

(Total for Question 3 is 11 marks)
The network in Figure 3 shows the roads linking a depot, D, and three collection points A, B and C. The number on each arc represents the length, in miles, of the corresponding road. The road from B to D is a one-way road, as indicated by the arrow.

(a) Explain clearly if Dijkstra’s algorithm can be used to find a route from D to A.

The initial distance and route tables for the network are given in the answer book.

(b) Use Floyd’s algorithm to find a table of least distances. You should show both the distance table and the route table after each iteration.

(c) Explain how the final route table can be used to find the shortest route from D to B. State this route.

There are items to collect at A, B and C. A van will leave D to make these collections in any order and then return to D. A minimum route is required.

Using the final distance table and the Nearest Neighbour algorithm starting at D,

(d) find a minimum route and state its length.

Floyd’s algorithm and Dijkstra’s algorithm are applied to a network. Each will find the shortest distance between vertices of the network.

(e) Describe how the results of these algorithms differ.

(Total for Question 4 is 14 marks)
3. Algorithms on graphs II

Structure and calculation

<table>
<thead>
<tr>
<th>3.1</th>
<th>Algorithm for finding the shortest route around a network, travelling along every edge at least once and ending at the start vertex (The Route Inspection Algorithm).</th>
</tr>
</thead>
</table>
| 3.2 | The practical and classical Travelling Salesman problems.  
The classical problem for complete graphs satisfying the triangle inequality.  
Determination of upper and lower bounds using minimum spanning tree methods.  
The nearest neighbour algorithm. |

Sample questions

AS Further Mathematics SAMs Paper 2K Q1 (2.2, 3.1)

Figure 1

*The total weight of the network is 189*

Figure 1 represents a network of pipes in a building. The number on each arc is the length, in metres, of the corresponding pipe.

(a) Use Dijkstra’s algorithm to find the shortest path from A to F. State the path and its length.  
(5)

On a particular day, Gabriel needs to check each pipe. A route of minimum length, which traverses each pipe at least once and which starts and finishes at A, needs to be found.

(b) Use an appropriate algorithm to find the pipes that will need to be traversed twice.  
You must make your method and working clear.  
(4)

(c) State the minimum length of Gabriel’s route.  
(1)

A new pipe, BG, is added to the network. A route of minimum length that traverses each pipe, including BG, needs to be found. The route must start and finish at A. Gabriel works out that the addition of the new pipe increases the length of the route by twice the length of BG.

(d) Calculate the length of BG. You must show your working.  
(2)

(Total for Question 1 is 12 marks)
(a) Explain clearly the difference between the classical travelling salesperson problem and the practical travelling salesperson problem.

(b) Starting by deleting B and all of its arcs, find a lower bound for Preety’s route.

(c) find \( x \), making your method clear.

(d) Write down the smallest interval that you can be confident contains the optimal length of Preety’s route. Give your answer as an inequality.

(Total for Question 3 is 11 marks)
The network in Figure 3 shows the roads linking a depot, D, and three collection points A, B and C. The number on each arc represents the length, in miles, of the corresponding road. The road from B to D is a one-way road, as indicated by the arrow.

(a) Explain clearly if Dijkstra’s algorithm can be used to find a route from D to A.

(b) Use Floyd’s algorithm to find a table of least distances. You should show both the distance table and the route table after each iteration.

(c) Explain how the final route table can be used to find the shortest route from D to B. State this route.

There are items to collect at A, B and C. A van will leave D to make these collections in any order and then return to D. A minimum route is required.

Using the final distance table and the Nearest Neighbour algorithm starting at D,

(d) find a minimum route and state its length.

Floyd’s algorithm and Dijkstra’s algorithm are applied to a network. Each will find the shortest distance between vertices of the network.

(e) Describe how the results of these algorithms differ.

(Total for Question 4 is 14 marks)
4. Critical path analysis

Structure and calculation

4.1 Modelling of a project by an activity network, from a precedence table.
4.2 Completion of the precedence table for a given activity network.
4.3 Algorithm for finding the critical path. Earliest and latest event times. Earliest and latest start and finish times for activities.
   Identification of critical activities and critical path(s).
4.4 Calculation of the total float of an activity. Construction of Gantt (cascade) charts.
4.5 Construct resource histograms (including resource levelling) based on the number of workers required to complete each activity.
4.6 Scheduling the activities using the least number of workers required to complete the project.

Sample questions

AS Further Mathematics SAMs Paper 2K Q3 (4.1, 4.3)

<table>
<thead>
<tr>
<th>Activity</th>
<th>Time taken (days)</th>
<th>Immediately preceding activities</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>5</td>
<td>–</td>
</tr>
<tr>
<td>B</td>
<td>7</td>
<td>–</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
<td>–</td>
</tr>
<tr>
<td>D</td>
<td>4</td>
<td>A, B</td>
</tr>
<tr>
<td>E</td>
<td>4</td>
<td>D</td>
</tr>
<tr>
<td>F</td>
<td>2</td>
<td>B</td>
</tr>
<tr>
<td>G</td>
<td>4</td>
<td>B</td>
</tr>
<tr>
<td>H</td>
<td>5</td>
<td>C, G</td>
</tr>
<tr>
<td>I</td>
<td>10</td>
<td>C, G</td>
</tr>
</tbody>
</table>

The table above shows the activities required for the completion of a building project.

For each activity, the table shows the time taken in days to complete the activity and the immediately preceding activities. Each activity requires one worker. The project is to be completed in the shortest possible time.

(a) Draw the activity network described in the table, using activity on arc. Your activity network must contain the minimum number of dummies only.

(3)

(b) (i) Show that the project can be completed in 21 days, showing your working.

(ii) Identify the critical activities.

(4)

(Total for Question 3 is 7 marks)
A project is modelled by the activity network shown in Figure 4. The activities are represented by the arcs. The number in brackets on each arc gives the time, in days, to complete that activity. Each activity requires one worker. The project is to be completed in the shortest possible time.

(a) Calculate the early time and the late time for each event, using Diagram 1 in the answer book. (3)

(b) On Grid 1 in the answer book, complete the cascade (Gantt) chart for this project. (3)

(c) On Grid 2 in the answer book, draw a resource histogram to show the number of workers required each day when each activity begins at its earliest time. (3)

The supervisor of the project states that only three workers are required to complete the project in the minimum time.

(d) Use Grid 2 to determine if the project can be completed in the minimum time by only three workers. Give reasons for your answer. (3)

(Total for Question 6 is 12 marks)
5. Linear programming

Structure and calculation

5.1 **Formulation of problems as linear programs** including the meaning and use of slack, surplus and artificial variables.

5.2 **Graphical solution of two variable problems using objective line and vertex methods** including cases where integer solutions are required.

5.3 The Simplex algorithm and tableau for maximising and minimising problems with \( \leq \) constraints.

5.4 The two-stage Simplex and big-M methods for maximising and minimising problems which may include both \( \leq \) and \( \geq \) constraints.

Sample questions

AS Further Mathematics SAMs Paper 2K Q2 (5.2)
5. Linear programming

A teacher buys pens and pencils. The number of pens, \( x \), and the number of pencils, \( y \), that he buys can be represented by a linear programming problem as shown in Figure 2, which models the following constraints:

\[
\begin{align*}
8x + 3y &\leq 480 \\
8x + 7y &\geq 560 \\
y &\geq 4x \\
x, y &\geq 0
\end{align*}
\]

The total cost, in pence, of buying the pens and pencils is given by \( C = 12x + 15y \).

Determine the number of pens and the number of pencils which should be bought in order to minimise the total cost. You should make your method and working clear.

(Total for Question 2 is 7 marks)

---

AS Further Mathematics SAMs Paper 2K Q5 (5.1)

Jonathan makes two types of information pack for an event, Standard and Value.

Each Standard pack contains 25 posters and 500 flyers.

Each Value pack contains 15 posters and 800 flyers.

He must use at least 150 000 flyers.

Between 35% and 65% of the packs must be Standard packs.

Posters cost 20p each and flyers cost 4p each.

Jonathan wishes to minimise his costs.

Let \( x \) and \( y \) represent the number of Standard packs and Value packs produced respectively.

Formulate this as a linear programming problem, stating the objective and listing the constraints as simplified inequalities with integer coefficients.

You should not attempt to solve the problem.

(Total for Question 5 is 5 marks)
A garden centre makes hanging baskets to sell to its customers. Three types of hanging basket are made, *Sunshine*, *Drama* and *Peaceful*. The plants used are categorised as *Impact*, *Flowering* or *Trailing*.


The garden centre can use at most 80 *Impact* plants, at most 140 *Flowering* plants and at most 96 *Trailing* plants each day.

The profit on *Sunshine*, *Drama* and *Peaceful* baskets are £12, £20 and £16 respectively. The garden centre wishes to maximise its profit.

Let $x$, $y$ and $z$ be the number of *Sunshine*, *Drama* and *Peaceful* baskets respectively, produced each day.

(a) Formulate this situation as a linear programming problem, giving your constraints as inequalities. (5)

(b) State the further restriction that applies to the values of $x$, $y$ and $z$ in this context. (1)

The Simplex algorithm is used to solve this problem. After one iteration, the tableau is

<table>
<thead>
<tr>
<th>b.v.</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
<th>$r$</th>
<th>$s$</th>
<th>$t$</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>$\frac{1}{4}$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>0</td>
<td>$\frac{3}{4}$</td>
<td>8</td>
</tr>
<tr>
<td>$s$</td>
<td>$\frac{5}{2}$</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>92</td>
</tr>
<tr>
<td>$y$</td>
<td>$\frac{3}{4}$</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{4}$</td>
<td>24</td>
</tr>
<tr>
<td>$P$</td>
<td>3</td>
<td>0</td>
<td>$-6$</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>480</td>
</tr>
</tbody>
</table>

(c) State the variable that was increased in the first iteration. Justify your answer. (2)

(d) Determine how many plants in total are being used after only one iteration of the Simplex algorithm. (1)

(e) Explain why for a second iteration of the Simplex algorithm the 2 in the $z$ column is the pivot value. (2)
5. Linear programming

After a second iteration, the tableau is

<table>
<thead>
<tr>
<th>b.v.</th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>r</th>
<th>s</th>
<th>t</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>r</td>
<td>3/8</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1/4</td>
<td>7/8</td>
<td>31</td>
</tr>
<tr>
<td>s</td>
<td>5/4</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1/2</td>
<td>1/4</td>
<td>46</td>
</tr>
<tr>
<td>y</td>
<td>1/8</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1/4</td>
<td>3/8</td>
<td>1</td>
</tr>
<tr>
<td>P</td>
<td>21/2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>7/2</td>
<td>756</td>
</tr>
</tbody>
</table>

(f) Use algebra to explain why this tableau is optimal. (1)

(g) State the optimal number of each type of basket that should be made. (1)

The manager of the garden centre is able to increase the number of Impact plants available each day from 80 to 100. She wants to know if this would increase her profit.

(h) Use your final tableau to determine the effect of this increase. (You should not carry out any further calculations.) (2)

(Total for Question 5 is 15 marks)

A level Further Mathematics SAMs Paper 3D/4D Q7 (5.1, 5.4)

A linear programming problem in x, y and z is described as follows.

Maximise \( P = 3x + 2y + 2z \)

subject to \( 2x + 2y + z \leq 25 \)
\( x + 4y \leq 15 \)
\( x \geq 3 \)

(a) Explain why the Simplex algorithm cannot be used to solve this linear programming problem. (1)

The big-M method is to be used to solve this linear programming problem.

(b) Define, in this context, what M represents. You must use correct mathematical language in your answer. (1)
5. Linear programming

The initial tableau for a big-M solution to the problem is shown below.

<table>
<thead>
<tr>
<th>b.v.</th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>s1</th>
<th>s2</th>
<th>s3</th>
<th>t1</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>s1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>25</td>
</tr>
<tr>
<td>s2</td>
<td>1</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>15</td>
</tr>
<tr>
<td>t1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>–1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>P</td>
<td>–(3 + M)</td>
<td>–2</td>
<td>–2</td>
<td>0</td>
<td>0</td>
<td>M</td>
<td>0</td>
<td>–3M</td>
</tr>
</tbody>
</table>

(c) Explain clearly how the equation represented in the b.v. \( t_1 \) row was derived.  

(1)

(d) Show how the equation represented in the b.v. \( P \) row was derived.  

(2)

The tableau obtained from the first iteration of the big-M method is shown below.

<table>
<thead>
<tr>
<th>b.v.</th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>s1</th>
<th>s2</th>
<th>s3</th>
<th>t1</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>s1</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>–2</td>
<td>19</td>
</tr>
<tr>
<td>s2</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>–1</td>
<td>12</td>
</tr>
<tr>
<td>x</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>–1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>P</td>
<td>0</td>
<td>–2</td>
<td>–2</td>
<td>0</td>
<td>0</td>
<td>–3</td>
<td>3 + M</td>
<td>9</td>
</tr>
</tbody>
</table>

(e) Solve the linear programming problem, starting from this second tableau. You must

- give a detailed explanation of your method by clearly stating the row operations you use and
- state the solution by deducing the final values of \( P, x, y \) and \( z \).

(7)

(Total for Question 7 is 12 marks)
1. Transportation problems

Structure and calculation

1.1 The north-west corner method for finding an initial basic feasible solution.
1.2 Use of the steppingstone method for obtaining an improved solution. Improvement indices.
1.3 Formulation of the transportation problem as a linear programming problem.

Sample questions

A level Further Mathematics SAMs Paper 4G Q2 (1.1, 1.2)

<table>
<thead>
<tr>
<th></th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>Available</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>15</td>
<td>19</td>
<td>9</td>
<td>25</td>
</tr>
<tr>
<td>B</td>
<td>11</td>
<td>18</td>
<td>10</td>
<td>55</td>
</tr>
<tr>
<td>C</td>
<td>11</td>
<td>12</td>
<td>18</td>
<td>20</td>
</tr>
<tr>
<td>Required</td>
<td>38</td>
<td>24</td>
<td>38</td>
<td></td>
</tr>
</tbody>
</table>

A company has three factories, A, B and C. It supplies mattresses to three shops, D, E and F. The table shows the transportation cost, in pounds, of moving one mattress from each factory to each shop. It also shows the number of mattresses available at each factory and the number of mattresses required at each shop. A minimum cost solution is required.

(a) Use the north-west corner method to obtain an initial solution.

(b) Show how the transportation algorithm is used to solve this problem.

You must state, at each appropriate step, the
- shadow costs,
- improvement indices,
- route,
- entering cell and exiting cell,
and explain clearly how you know that your final solution is optimal.

(Total for Question 2 is 12 marks)
2. Allocation (assignment) problems

Structure and calculation

2.1  Cost matrix reduction.
   Use of the Hungarian algorithm to find a least cost allocation.
   Modification of the Hungarian algorithm to deal with a maximum profit allocation.

2.2  Formulation of the Hungarian algorithm as a linear programming problem.

Sample questions

AS Further Mathematics SAMs Paper 2K Q6 (2.1)

Six workers, A, B, C, D, E and F, are to be assigned to five tasks, P, Q, R, S and T.

Each worker can be assigned to at most one task and each task must be done by just one worker.

The time, in minutes, that each worker takes to complete each task is shown in the table below.

<table>
<thead>
<tr>
<th></th>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>S</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>32</td>
<td>32</td>
<td>35</td>
<td>34</td>
<td>33</td>
</tr>
<tr>
<td>B</td>
<td>28</td>
<td>35</td>
<td>31</td>
<td>37</td>
<td>40</td>
</tr>
<tr>
<td>C</td>
<td>35</td>
<td>29</td>
<td>33</td>
<td>36</td>
<td>35</td>
</tr>
<tr>
<td>D</td>
<td>36</td>
<td>30</td>
<td>34</td>
<td>33</td>
<td>35</td>
</tr>
<tr>
<td>E</td>
<td>30</td>
<td>31</td>
<td>29</td>
<td>37</td>
<td>36</td>
</tr>
<tr>
<td>F</td>
<td>29</td>
<td>28</td>
<td>32</td>
<td>31</td>
<td>34</td>
</tr>
</tbody>
</table>

Reducing rows first, use the Hungarian algorithm to obtain an allocation which minimises the total time. You must explain your method and show the table after each stage.

(Total for Question 6 is 9 marks)
A level Further Mathematics SAMs Paper 4G Q3 (2.1, 2.2)

Four workers, A, B, C and D, are to be assigned to four tasks, P, Q, R and S.

Each worker must be assigned to at most one task and each task must be done by just one worker.

The amount, in pounds, that each worker would earn while assigned to each task is shown in the table below.

<table>
<thead>
<tr>
<th></th>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>32</td>
<td>32</td>
<td>33</td>
<td>35</td>
</tr>
<tr>
<td>B</td>
<td>28</td>
<td>35</td>
<td>31</td>
<td>37</td>
</tr>
<tr>
<td>C</td>
<td>35</td>
<td>29</td>
<td>33</td>
<td>36</td>
</tr>
<tr>
<td>D</td>
<td>36</td>
<td>30</td>
<td>36</td>
<td>33</td>
</tr>
</tbody>
</table>

The Hungarian algorithm is to be used to find the maximum total amount which may be earned by the four workers.

(a) Explain how the table should be modified. 

(b) Reducing rows first, use the Hungarian algorithm to obtain an allocation which maximises the total earnings, stating how each table was formed.

(c) Formulate the problem as a linear programming problem. You must define your decision variables and make your objective function and constraints clear.

(Total for Question 3 is 13 marks)
3. Flows in networks

Structure and calculation

3.1 Cuts and their capacity.
3.2 Use of the labelling procedure to augment a flow to determine the maximum flow in a network.
3.3 Use of the max–flow min–cut theorem to prove that a flow is a maximum flow.
3.4 Multiple sources and sinks. Vertices with restricted capacity.
3.5 Determine the optimal flow rate in a network, subject to given constraints.

Sample questions

AS Further Mathematics SAMs Paper 2K Q8 (3.1, 3.2, 3.3)

Figure 5

Figure 5 represents a network of corridors in a school. The number on each arc represents the maximum number of students, per minute, that may pass along each corridor at any one time.

At 11 a.m. on Friday morning, all students leave the hall (S) after assembly and travel to the cybercafe (T). The numbers in circles represent the initial flow of students recorded at 11 a.m. one Friday.

(a) State an assumption that has been made about the corridors in order for this situation to be modelled by a directed network.

(b) Find the value of x and the value of y, explaining your reasoning.

(1)
3. Flows in networks

Five new students also attend the assembly in the hall the following Friday. They too need to travel to the cybercafe at 11 a.m. They wish to travel together so that they do not get lost. You may assume that the initial flow of students through the network is the same as that shown in Figure 5 above.

(c) (i) List all the flow augmenting routes from $S$ to $T$ that increase the flow by at least 5.

(ii) State which route the new students should take, giving a reason for your answer.

(d) Use the answer to part (c) to find a maximum flow pattern for this network.

(e) Prove that the answer to part (d) is optimal.

The school is intending to increase the number of students it takes but has been informed it cannot do so until it improves the flow of students at peak times. The school can widen corridors to increase their capacity, but can only afford to widen one corridor in the coming term.

(f) State, explaining your reasoning,

(i) which corridor they should widen,

(ii) the resulting increase of flow through the network.

(Total for Question 8 is 14 marks)
4. Dynamic programming

4. Dynamic programming

Structure and calculation

Stage variables and State variables. Use of tabulation to solve maximum, minimum, minimax or maximin problems.

Sample questions

A level Further Mathematics SAMs Paper 4G Q7 (4.1)

A company assembles boats.

They can assemble up to five boats in any one month, but if they assemble more than three they will have to hire additional space at a cost of £800 per month.

The company can store up to two boats at a cost of £350 each per month.
The overhead costs are £1500 in any month in which work is done.

Boats are delivered at the end of each month. There are no boats in stock at the beginning of January and there must be none in stock at the end of May.

The order book for boats is

<table>
<thead>
<tr>
<th>Month</th>
<th>January</th>
<th>February</th>
<th>March</th>
<th>April</th>
<th>May</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number ordered</td>
<td>3</td>
<td>2</td>
<td>6</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

Use dynamic programming to determine the production schedule which minimises the costs to the company. Show your working in the table provided in the answer book and state the minimum production cost.

(Total for Question 7 is 12 marks)
5. Game theory

Structure and calculation

<table>
<thead>
<tr>
<th>5.1</th>
<th>Two person zero-sum games and the pay-off matrix.</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.2</td>
<td>Identification of play safe strategies and stable solutions (saddle points).</td>
</tr>
<tr>
<td>5.3</td>
<td>Reduction of pay-off matrices using dominance arguments.</td>
</tr>
<tr>
<td>5.4</td>
<td>Optimal mixed strategies for a game with no stable solution by use of graphical methods for $2 \times n$ or $n \times 2$ games where $n = 1, 2, 3$ or $4$.</td>
</tr>
<tr>
<td>5.5</td>
<td>Optimal mixed strategies for a game with no stable solution by converting games to linear programming problems that can be solved by the Simplex algorithm.</td>
</tr>
</tbody>
</table>

Sample questions

AS Further Mathematics SAMs Paper 2K Q9 (5.1, 5.2, 5.4)

A two person zero-sum game is represented by the following pay-off matrix for player $A$.

<table>
<thead>
<tr>
<th></th>
<th>$B$ plays 1</th>
<th>$B$ plays 2</th>
<th>$B$ plays 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$ plays 1</td>
<td>4</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$A$ plays 2</td>
<td>2</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

(a) Verify that there is no stable solution. (3)

(b) (i) Find the best strategy for player $A$.

(ii) Find the value of the game to her. (9)

(Total for Question 9 is 12 marks)

A level Further Mathematics SAMs Paper 4G Q5 (5.2, 5.3, 5.5)

<table>
<thead>
<tr>
<th></th>
<th>$B$ plays 1</th>
<th>$B$ plays 2</th>
<th>$B$ plays 3</th>
<th>$B$ plays 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$ plays 1</td>
<td>4</td>
<td>$-$2</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>$A$ plays 2</td>
<td>3</td>
<td>$-$1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$A$ plays 3</td>
<td>$-$1</td>
<td>2</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>

A two person zero-sum game is represented by the pay-off matrix for player $A$ given above.

(a) Explain, with justification, how this matrix may be reduced to a $3 \times 3$ matrix. (2)

(b) Find the play-safe strategy for each player and verify that there is no stable solution to this game. (4)
5. Game theory

The game is formulated as a linear programming problem for player A.

The objective is to maximise $P = V$, where $V$ is the value of the game to player A.

One of the constraints is that $p_1 + p_2 + p_3 \leq 1$, where $p_1, p_2, p_3$ are the probabilities that player A plays 1, 2, 3 respectively.

(c) Formulate the remaining constraints for this problem. Write these constraints as inequalities.

The Simplex algorithm is used to solve the linear programming problem.

The solution obtained is $p_1 = 0, p_2 = \frac{3}{7}, p_3 = \frac{4}{7}$

(d) Calculate the value of the game to player A.

(Total for Question 5 is 12 marks)

A level Further Mathematics SAMs Paper 4G Q6 (5.2, 5.3, 5.5)

Figure 1 shows a capacitated, directed network. The number on each arc $(x, y)$ represents the lower ($x$) capacity and upper ($y$) capacity of that arc.

(a) Calculate the value of the cut $C_1$ and cut $C_2$

(b) Explain why the flow through the network must be at least 12 and at most 16

(c) Explain why arcs DG, AG, EG and FG must all be at their lower capacities.

(d) Determine a maximum flow pattern for this network and draw it on Diagram 1 in the answer book. You do not need to use the labelling procedure.
5. Game theory

(e) (i) State the value of the maximum flow through the network.

(ii) Explain why the value of the maximum flow is equal to the value of the minimum flow through the network.

Node E becomes blocked and no flow can pass through it. To maintain the maximum flow through the network the upper capacity of exactly one arc is increased.

(f) Explain how it is possible to maintain the maximum flow found in (d).

(Total for Question 6 is 12 marks)
6. Recurrence relations

Structure and calculation

<table>
<thead>
<tr>
<th>6.1</th>
<th>Use of recurrence relations to model appropriate problems.</th>
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<tbody>
<tr>
<td>6.2</td>
<td>Solution of first and second order linear homogeneous and nonhomogeneous recurrence relations.</td>
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</table>

Sample questions

AS Further Mathematics SAMs Paper 2K Q7 (6.1)

In two-dimensional space, lines divide a plane into a number of different regions.

It is known that:

- One line divides a plane into 2 regions, as shown in Figure 1,
- Two lines divide a plane into a maximum of 4 regions, as shown in Figure 2,
- Three lines divide a plane into a maximum of 7 regions, as shown in Figure 3,
- Four lines divide a plane into a maximum of 11 regions, as shown in Figure 4.

(a) Complete the table in the answer book to show the maximum number of regions when five, six and seven lines divide a plane.

(b) Find, in terms of $u_n$, the recurrence relation for $u_{n+1}$, the maximum number of regions when a plane is divided by $(n+1)$ lines, where $n \geq 1$.

(c) (i) Solve the recurrence relation for $u_n$.

(ii) Hence determine the maximum number of regions created when 200 lines divide a plane.

(Total for Question 7 is 5 marks)
6. Recurrence relations

A level Further Mathematics SAMs Paper 4G Q1 (6.2, A level Mathematics Pure Mathematics 2.2, 2.4)

(a) Find the general solution of the recurrence relation

\[ u_{n+2} = u_{n+1} + u_n , \quad n \geq 1 \]

Given that \( u_1 = 1 \) and \( u_2 = 1 \)

(b) Find the particular solution of the recurrence relation.

(Total for Question 1 is 6 marks)
7. Decision analysis

Structure and calculation

<table>
<thead>
<tr>
<th>7.1</th>
<th>Use, construct and interpret decision trees.</th>
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<tr>
<td>7.2</td>
<td>Use of expected monetary values (EMVs) and utility to compare alternative courses of action.</td>
</tr>
</tbody>
</table>

Sample questions

A level Further Mathematics SAMs Paper 4G Q4 (7.1, 7.2)

A game uses a standard pack of 52 playing cards. A player gives 5 tokens to play and then picks a card. If they pick a 2, 3, 4, 5 or 6 then they gain 15 tokens. If any other card is picked they lose.

If they lose, the card is replaced and they can choose to pick again for another 5 tokens. This time if they pick either an ace or a king they gain 40 tokens. If any other card is picked they lose.

Daniel is deciding whether to play this game.

(a) Draw a decision tree to model Daniel’s possible decisions and the possible outcomes.  

(b) Calculate Daniel’s optimal EMV and state the optimal strategy indicated by the decision tree.  

(Total for Question 4 is 8 marks)