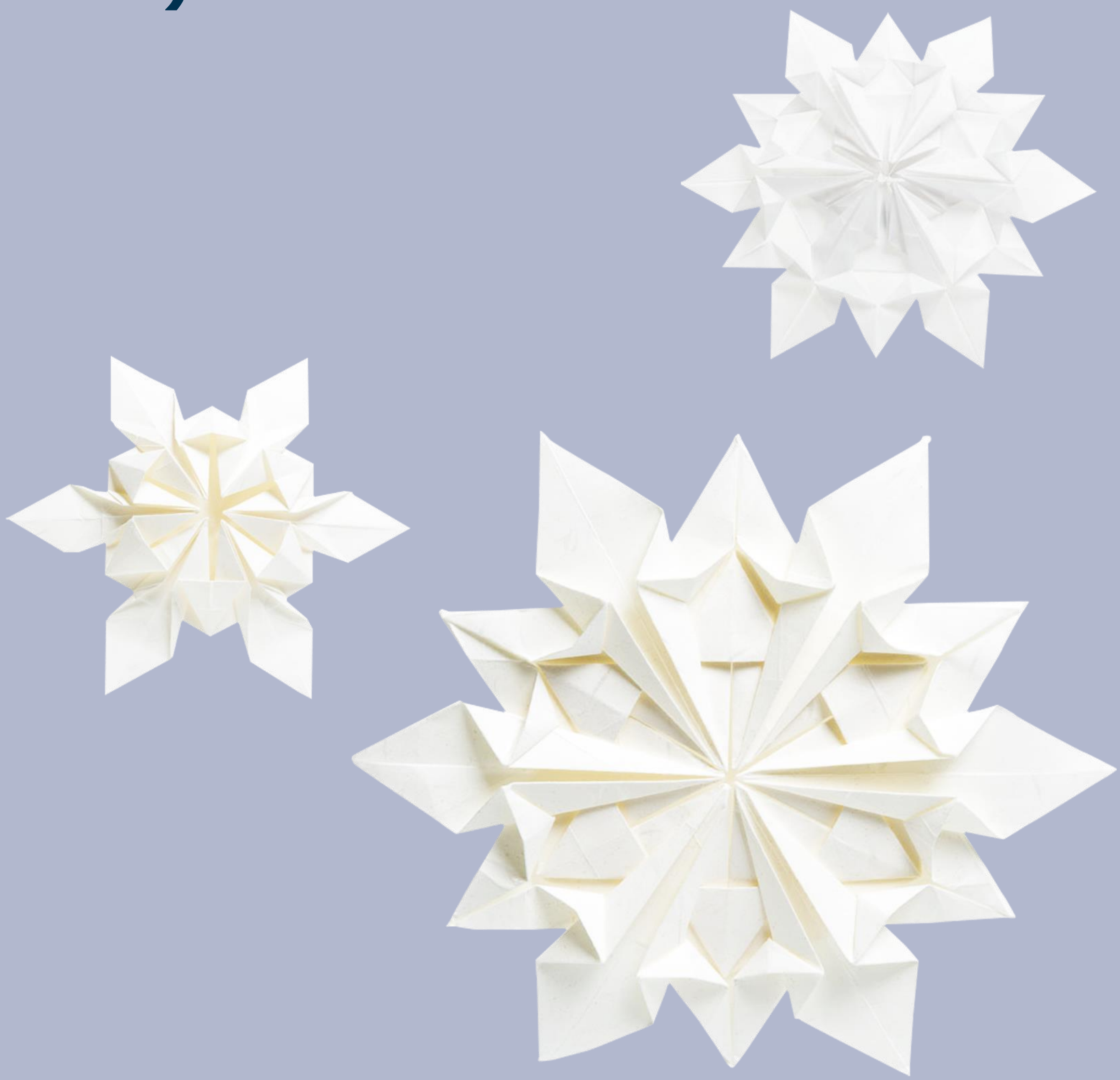
# Pearson Edexcel
# Level 3 Advanced Subsidiary GCE in Further Mathematics (8FM0)



## Sample Assessment Materials Exemplification

First teaching from September 2017
First certification from June 2018

# Contents

# About this booklet

This booklet has been produced to support mathematics teachers delivering the new Pearson Edexcel Level 3 Advanced Subsidiary GCE in Further Mathematics (8FM0) (first assessment summer 2018).

The booklet provides additional information on all the questions in the Sample Assessment Materials, accredited by Ofqual in 2017. It details the content references and Assessment Objectives being assessed in each question or question part.

## How to use this booklet

Callouts have been added to each question in the accredited Sample Assessment Materials. In the callouts, the following information has been presented, as relevant to the question:

- **Specification References**;
- **Assessment Objectives**.

Where content references or Assessment Objectives are being assessed across all the parts of a question, these are referred to by a single callout at the end of the question rather than by a callout for each question part.

# AS Further Mathematics Paper 1 (Core Pure Mathematics 1)

**1.** $$f(z) = z^3 + pz^2 + qz - 15,$$

where $p$ and $q$ are real constants.

Given that the equation $f(z) = 0$ has roots

$$\alpha, \frac{5}{\alpha} \text{ and } \left(\alpha + \frac{5}{\alpha} - 1\right),$$

(a) solve completely the equation $f(z) = 0$.

(b) Hence find the value of $p$.

**Specification reference (2.1, 2.2, 2.3, 4.1):**
Solve cubic or quartic equations with real coefficients.
Add, subtract, multiply and divide complex numbers in the form $x + \mathrm{i}y$ with $x$ and $y$ real.
Understand and use the terms 'real part' and 'imaginary part'.
Understand and use the complex conjugate.
Know that non-real roots of polynomial equations with real coefficients occur in conjugate pairs.
Understand and use the relationship between roots and coefficients of polynomial equations up to quartic equations.

**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (4 marks)
**AO2.2a:** Make deductions and inferences: make deductions (1 mark)
**AO3.1a:** Translate problems in mathematical and non-mathematical contexts into mathematical processes: translate problems in mathematical contexts into mathematical processes (2 marks)

**(Total for Question 1 is 7 marks)**

**2.** The plane $\Pi$ passes through the point $A$ and is perpendicular to the vector $\mathbf{n}$.

Given that

$$\overrightarrow{OA} = \begin{pmatrix} 5 \\ -4 \\ -4 \end{pmatrix} \text{ and } \mathbf{n} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix},$$

where $O$ is the origin,

(a) find a Cartesian equation of $\Pi$.

With respect to the fixed origin $O$, the line $l$ is given by the equation

$$\mathbf{r} = \begin{pmatrix} 7 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -5 \\ 3 \end{pmatrix}.$$

The line $l$ intersects the plane $\Pi$ at the point $X$.

(b) Show that the acute angle between the plane $\Pi$ and the line $l$ is 21.2°, correct to one decimal place.

(c) Find the coordinates of the point $X$.

**Specification reference (6.1, 6.2, 6.3, 6.5):**

Understand and use the vector and Cartesian forms of an equation of a straight line in 3-D.

Understand and use the vector and Cartesian forms of the equation of a plane.

Calculate the scalar product and use it to express the equation of a plane, and to calculate the angle between two lines, the angle between two planes and the angle between a line and a plane.

Find the intersection of a line and a plane.

**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (7 marks)

**AO2.1:** Construct rigorous mathematical arguments (including proofs) (1 mark)

**AO2.5:** Use mathematical language and notation correctly (1 mark)

**AO3.1a:** Translate problems in mathematical and non-mathematical contexts into mathematical processes: translate problems in mathematical contexts into mathematical processes (1 mark)

**(Total for Question 2 is 10 marks)**

**3.** Tyler invested a total of £5000 across three different accounts; a savings account, a property bond account and a share dealing account.

Tyler invested £400 more in the property bond account than in the savings account.

After one year

- the savings account had increased in value by 1.5%,
- the property bond account had increased in value by 3.5%,
- the share dealing account had **decreased** in value by 2.5%,
- the total value across Tyler's three accounts had increased by £79.

Form and solve a matrix equation to find out how much money was invested by Tyler in each account.

**Specification reference (3.1, 3.6, 3.7):**

Add, subtract and multiply conformable matrices.

Multiply a matrix by a scalar.

Understand and use singular and non-singular matrices.

Properties of inverse matrices.

Calculate and use the inverse of non-singular $2 \times 2$ matrices and $3 \times 3$ matrices.

Solve three linear simultaneous equations in three variables by use of the inverse matrix.

**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (4 marks)

**AO3.1a:** Translate problems in mathematical and non-mathematical contexts into mathematical processes: translate problems in mathematical contexts into mathematical processes (1 mark)

**AO3.1b:** Translate problems in mathematical and non-mathematical contexts into mathematical processes: translate problems in non-mathematical contexts into mathematical processes (1 mark)

**AO3.2a:** Interpret solutions to problems in their original context, and, where appropriate evaluate their accuracy and limitations: interpret solutions to problems in their original context (1 mark)

**(Total for Question 3 is 7 marks)**

**4.** The cubic equation

$$x^3 + 3x^2 - 8x + 6 = 0$$

has roots $\alpha$, $\beta$ and $\gamma$.

Without solving the equation, find the cubic equation whose roots are $(\alpha - 1)$, $(\beta - 1)$ and $(\gamma - 1)$, giving your answer in the form $w^3 + pw^2 + qw + r = 0$, where $p$, $q$ and $r$ are integers to be found.

**Specification reference (4.1 or 4.2** depending on which method is used for solution**):**

Understand and use the relationship between roots and coefficients of polynomial equations up to quartic equations.

Form a polynomial equation whose roots are a linear transformation of the roots of a given polynomial equation (of at least cubic degree).

**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (3 marks)

**AO3.1a:** Translate problems in mathematical and non-mathematical contexts into mathematical processes: translate problems in mathematical contexts into mathematical processes (2 marks)

**(Total for Question 4 is 5 marks)**

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**5.**

$$\mathbf{M} = \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$$

(a) Show that **M** is non-singular.

The hexagon $R$ is transformed to the hexagon $S$ by the transformation represented by the matrix **M**.

Given that the area of hexagon $R$ is 5 square units,

(b) find the area of hexagon $S$.

The matrix **M** represents an enlargement, with centre (0, 0) and scale factor $k$, where $k > 0$, followed by a rotation anti-clockwise through an angle $\theta$ about (0, 0).

(c) Find the value of $k$.

(d) Find the value of $\theta$.

**Specification reference (3.3, 3.5, 3.6):**

Use matrices to represent linear transformations in 2-D.

Successive transformations.

Calculate determinants of: 2 × 2 and 3 × 3 matrices and interpret as scale factors, including the effect on orientation.

Understand and use singular and non-singular matrices.

**AO1.1a:** Select and correctly carry out routine procedures: select routine procedures (1 mark)

**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (4 marks)

**AO1.2:** Accurately recall facts, terminology and definitions (1 mark)

**AO2.4:** Explain their reasoning (1 mark)

**(Total for Question 5 is 7 marks)**

**6.** (a) Prove by induction that for all positive integers $n$,

$$\sum_{r=1}^{n} r^2 \quad \frac{1}{6}n(n+1)(2n+1)$$

(b) Use the standard results for $\sum_{r=1}^{n} r^3$ and $\sum_{r=1}^{n} r$ to show that for all positive integers $n$,

$$\sum_{r=1}^{n} r^2 = r(r+6)(r-6) = \frac{1}{4}n(n+1)(n-8)(n+9).$$

(c) Hence find the value of $n$ that satisfies

$$\sum_{r=1}^{n} r(r+6)(r-6) = 17\sum_{r=1}^{n} r^2 .$$

| |
|---|
| **Specification reference (1.1, 4.3):** |
| Construct proofs using mathematical induction. |
| Contexts include sums of series, divisibility and powers of matrices. |
| Understand and use formulae for the sums of integers, squares and cubes and use these to sum other series. |

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| **AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (9 marks) |
| **AO2.1:** Construct rigorous mathematical arguments (including proofs) (2 marks) |
| **AO2.2a:** Make deductions and inferences: make deductions (1 mark) |
| **AO2.3:** Assess the validity of mathematical arguments (1 mark) |
| **AO2.4:** Explain their reasoning (2 marks) |

**(Total for Question 6 is 15 marks)**

**7.**

Diagrams not drawn to scale

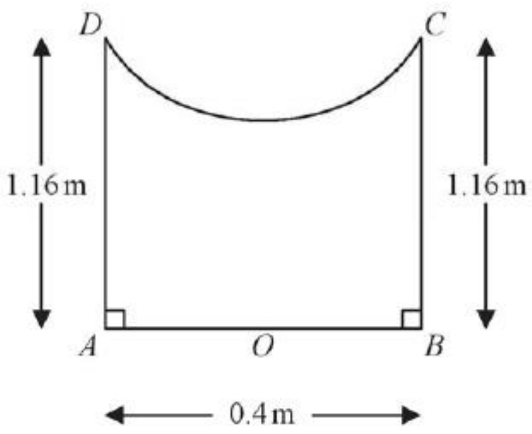


**Figure 1**

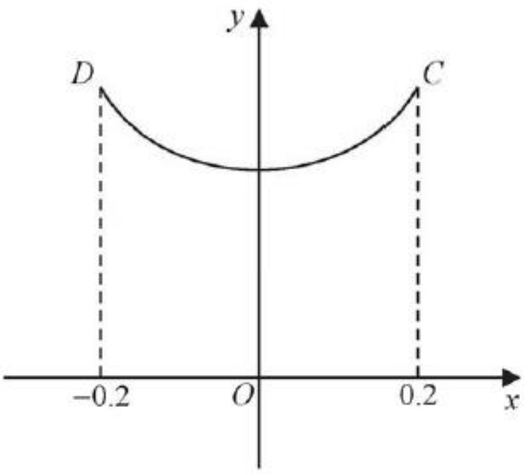


**Figure 2**

Figure 1 shows the central cross-section *AOBCD* of a circular birdbath, which is made of concrete. Measurements of the height and diameter of the birdbath, and the depth of the bowl of the birdbath have been taken in order to estimate the amount of concrete that was required to make this birdbath.

Using these measurements, the cross-sectional curve *CD*, shown in Figure 2, is modelled as a curve with equation

$$y = 1 + kx^2, \quad -0.2 \leq x \leq 0.2,$$

where $k$ is a constant and where $O$ is the fixed origin.

The height of the bird bath measured 1.16 m and the diameter, *AB*, of the base of the birdbath measured 0.40 m, as shown in Figure 1.

(a) Suggest the maximum depth of the birdbath.

(b) Find the value of $k$.

(c) Hence find the volume of concrete that was required to make the birdbath according to this model. Give your answer, in $m^3$, correct to 3 significant figures.

(d) State a limitation of the model.

It was later discovered that the volume of concrete used to make the birdbath was 0.127 $m^3$ correct to 3 significant figures.

(e) Using this information and the answer to part (*c*), evaluate the model, explaining your reasoning.

**Specification reference (5.1):**

Derive formulae for and calculate volumes of revolution.

**AO1.1a:** Select and correctly carry out routine procedures: select routine procedures (1 mark)

**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (5 marks)

**AO2.2b:** Make deductions and inferences: make inferences (1 mark)

**AO3.3:** Translate situations in context into mathematical models (2 marks)

**AO3.4:** Use mathematical models (1 mark)

**AO3.5a:** Evaluate the outcomes of modelling in context, recognise the limitations of models and, where appropriate, explain how to refine them: evaluate the outcomes of modelling in context (1 mark)

**AO3.5b:** Evaluate the outcomes of modelling in context, recognise the limitations of models and, where appropriate, explain how to refine them: recognise the limitations of models (1 mark)

**(Total for Question 7 is 12 marks)**

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**8.** (a) Shade on an Argand diagram the set of points

$$\{z \in \mathbb{C}: |z - 4\text{i}| \leq 3\} \cap \{-\frac{\pi}{2} < \arg(z + 3 - 4\text{i} \leq \frac{\pi}{4}\}.$$

The complex number $w$ satisfies $|w - 4\text{i}| = 3$.

(b) Find the maximum value of arg $w$ in the interval $(-\pi, \pi]$.
Give your answer in radians correct to 2 decimal places.

**Specification reference (2.4, 2.7):**

Use and interpret Argand diagrams.

Construct and interpret simple loci in the argand diagram such as $z - a > r$ and $\arg(z - a) = \theta$

Knowledge of radians is assumed.

**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (5 marks)

**AO2.2a:** Make deductions and inferences: make deductions (1 mark)

**AO3.1a:** Translate problems in mathematical and non-mathematical contexts into mathematical processes: translate problems in mathematical contexts into mathematical processes (2 marks)

**(Total for Question 8 is 8 marks)**

**9.** An octopus is able to catch any fish that swim within a distance of 2 m from the octopus's position.

A fish $F$ swims from a point $A$ to a point $B$.

The octopus is modelled as a fixed particle at the origin $O$.

Fish $F$ is modelled as a particle moving in a straight line from $A$ to $B$.

Relative to $O$, the coordinates of $A$ are $(-3, 1, -7)$ and the coordinates of $B$ are $(9, 4, 11)$, where the unit of distance is metres.

(a) Use the model to determine whether or not the octopus is able to catch fish $F$.

(b) Criticise the model in relation to fish $F$.

(c) Criticise the model in relation to the octopus.

**Specification reference (6.1, 6.3, 6.4, 6.5):**

Understand and use the vector and Cartesian forms of an equation of a straight line in 3-D.

Calculate the scalar product and use it to express the equation of a plane, and to calculate the angle between two lines, the angle between two planes and the angle between a line and a plane.

Check whether vectors are perpendicular by using the scalar product.

Find the intersection of a line and a plane.

Calculate the perpendicular distance between two lines, from a point to a line and from a point to a plane.

**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (4 marks)

**AO3.1a:** Translate problems in mathematical and non-mathematical contexts into mathematical processes: translate problems in mathematical contexts into mathematical processes (2 marks)

**AO3.2a:** Interpret solutions to problems in their original context, and, where appropriate evaluate their accuracy and limitations: interpret solutions to problems in their original context (1 mark)

**(Total for Question 9 is 9 marks)**

# AS Further Mathematics Paper 2A (Further Pure Mathematics 1&2)

**1.** (a) Use the substitution $t = \tan \frac{x}{2}$ to show that

$$\sec x - \tan x = \frac{1-t}{1+t}, \qquad x \neq (2n+1)\frac{\pi}{2}, \quad n \in \mathbb{Z}.$$

(b) Use the substitution $t = \tan \frac{x}{2}$ and the answer to part (a) to prove that

$$\frac{1-\sin x}{1+\sin x} \equiv (\sec x - \tan x)^2, \qquad x \neq (2n+1)\frac{\pi}{2}, \quad n \in \mathbb{Z}.$$

**Specification reference (1.1, 1.2):**

The *t*-formulae

$\sin\theta \equiv \frac{2t}{1+t^2}, \cos\theta \equiv \frac{1-t}{1+t^2}$

$\tan\theta \equiv \frac{2t}{1-t^2}$, where $t = \tan\frac{\theta}{2}$

Applications of *t*-formulae to trigonometric identities.

**AO1.1a:** Select and correctly carry out routine procedures: select routine procedures (1 mark)

**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (2 marks)

**AO2.1:** Construct rigorous mathematical arguments (including proofs) (3 marks)

**(Total for Question 1 is 6 marks)**

**2.** The value, $V$ hundred pounds, of a particular stock $t$ hours after the opening of trading on a given day is modelled by the differential equation

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{V^2 - 1}{t^2 + tV}, \qquad 0 < t < 8.5.$$

A trader purchases £300 of the stock one hour after the opening of trading.

Use two iterations of the approximation formula $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 \approx \frac{y_1 - y_0}{h}$ to estimate, to the nearest £, the value of the trader's stock half an hour after it was purchased.

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| **Specification reference (4.1):**<br>Numerical solution of first order and second order differential equations. |

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| **AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (3 marks)<br>**AO3.2a:** Interpret solutions to problems in their original context, and, where appropriate evaluate their accuracy and limitations: interpret solutions to problems in their original context (1 mark)<br>**AO3.3:** Translate situations in context into mathematical models (1 mark)<br>**AO3.4:** Use mathematical models (1 mark) |

**(Total for Question 2 is 6 marks)**

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**3.** Use algebra to find the set of values of $x$ for which

$$\frac{1}{x} < \frac{x}{x+2}$$

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| **Specification reference (5.1):**<br>The manipulation and solution of algebraic inequalities and inequations. |

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| **AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (3 marks)<br>**AO2.1:** Construct rigorous mathematical arguments (including proofs) (1 mark)<br>**AO2.2a:** Make deductions and inferences: make deductions (1 mark)<br>**AO2.5:** Use mathematical language and notation correctly (1 mark) |

**(Total for Question 3 is 6 marks)**

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**4.**

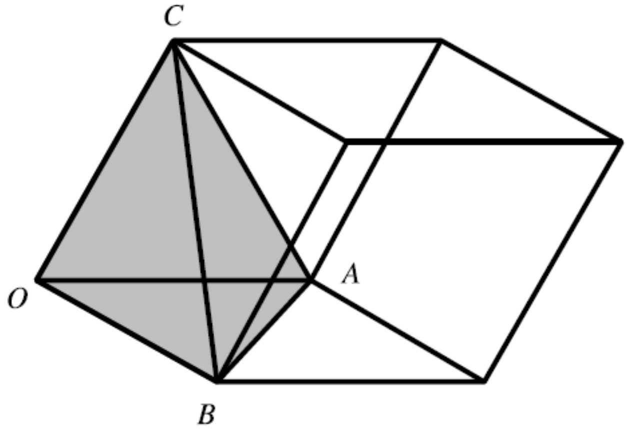


**Figure 1**

Figure 1 shows a sketch of a solid sculpture made of glass and concrete. The sculpture is modelled as a parallelepiped.

The sculpture is made up of a concrete solid in the shape of a tetrahedron, shown shaded in Figure 1, whose vertices are $O(0, 0, 0)$, $A(2, 0, 0)$, $B(0, 3, 1)$ and $C(1, 1, 2)$,where the units are in metres. The rest of the solid parallelepiped is made of glass which is glued to the concrete tetrahedron.

(a) Find the surface area of the glued face of the tetrahedron.

(b) Find the volume of glass contained in this parallelepiped.

(c) Give a reason why the volume of concrete predicted by this model may not be an accurate value for the volume of concrete that was used to make the sculpture.

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| **Specification reference (3.1, 3.2, 3.3):** |
| The vector product $a \times b$ of two vectors |
| The scalar triple product $a.b \times c$ |
| Applications of the vector product. |

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| **AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (6 marks) |
| **AO3.1a:** Translate problems in mathematical and non-mathematical contexts into mathematical processes: translate problems in mathematical contexts into mathematical processes (3 marks) |
| **AO3.2b:** Interpret solutions to problems in their original context, and, where appropriate evaluate their accuracy and limitations: where appropriate, evaluation the accuracy and limitations of solutions to problems (1 mark) |

**(Total for Question 4 is 10 marks)**

**5.**

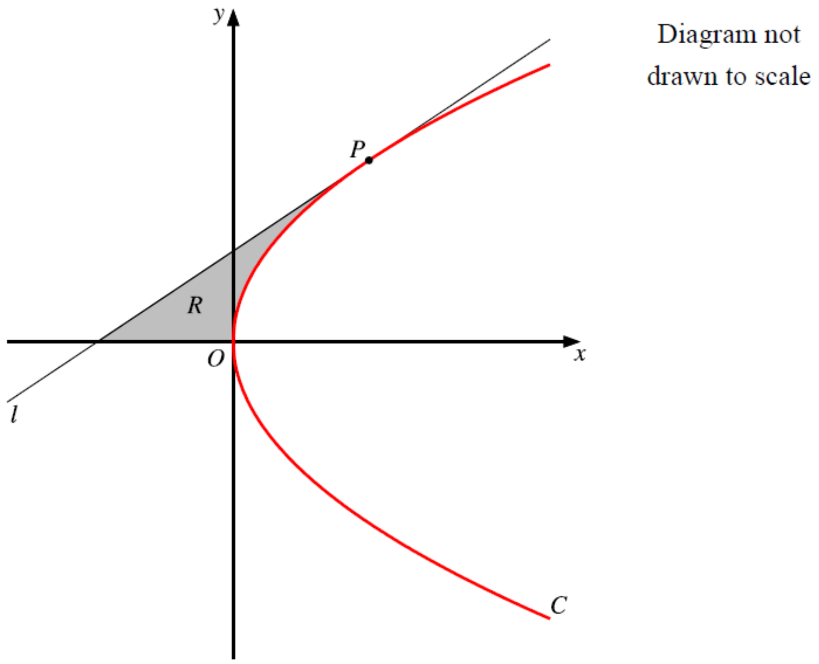


**Figure 2**

[You may quote without proof that for the general parabola $y^2 = 4ax$, $\dfrac{dy}{dx} = \dfrac{2a}{y}$ ]

The parabola $C$ has equation $y^2 = 16x$.

(a) Deduce that the point $P(4p^2, 8p)$ is a general point on $C$.

The line $l$ is the tangent to $C$ at the point $P$.

(b) Show that an equation for $l$ is

$$py = x + 4p^2$$

The finite region $R$, shown shaded in Figure 2, is bounded by the line $l$, the $x$-axis and the parabola $C$.

The line $l$ intersects the directrix of $C$ at the point $B$, where the $y$ coordinate of $B$ is $\dfrac{10}{3}$

Given that $p > 0$

(c) show that the area of $R$ is 36

**Specification reference (2.1, 2.2, 2.3, 2.4, AS Mathematics Pure Mathematics 8.2, 8.3):**

Cartesian equations for the parabola and rectangular hyperbola.

Parametric equations for the parabola and rectangular hyperbola.

The focus-directrix property of the parabola.

Tangents and normals to these curves.

Integrate $x^n$ (excluding $n = -1$) and related sums, differences and constant multiples.

Integrate $e^{kx}, \frac{1}{x}, \sin kx, \cos kx$ and related sums, differences and constant multiples.

**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (6 marks)

**AO2.1:** Construct rigorous mathematical arguments (including proofs) (3 marks)

**AO2.2a:** Make deductions and inferences: make deductions (2 marks)

**AO3.1a:** Translate problems in mathematical and non-mathematical contexts into mathematical processes: translate problems in mathematical contexts into mathematical processes (1 mark)

**(Total for Question 5 is 12 marks)**

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**6.** Given that

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 6 & 4 \end{pmatrix},$$

(a) find the characteristic equation of the matrix **A**.

(b) Hence show that $\mathbf{A}^3 = 43\mathbf{A} - 42\mathbf{I}$ .

**Specification reference (2.1, 2.3):**
Eigenvalues and eigenvectors of 2 × 2 matrices.
The use of the CayleyHamilton theorem.

**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (4 marks)
**AO3.1a:** Translate problems in mathematical and non-mathematical contexts into mathematical processes: translate problems in mathematical contexts into mathematical processes (1 mark)

**(Total for Question 6 is 5 marks)**

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**7.** (i) Without performing any division, explain why 8184 is divisible by 6.

(ii) Use the Euclidean algorithm to find integers $a$ and $b$ such that $27a + 31b = 1$.

**Specification reference (4.1, 4.2, 4.4):**
An understanding of the division theorem and its application to the Euclidean Algorithm and congruences.
Bezout's identity.
Divisibility Tests.

**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (5 marks)
**AO1.2:** Accurately recall facts, terminology and definitions (1 mark)

**(Total for Question 7 is 6 marks)**

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**8.** A curve $C$ is described by the equation

$$|z - 9 + 12\mathrm{i}| = 2|z|.$$

(a) Show that $C$ is a circle, and find its centre and radius.

(b) Sketch $C$ on an Argand diagram.

Given that $w$ lies on $C$,

(c) find the largest value of $a$ and the smallest value of $b$ that must satisfy $a \leq \mathrm{Re}(w) \leq b$.

**Specification reference (3.1):**
Further loci and regions in the Argand diagram.

**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (5 marks)
**AO2.1:** Construct rigorous mathematical arguments (including proofs) (1 mark)
**AO2.2a:** Make deductions and inferences: make deductions (1 mark)
**AO3.1a:** Translate problems in mathematical and non-mathematical contexts into mathematical processes: translate problems in mathematical contexts into mathematical processes (1 mark)

**(Total for Question 8 is 8 marks)**

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**9.** The operation * is defined on the set $S = \{0, 2, 3, 4, 5, 6\}$ by $x*y = x + y = xy \pmod 7$.

| * | 0 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 0 | | | | | | |
| 2 | | 0 | | | | |
| 3 | | | | | | 5 |
| 4 | | | | | | |
| 5 | | 4 | | | | |
| 6 | | | | | | |

(a) (i) Copy and complete the Cayley table shown above

(ii) Show that $S$ is a group under the operation *

(You may assume the associative law is satisfied.)

(b) Show that the element 4 has order 3.

(c) Find an element which generates the group and express each of the elements in terms of this generator.

**Specification reference (1.1, 1.2, 1.3, 2.3):**
The Axioms of a group.
Examples of groups. Cayley tables. Cyclic groups.
The order of a group and the order of an element. Subgroups.
The use of the CayleyHamilton theorem.

**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (6 marks)
**AO2.1:** Construct rigorous mathematical arguments (including proofs) (2 marks)
**AO2.2a:** Make deductions and inferences: make deductions (1 mark)
**AO2.5:** Use mathematical language and notation correctly (1 mark)
**AO3.1a:** Translate problems in mathematical and non-mathematical contexts into mathematical processes: translate problems in mathematical contexts into mathematical processes (1 mark)

**(Total for Question 9 is 11 marks)**

**10.** A population of deer on a large estate is assumed to increase by 10% during each year due to natural causes.

The population is controlled by removing a constant number, $Q$, of the deer from the estate at the end of each year.

At the start of the first year there are 5000 deer on the estate.

Let $P_n$ be the population of deer at the end of year $n$.

(a) Explain, in the context of the problem, the reason that the deer population is modelled by the recurrence relation

$$P_n = 1.1\,P_{n-1} - Q, \qquad P_0 = 5000, \qquad n \in \mathbb{Z}^+.$$

(b) Prove by induction that $P_n = (1.1)^n(5000 - 10Q) + 10Q, \quad n \geq 0$.

(c) Explain how the long term behaviour of this population varies for different values of $Q$.

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| **Specification reference (5.1, 5.2, 5.3):** |
| First order recurrence relations. |
| The solution of recurrence relations to obtain closed forms. |
| Proof by induction of closed forms. |

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| **AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (3 marks) |
| **AO2.1:** Construct rigorous mathematical arguments (including proofs) (1 mark) |
| **AO2.2a:** Make deductions and inferences: make deductions (1 mark) |
| **AO2.4:** Explain their reasoning (1 mark) |
| **AO3.3:** Translate situations in context into mathematical models (1 mark) |
| **AO3.4:** Use mathematical models (3 mark) |

**(Total for Question 10 is 10 marks)**

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# AS Further Mathematics Paper 2G (Further Statistics 1&2)

**1.** A university foreign language department carried out a survey of prospective students to find out which of three languages they were most interested in studying.

A random sample of 150 prospective students gave the following results.

| | | **Language** | | |
|---|---|---|---|---|
| | | **French** | **Spanish** | **Mandarin** |
| **Gender** | **Male** | 23 | 22 | 20 |
| | **Female** | 38 | 32 | 15 |

A test is carried out at the 1% level of significance to determine whether or not there is an association between gender and choice of language.

(a) State the null hypothesis for this test.

(b) Show that the expected frequency for females choosing Spanish is 30.6.

(c) Calculate the test statistic for this test, stating the expected frequencies you have used.

(d) State whether or not the null hypothesis is rejected. Justify your answer.

(e) Explain whether or not the null hypothesis would be rejected if the test was carried out at the 10% level of significance.

**Specification reference (3.1):**

Goodness of fit tests and Contingency Tables.

The null and alternative hypotheses.

The use of $\sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$ as an approximate $\chi^2$ statistic.

Degrees of freedom.

**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (3 marks)

**AO1.2:** Accurately recall facts, terminology and definitions (1 mark)

**AO2.1:** Construct rigorous mathematical arguments (including proofs) (1 mark)

**AO2.2b:** Make deductions and inferences: Where appropriate, evaluation accuracy and limitations of solutions to problems (1 mark)

**AO2.4:** Explain their reasoning (1 mark)

**AO3.1b:** Translate problems in mathematical and non-mathematical contexts into mathematical processes: translate problems in mathematical contexts into mathematical processes (1 mark)

**(Total for Question 1 is 8 marks)**

**2.** The discrete random variable $X$ has probability distribution given by

| $x$ | –1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $\mathrm{P}(X = x)$ | $c$ | $a$ | $a$ | $b$ | $c$ |

The random variable $Y = 2 - 5X$.

Given that $\mathrm{E}(Y) = -4$ and $\mathrm{P}(Y \geq -3) = 0.45$.

(a) find the probability distribution of $X$.

Given also that $\mathrm{E}(Y^2) = 75$,

(b) find the exact value of Var ($X$).

(c) Find $\mathrm{P}(Y > X)$.

**Specification reference (1.1, AS Further Mathematics Core Pure 3.7):**
Calculation of the mean and variance of discrete probability distributions.
Extension of expected value function to include $\mathrm{E}(\mathrm{g}(X))$.
Solve three linear simultaneous equations in three variables by use of the inverse matrix.

**AO1.1a:** Select and correctly carry out routine procedures: select routine procedures (1 mark)
**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (6 marks)
**AO1.2:** Accurately recall facts, terminology and definitions (1 mark)
**AO2.1:** Construct rigorous mathematical arguments (including proofs) (1 mark)
**AO3.1a:** Translate problems in mathematical and non-mathematical contexts into mathematical processes: translate problems in mathematical contexts into mathematical processes (2 marks)

**(Total for Question 2 is 11 marks)**

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**3.** Two car hire companies hire cars independently of each other.

Car Hire A hires cars at a rate of 2.6 cars per hour.

Car Hire B hires cars at a rate of 1.2 cars per hour.

(a) In a one-hour period, find the probability that each company hires exactly 2 cars.

(b) In a one-hour period, find the probability that the total number of cars hired by the two companies is 3.

(c) In a 2 hour period, find the probability that the total number of cars hired by the two companies is less than 9.

On average, 1 in 250 new cars produced at a factory has a defect.

In a random sample of 600 new cars produced at the factory,

(d) (i) find the mean of the number of cars with a defect,

(ii) find the variance of the number of cars with a defect.

(e) (i) Use a Poisson approximation to find the probability that no more than 4 of the cars in the sample have a defect.

(ii) Give a reason to support the use of a Poisson approximation.

**Specification reference (2.1, 2.2, 2.3):**
The Poisson distribution.
The additive property of Poisson distributions.
The mean and variance of the binomial distribution and the Poisson distribution.
The use of the Poisson distribution as an approximation to the binomial distribution.

**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (6 marks)
**AO2.4:** Explain their reasoning (1 mark)
**AO3.3:** Translate situations in context into mathematical models (2 marks)
**AO3.4:** Use mathematical models (1 mark)

**(Total for Question 3 is 10 marks)**

**4.** The discrete random variable $X$ follows a Poisson distribution with mean 1.4

(a) Write down the value of

(i) $P(X = 1)$,

(ii) $P(X \leq 4)$.

The manager of a bank recorded the number of mortgages approved each week over a 40 week period.

| **Number of mortgages approved** | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| **Frequency** | 10 | 16 | 7 | 4 | 2 | 0 | 1 |

(b) Show that the mean number of mortgages approved over the 40 week period is 1.4.

The bank manager believes that the Poisson distribution may be a good model for the number of mortgages approved each week.

She uses a Poisson distribution with a mean of 1.4 to calculate expected frequencies as follows.

| **Number of mortgages approved** | 0 | 1 | 2 | 3 | 4 | 5 or more |
|---|---|---|---|---|---|---|
| **Expected frequency** | 9.86 | $r$ | 9.67 | 4.51 | 1.58 | $s$ |

(c) Find the value of $r$ and the value of $s$, giving your answers to 2 decimal places.

The bank manager will test, at the 5% level of significance, whether or not the data can be modelled by a Poisson distribution.

(d) Calculate the test statistic and state the conclusion for this test. State clearly the degrees of freedom and the hypotheses used in the test.

**Specification reference (1.1, 2.1, 3.1):**
Calculation of the mean and variance of discrete probability distributions.
The Poisson distribution. The null and alternative hypotheses. Degrees of freedom

**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (6 marks)
**AO2.1:** Construct rigorous mathematical arguments (including proofs) (1 mark)
**AO3.1b:** Translate problems in mathematical and non-mathematical contexts into mathematical processes: translate problems in mathematical contexts into mathematical processes (1 mark)
**AO3.4:** Use mathematical models (2 marks)
**AO3.5a:** Evaluate the outcomes of modelling in context, recognise the limitations of models and, where appropriate, explain how to refine them: recognise the limitations of models (1 mark)

**(Total for Question 4 is 11 marks)**

**5.** In a gymnastics competition, two judges scored each of 8 competitors on the vault.

| **Competitor** | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| **Judge 1's scores** | 4.6 | 9.1 | 8.4 | 8.8 | 9.0 | 9.5 | 9.2 | 9.4 |
| **Judge 2's scores** | 7.8 | 8.8 | 8.6 | 8.5 | 9.1 | 9.6 | 9.0 | 9.3 |

(a) Calculate Spearman's rank correlation coefficient for these data.

(b) Stating your hypotheses clearly, test at the 1% level of significance, whether or not the two judges are generally in agreement.

(c) Give a reason to support the use of Spearman's rank correlation coefficient in this case.

The judges also scored the competitors on the beam.

Spearman's rank correlation coefficient for their ranks on the beam was found to be 0.952.

(d) Compare the judges' ranks on the vault with their ranks on the beam.

**Specification reference (3.1, 3.2, 3.3):**

The null and alternative hypotheses.

Spearman's rank correlation coefficient, its use and interpretation.

Testing the hypothesis that a correlation is zero using either Spearman's rank correlation or the product moment correlation coefficient.

**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (5 marks)

**AO2.1:** Construct rigorous mathematical arguments (including proofs) (1 mark)

**AO2.2b:** Make deductions and inferences: Where appropriate, evaluation accuracy and limitations of solutions to problems (2 marks)

**AO2.4:** Explain their reasoning (1 mark)

**AO2.5:** Use mathematical language and notation correctly (1 mark)

**(Total for Question 5 is 10 marks)**

**6.** The continuous random variable $X$ has probability density function

$$f(x) = \begin{cases} \frac{1}{18}(11-2x) & 1 \leq x \leq 4 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $P(X < 3)$

(b) State, giving a reason, whether the upper quartile of X is greater than 3, less than 3 or equal to 3.

Given that $E(X) = \frac{9}{4}$,

(c) use algebraic integration to find $\text{Var}(X)$.

The cumulative distribution function of $X$ is given by

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{18}(11-2x) & 1 \leq x \leq 4 \\ 1 & \text{otherwise.} \end{cases}$$

(d) Show that $c = –10$.

(e) Find the median of $X$, giving your answer to 3 significant figures.

| |
|---|
| **Specification reference (2.1, 2.3):** |
| The concept of a continuous random variable. |
| The probability density function and the cumulative distribution function for a continuous random variable. |
| Mean and variance of continuous random variables. |
| Mode, median and percentiles of continuous random variables. |

| |
|---|
| **AO1.1a:** Select and correctly carry out routine procedures: select routine procedures (1 mark) |
| **AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (6 marks) |
| **AO1.2:** Accurately recall facts, terminology and definitions (1 mark) |
| **AO2.1:** Construct rigorous mathematical arguments (including proofs) (1 mark) |
| **AO2.2a:** Make deductions and inferences: make deductions (2 marks) |

**(Total for Question 6 is 11 marks)**

**7.** A scientist wants to develop a model to describe the relationship between the average daily temperature, $x$ °C, and a household's daily energy consumption, $y$ kWh, in winter.

A random sample of the average temperature and energy consumption are taken from 10 winter days and are summarised below.

$$\sum x = 12 \qquad \sum x^2 = 24.76 \qquad \sum y = 251 \qquad \sum y^2 = 6341 \qquad \sum xy = 284.8$$

$$S_{xx} = 10.36 \qquad S_{yy} = 40.9$$

(a) Find the product moment correlation coefficient between $y$ and $x$.

(b) Find the equation of the regression line of $y$ on $x$ in the form $y = a + bx$.

(c) Use your equation to estimate the daily energy consumption when the average daily temperature is 2 °C.

(d) Calculate the residual sum of squares (RSS).

The table shows the residual for each value of $x$.

| $x$ | –0.4 | –0.2 | 0.3 | 0.8 | 1.1 | 1.4 | 1.8 | 2.1 | 2.5 | 2.6 |
|---|---|---|---|---|---|---|---|---|---|---|
| **Residual** | –0.63 | –0.32 | –0.52 | –0.73 | 0.74 | 2.22 | 1.84 | 0.32 | $f$ | –1.88 |

(e) Find the value of $f$.

(f) By considering the signs of the residuals, explain whether or not the linear regression model is a suitable model for these data.

**Specification reference (1.1, 1.2, 3.1):**

Least squares linear regression. The concept of residuals and minimising the sum of squares of residuals.

Residuals.

The residual sum of squares (RSS).

Use of formulae to calculate the product moment correlation coefficient.

Knowledge of the conditions for the use of the product moment correlation coefficient.

**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (7 marks)

**AO3.1a:** Translate problems in mathematical and non-mathematical contexts into mathematical processes: translate problems in mathematical contexts into mathematical processes (1 mark)

**AO3.3:** Translate situations in context into mathematical models (1 mark)

**AO3.4:** Use mathematical models (1 mark)

**AO3.5b:** Evaluate the outcomes of modelling in context, recognise the limitations of models and, where appropriate, explain how to refine them: recognise the limitations of models (1 mark)

**(Total for Question 7 is 11 marks)**

**8.** The continuous random variable $X$ is uniformly distributed over the interval [–3, 5].

(a) Sketch the probability density function f($x$) of $X$.

(b) Find the value of k such that $P(X < 2[k - X]) = 0.25$.

(c) Use algebraic integration to show that $E(X^3) = 17$.

| |
|---|
| **Specification reference (2.3, 2.4):**<br>Mean and variance of continuous random variables.<br>Extension of expected value function to include E(g(X)).<br>The continuous uniform (rectangular) distribution. |

| |
|---|
| **AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (6 marks)<br>**AO2.1:** Construct rigorous mathematical arguments (including proofs) (1 mark)<br>**AO3.1a:** Translate problems in mathematical and non-mathematical contexts into mathematical processes: translate problems in mathematical contexts into mathematical processes (1 mark) |

**(Total for Question 8 is 8 marks)**

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# AS Further Mathematics Paper 2J (Further Mechanics 1&2)

Unless otherwise indicated, whenever a numerical value of g is required, take g = 9.8 m s$^{-2}$ and give your answer to either 2 significant figures or 3 significant figures.

**1.** A small ball of mass 0.1 kg is dropped from a point which is 2.4 m above a horizontal floor. The ball falls freely under gravity, strikes the floor and bounces to a height of 0.6 m above the floor. The ball is modelled as a particle.

(a) Show that the coefficient of restitution between the ball and the floor is 0.5.

(b) Find the height reached by the ball above the floor after it bounces on the floor for the second time.

(c) By considering your answer to part (*b*), describe the subsequent motion of the ball.

**Specification reference (3.1, 3.2):**

Kinematics of a particle moving in a straight line when the acceleration is a function of the time ($t$) or velocity ($v$).

Direct impact of elastic spheres. Newton's law of restitution. Loss of kinetic energy due to impact.

Successive direct impacts of spheres and/or a sphere with a smooth plane surface.

**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (4 marks)

**AO2.2b:** Make deductions and inferences: Where appropriate, evaluation accuracy and limitations of solutions to problems (1 mark)

**AO3.1b:** Translate problems in mathematical and non-mathematical contexts into mathematical processes: translate problems in mathematical contexts into mathematical processes (1 mark)

**AO3.4:** Use mathematical models (4 marks)

**(Total for Question 1 is 10 marks)**

**2.** A small stone of mass 0.5 kg is thrown vertically upwards from a point *A* with an initial speed of 25 m s$^{-1}$. The stone first comes to instantaneous rest at the point *B* which is 20 m vertically above the point *A*. As the stone moves it is subject to air resistance. The stone is modelled as a particle.

(a) Find the energy lost due to air resistance by the stone, as it moves from *A* to *B*.

The air resistance is modelled as a constant force of magnitude *R* newtons.

(b) Find the value of *R*.

(c) State how the model for air resistance could be refined to make it more realistic.

**Specification reference (2.1):**
Kinetic and potential energy, work and power. The work-energy principle. The principle of conservation of mechanical energy.

**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (3 marks)

**AO3.3:** Translate situations in context into mathematical models (2 marks)

**AO3.5c:** Evaluate the outcomes of modelling in context, recognise the limitations of models and, where appropriate, explain how to refine them: recognise the limitations of models (1 mark)

**(Total for Question 2 is 6 marks)**

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**3.** [In this question use $g = 10 \text{ m s}^{-2}$]

A jogger of mass 60 kg runs along a straight horizontal road at a constant speed of $4 \text{ m s}^{-1}$. The total resistance to the motion of the jogger is modelled as a constant force of magnitude 30 N.

(a) Find the rate at which the jogger is working.

The jogger now comes to a hill which is inclined to the horizontal at an angle $\alpha$, where $\sin \alpha = \frac{1}{15}$. Because of the hill, the jogger reduces her speed to $3 \text{ m s}^{-1}$ and maintains this constant speed as she runs up the hill. The total resistance to the motion of the jogger from non-gravitational forces continues to be modelled as a constant force of magnitude 30 N.

(b) Find the rate at which she has to work in order to run up the hill at $3 \text{ m s}^{-1}$.

**Specification reference (2.1):**
Kinetic and potential energy, work and power. The work-energy principle. The principle of conservation of mechanical energy.

**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (6 marks)
**AO3.1b:** Translate problems in mathematical and non-mathematical contexts into mathematical processes: translate problems in mathematical contexts into mathematical processes (2 marks)

**(Total for Question 3 is 8 marks)**

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**4.** A particle $P$ of mass $3m$ is moving in a straight line on a smooth horizontal table.

A particle $Q$ of mass $m$ is moving in the opposite direction to $P$ along the same straight line. The particles collide directly. Immediately before the collision the speed of $P$ is $u$ and the speed of $Q$ is $2u$. The velocities of $P$ and $Q$ immediately after the collision, measured in the direction of motion of $P$ before the collision, are $v$ and $w$ respectively. The coefficient of restitution between $P$ and $Q$ is $e$.

(a) Find an expression for $v$ in terms of $u$ and $e$.

Given that the direction of motion of $P$ is changed by the collision,

(a) find the range of possible values of $e$.

(c) Show that w = $\frac{u}{4}(1 + 9e)$.

Following the collision with $P$, the particle $Q$ then collides with and rebounds from a fixed vertical wall which is perpendicular to the direction of motion of $Q$. The coefficient of restitution between $Q$ and the wall is $f$.

Given that $e = \frac{5}{9}$, and that $P$ and $Q$ collide again in the subsequent motion,

(d) find the range of possible values of $f$ .

**Specification reference (1.1, 3.1, 3.2, AS Pure Mathematics 2.4):**

Momentum and impulse. The impulse-momentum principle. The principle of conservation of momentum applied to two spheres colliding directly.

Kinematics of a particle moving in a straight line when the acceleration is a function of the time ($t$) or velocity ($v$).

Direct impact of elastic spheres. Newton's law of restitution. Loss of kinetic energy due to impact.

Successive direct impacts of spheres and/or a sphere with a smooth plane surface.

Solve simultaneous equations in two variables by elimination and by substitution, including one linear and one quadratic equation.

**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (10 marks)

**AO2.1:** Construct rigorous mathematical arguments (including proofs) (1 mark)

**AO3.1a:** Translate problems in mathematical and non-mathematical contexts into mathematical processes: translate problems in mathematical contexts into mathematical processes (2 marks)

**AO3.1b:** Translate problems in mathematical and non-mathematical contexts into mathematical processes: translate problems in mathematical contexts into mathematical processes (2 marks)

**AO3.4:** Use mathematical models (1 mark)

**(Total for Question 4 is 16 marks)**

Unless otherwise indicated, whenever a numerical value of g is required, take $g = 9.8$ m s$^{-2}$ and give your answer to either 2 significant figures or 3 significant figures.

**5.** A particle $P$ moves on the $x$-axis. At time $t$ seconds the velocity of $P$ is $v$ m s$^{-1}$ in the direction of $x$ increasing, where

$$v = (t - 2)(3t - 10), \qquad t \geq 0.$$

When $t = 0$, $P$ is at the origin $O$.

(a) Find the acceleration of $P$ at time $t$ seconds.

(b) Find the total distance travelled by $P$ in the first 2 seconds of its motion.

(c) Show that $P$ never returns to $O$, explaining your reasoning.

**Specification reference (3.1):**
Kinematics of a particle moving in a straight line when the acceleration is a function of the time ($t$) or velocity ($v$).

**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (5 marks)
**AO2.1:** Construct rigorous mathematical arguments (including proofs) (1 mark)
**AO2.4:** Explain their reasoning (1 mark)
**AO3.2a:** Interpret solutions to problems in their original context, and, where appropriate evaluate their accuracy and limitations: interpret solutions to problems in their original context (1 mark)

**(Total for Question 5 is 8 marks)**

**6.** A light inextensible string has length $7a$. One end of the string is attached to a fixed point $A$ and the other end of the string is attached to a fixed point $B$, with $A$ vertically above $B$ and $AB = 5a$.

A particle of mass $m$ is attached to a point $P$ on the string where $AP = 4a$. The particle moves in a horizontal circle with constant angular speed $\omega$, with both $AP$ and $BP$ taut.

(a) Show that

(i) the tension in $AP$ is $\frac{4m}{25}(9a\omega^2 + 5g)$,

(ii) the tension in $BP$ is $\frac{3m}{25}(16a\omega^2 - 5g)$.

The string will break if the tension in it reaches a magnitude of $4mg$.
The time for the particle to make one revolution is $S$.

(b) Show that

$$3\pi\sqrt{\frac{a}{5g}} < S < 8\pi\sqrt{\frac{a}{5g}}$$

(c) State how in your calculations you have used the assumption that the string is light.

| **Specification reference (1.1):** |
| --- |
| Angular speed. $v = r\omega$ |
| Radial acceleration in circular motion. |
| The forms $rw^2$ and $\frac{v^2}{r}$ are required. |
| Uniform motion of a particle moving in a horizontal circle |

| |
| --- |
| **AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (9 marks) |
| **AO2.1:** Construct rigorous mathematical arguments (including proofs) (3 marks) |
| **AO2.2a:** Make deductions and inferences: make deductions (1 mark) |
| **AO3.1b:** Translate problems in mathematical and non-mathematical contexts into mathematical processes: translate problems in mathematical contexts into mathematical processes (2 marks) |
| **AO3.5b:** Evaluate the outcomes of modelling in context, recognise the limitations of models and, where appropriate, explain how to refine them: recognise the limitations of models (1 mark) |

**(Total for Question 6 is 16 marks)**

**7.**

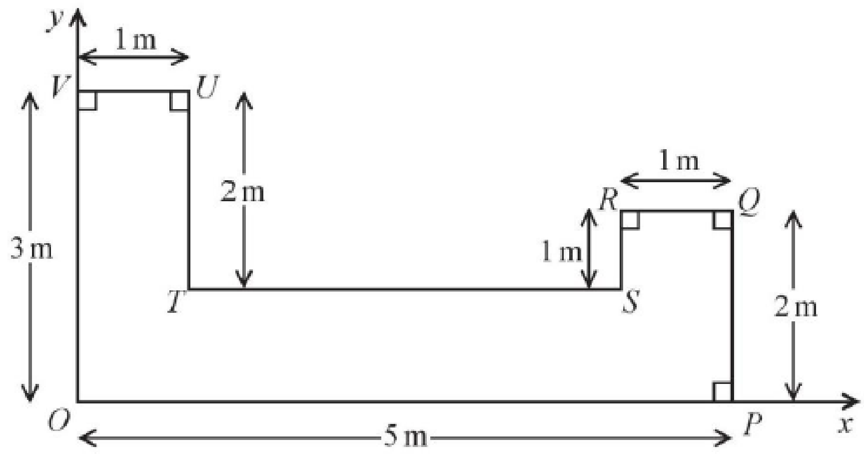



**Figure 1**

Figure 1 shows the shape and dimensions of a template *OPQRSTUV* made from thin uniform metal.

$OP = 5$ m, $PQ = 2$ m, $QR = 1$ m, $RS = 1$ m, $TU = 2$ m, $UV = 1$ m, $VO = 3$ m.

Figure 1 also shows a coordinate system with $O$ as origin and the $x$-axis and $y$-axis along $OP$ and $OV$ respectively. The unit of length on both axes is the metre. The centre of mass of the template has coordinates $(\bar{x}, \bar{y})$.

(a) (i) Show that $\bar{y} = 1$,

(ii) Find the value of $\bar{x}$.

A new design requires the template to have its centre of mass at the point (2.5, 1). In order to achieve this, two circular discs, each of radius $r$ metres, are removed from the template which is shown in Figure 1, to form a new template $L$. The centre of the first disc is (0.5, 0.5) and the centre of the second disc is (0.5, $a$) where $a$ is a constant.

(b) Find the value of $r$.

(c) (i) Explain how symmetry can be used to find the value of $a$.

(ii) Find the value of $a$.

The template $L$ is now freely suspended from the point $U$ and hangs in equilibrium.

(d) Find the size of the angle between the line $TU$ and the horizontal.

**Specification reference (2.1, 2,2):**

Moment of a force. Centre of mass of a discrete mass distribution in one and two dimensions.

Centre of mass of uniform plane figures, and simple cases of composite plane figures.

Centre of mass of frameworks.

Equilibrium of a plane lamina or framework under the action of coplanar forces.

**AO1.1a:** Select and correctly carry out routine procedures: select routine procedures (1 mark)

**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (7 marks)

**AO1.2:** Accurately recall facts, terminology and definitions (3 marks)

**AO2.1:** Construct rigorous mathematical arguments (including proofs) (2 marks)

**AO2.2a:** Make deductions and inferences: make deductions (1 mark)

**AO2.4:** Explain their reasoning (1 mark)

**AO3.1b:** Translate problems in mathematical and non-mathematical contexts into mathematical processes: translate problems in mathematical contexts into mathematical processes (1 mark)

**(Total for Question 7 is 16 marks)**

# AS Further Mathematics Paper 2K (Decision Mathematics 1&2)

**1.**

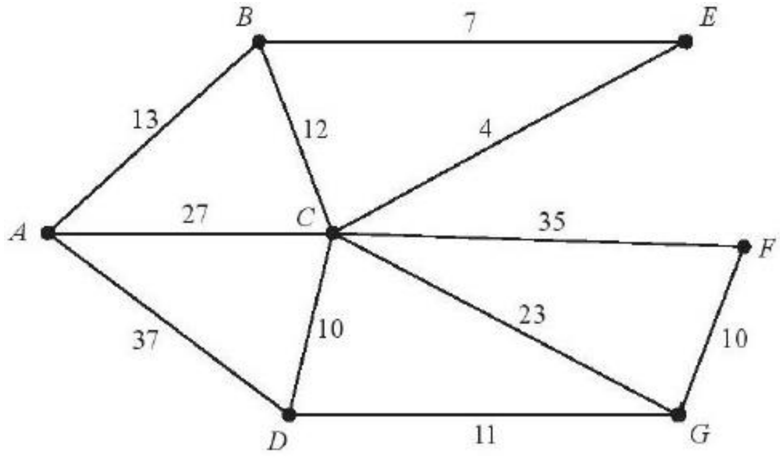



**Figure 1**
[*The total weight of the network is 189*]

Figure 1 represents a network of pipes in a building. The number on each arc is the length, in metres, of the corresponding pipe.

(a) Use Dijkstra's algorithm to find the shortest path from $A$ to $F$. State the path and its length.

On a particular day, Gabriel needs to check each pipe. A route of minimum length, which traverses each pipe at least once and which starts and finishes at $A$, needs to be found.

(b) Use an appropriate algorithm to find the pipes that will need to be traversed twice.
You must make your method and working clear.

(c) State the minimum length of Gabriel's route.

A new pipe, $BG$, is added to the network. A route of minimum length that traverses each pipe, including $BG$, needs to be found. The route must start and finish at $A$. Gabriel works out that the addition of the new pipe increases the length of the route by twice the length of $BG$.

(d) Calculate the length of $BG$. You must show your working.

**Specification reference (2.2, 3.1):**

Dijkstra's algorithm for finding the shortest path.

Algorithm for finding the shortest route around a network, travelling along every edge at least once and ending at the start vertex (The Route Inspection Algorithm).

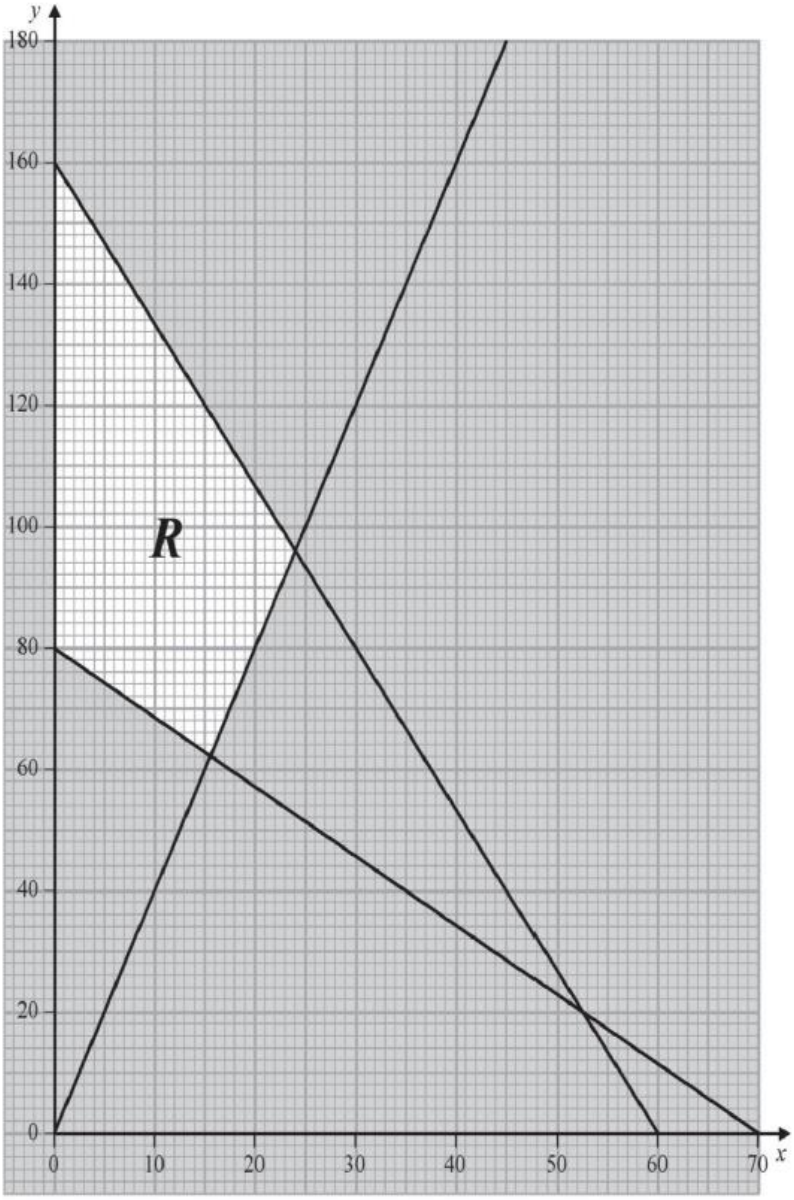
**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (10 marks)

**AO2.2a:** Make deductions and inferences: make deductions (1 mark)

**AO3.1b:** Translate problems in mathematical and non-mathematical contexts into mathematical processes: translate problems in mathematical contexts into mathematical processes (1 mark)

**(Total for Question 1 is 12 marks)**

**2.**




A teacher buys pens and pencils. The number of pens, $x$, and the number of pencils, $y$, that he buys can be represented by a linear programming problem as shown in Figure 2, which models the following constraints:

$$8x + 3y \leq 480$$

$$8x + 7y \geq 560$$

$$y \geq 4x$$

$$x, y \geq 0$$

The total cost, in pence, of buying the pens and pencils is given by $C = 12x + 15y$.

Determine the number of pens and the number of pencils which should be bought in order to minimise the total cost. You should make your method and working clear.

| **Specification reference (5.2):** Graphical solution of two variable problems using objective line and vertex methods including cases where integer solutions are required. |
|---|

| |
|---|
| **AO1.1a:** Select and correctly carry out routine procedures: select routine procedures (1 mark) |
| **AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (3 marks) |
| **AO2.2a:** Make deductions and inferences: make deductions (1 mark) |
| **AO3.1a:** Translate problems in mathematical and non-mathematical contexts into mathematical processes: translate problems in mathematical contexts into mathematical processes (1 mark) |
| **AO3.2a:** Interpret solutions to problems in their original context, and, where appropriate evaluate their accuracy and limitations: interpret solutions to problems in their original context (1 mark) |

**(Total for Question 2 is 7 marks)**

**3.**

| Activity | Time taken (days) | Immediately preceding activities |
|---|---|---|
| A | 5 | – |
| B | 7 | – |
| C | 3 | – |
| D | 4 | A, B |
| E | 4 | D |
| F | 2 | B |
| G | 4 | B |
| H | 5 | C, G |
| I | 10 | C, G |

The table above shows the activities required for the completion of a building project.

For each activity, the table shows the time taken in days to complete the activity and the immediately preceding activities. Each activity requires one worker. The project is to be completed in the shortest possible time.

(a) Draw the activity network described in the table, using activity on arc. Your activity network must contain the minimum number of dummies only.

(b) (i) Show that the project can be completed in 21 days, showing your working.

(ii) Identify the critical activities.

**Specification reference (4.1, 4.3):**

Modelling of a project by an activity network, from a precedence table.

Algorithm for finding the critical path. Earliest and latest event times. Earliest and latest start and finish times for activities. Identification of critical activities and critical path(s).

**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (5 marks)

**AO2.1:** Construct rigorous mathematical arguments (including proofs) (1 mark)

**AO2.2a:** Make deductions and inferences: make deductions (1 mark)

**(Total for Question 3 is 7 marks)**

**4.** (a) Explain why it is not possible to draw a graph with exactly 5 nodes with orders 1, 3, 4, 4 and 5.

A connected graph has exactly 5 nodes and contains 18 arcs. The orders of the 5 nodes are $2^{2x-1}$, $2^x$, $x+1$, $2^{x+1}-3$ and $11-x$.

(b) (i) Calculate $x$.

(ii) State whether the graph is Eulerian, semi-Eulerian or neither. You must justify your answer.

(c) Draw a graph which satisfies all of the following conditions:

- The graph has exactly 5 nodes.
- The nodes have orders 2, 2, 4, 4 and 4.
- The graph is not Eulerian.

| **Specification reference (1.3):** |
| --- |
| Use of the order of the nodes to determine whether a graph is Eulerian, semi-Eulerian or neither. |

| |
| --- |
| **AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (3 marks) |
| **AO2.1:** Construct rigorous mathematical arguments (including proofs) (1 mark) |
| **AO2.2a:** Make deductions and inferences: make deductions (2 marks) |
| **AO2.4:** Explain their reasoning (2 marks) |
| **AO2.5:** Use mathematical language and notation correctly (1 mark) |

**(Total for Question 4 is 9 marks)**

**5.** Jonathan makes two types of information pack for an event, Standard and Value.

Each Standard pack contains 25 posters and 500 flyers.

Each Value pack contains 15 posters and 800 flyers.

He must use at least 150 000 flyers.

Between 35% and 65% of the packs must be Standard packs.

Posters cost 20p each and flyers cost 4p each.

Jonathan wishes to minimise his costs.

Let $x$ and $y$ represent the number of Standard packs and Value packs produced respectively.

Formulate this as a linear programming problem, stating the objective and listing the constraints as simplified inequalities with integer coefficients.

You should not attempt to solve the problem.

**Specification reference (5.1):**
Formulation of problems as linear programs.

**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (1 mark)
**AO3.3:** Translate situations in context into mathematical models (4 marks)

**(Total for Question 5 is 5 marks)**

**6.** Six workers, *A*, *B*, *C*, *D*, *E* and *F*, are to be assigned to five tasks, *P*, *Q*, *R*, *S* and *T*.

Each worker can be assigned to at most one task and each task must be done by just one worker.

The time, in minutes, that each worker takes to complete each task is shown in the table below.

| | *P* | *Q* | *R* | *S* | *T* |
|---|---|---|---|---|---|
| *A* | 32 | 32 | 35 | 34 | 33 |
| *B* | 28 | 35 | 31 | 37 | 40 |
| *C* | 35 | 29 | 33 | 36 | 35 |
| *D* | 36 | 30 | 34 | 33 | 35 |
| *E* | 30 | 31 | 29 | 37 | 36 |
| *F* | 29 | 28 | 32 | 31 | 34 |

Reducing rows first, use the Hungarian algorithm to obtain an allocation which minimises the total time. You must explain your method and show the table after each stage.

**Specification reference (2.1):**

Cost matrix reduction.

Use of the Hungarian algorithm to find a least cost allocation.

**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (8 marks)

**AO2.2a:** Make deductions and inferences: make deductions (1 mark)

**(Total for Question 6 is 9 marks)**

**7.** In two-dimensional space, lines divide a plane into a number of different regions.

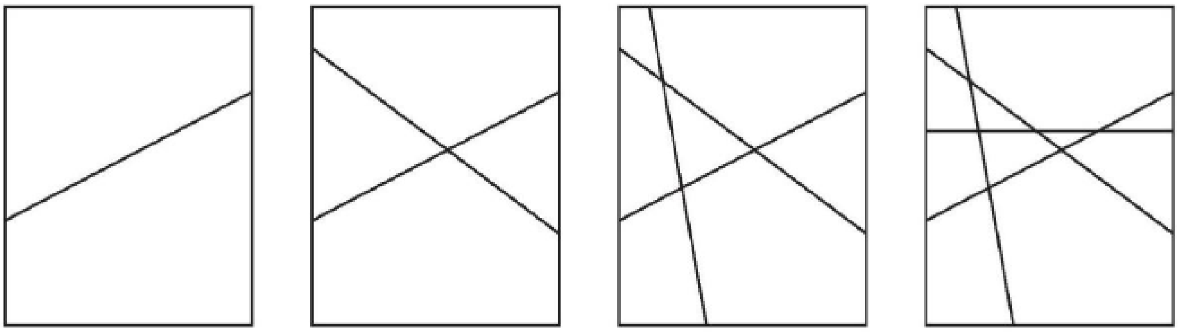


**Figure 1** **Figure 2** **Figure 3** **Figure 4**

It is known that:

- One line divides a plane into 2 regions, as shown in Figure 1,
- Two lines divide a plane into a maximum of 4 regions, as shown in Figure 2,
- Three lines divide a plane into a maximum of 7 regions, as shown in Figure 3,
- Four lines divide a plane into a maximum of 11 regions, as shown in Figure 4.

(a) Complete the table in the answer book to show the maximum number of regions when five, six and seven lines divide a plane.

(b) Find, in terms of $u_n$, the recurrence relation for $u_{n+1}$, the maximum number of regions when a plane is divided by $(n + 1)$ lines, where $n \geq 1$.

(c) (i) Solve the recurrence relation for $u_n$.

(ii) Hence determine the maximum number of regions created when 200 lines divide a plane.

| **Specification reference (4.1):** |
| --- |
| Use of recurrence relations to model appropriate problems. |

| **AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (4 marks) |
| --- |
| **AO3.3:** Translate situations in context into mathematical models (1 mark) |

**(Total for Question 7 is 5 marks)**

**8.**

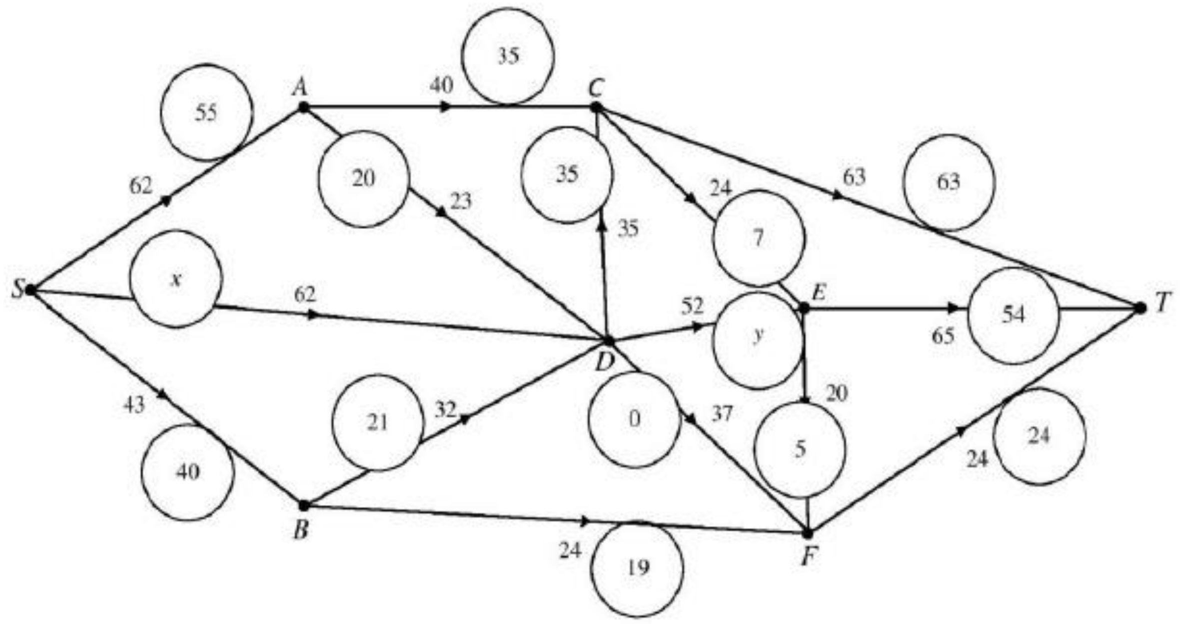



**Figure 5**

Figure 5 represents a network of corridors in a school. The number on each arc represents the maximum number of students, per minute, that may pass along each corridor at any one time.

At 11 a.m. on Friday morning, all students leave the hall ($S$) after assembly and travel to the cybercafe ($T$). The numbers in circles represent the initial flow of students recorded at 11 a.m. one Friday.

(a) State an assumption that has been made about the corridors in order for this situation to be modelled by a directed network.

(b) Find the value of $x$ and the value of $y$, explaining your reasoning.

Five new students also attend the assembly in the hall the following Friday. They too need to travel to the cybercafe at 11 a.m. They wish to travel together so that they do not get lost. You may assume that the initial flow of students through the network is the same as that shown in Figure 5 above.

(c) (i) List all the flow augmenting routes from $S$ to $T$ that increase the flow by at least 5.

(ii) State which route the new students should take, giving a reason for your answer.

(d) Use the answer to part (c) to find a maximum flow pattern for this network.

(e) Prove that the answer to part (d) is optimal.

The school is intending to increase the number of students it takes but has been informed it cannot do so until it improves the flow of students at peak times. The school can widen corridors to increase their capacity, but can only afford to widen one corridor in the coming term.

(f) State, explaining your reasoning,

(i) which corridor they should widen,

(ii) the resulting increase of flow through the network.

**Specification reference (2.1, 2.2, 2.3):**

Cuts and their capacity.

Use of the labelling procedure to augment a flow to determine the maximum flow in a network.

Use of the max–flow min–cut theorem to prove that a flow is a maximum flow.

**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (5 marks)

**AO2.1:** Construct rigorous mathematical arguments (including proofs) (2 marks)

**AO2.2a:** Make deductions and inferences: make deductions (3 marks)

**AO2.4:** Explain their reasoning (2 marks)

**AO3.1a:** Translate problems in mathematical and non-mathematical contexts into mathematical processes: translate problems in mathematical contexts into mathematical processes (1 mark)

**AO3.4:** Use mathematical models (1 mark)

**(Total for Question 8 is 14 marks)**

**9.** A two person zero-sum game is represented by the following pay-off matrix for player *A*.

| | *B* plays 1 | *B* plays 2 | *B* plays 3 |
|---|---|---|---|
| *A* plays 1 | 4 | 1 | 2 |
| *A* plays 2 | 2 | 4 | 3 |

(a) Verify that there is no stable solution.

(b) (i) Find the best strategy for player *A*.

(ii) Find the value of the game to her.

**Specification reference (3.1, 3.2, 3.4):**

Two person zero-sum games and the pay-off matrix.

Identification of play safe strategies and stable solutions (saddle points).

Optimal mixed strategies for a game with no stable solution by use of graphical methods for $2 \times n$ or $n \times 2$ games where $n = 1, 2, 3$ or 4.

**AO1.1b:** Select and correctly carry out routine procedures: correctly carry out routine procedures (8 marks)

**AO2.4:** Explain their reasoning (1 mark)

**AO3.1a:** Translate problems in mathematical and non-mathematical contexts into mathematical processes: translate problems in mathematical contexts into mathematical processes (1 mark)

**AO3.2a:** Interpret solutions to problems in their original context, and, where appropriate evaluate their accuracy and limitations: interpret solutions to problems in their original context (1 mark)

**AO3.3:** Translate situations in context into mathematical models (1 mark)

**(Total for Question 9 is 12 marks)**

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