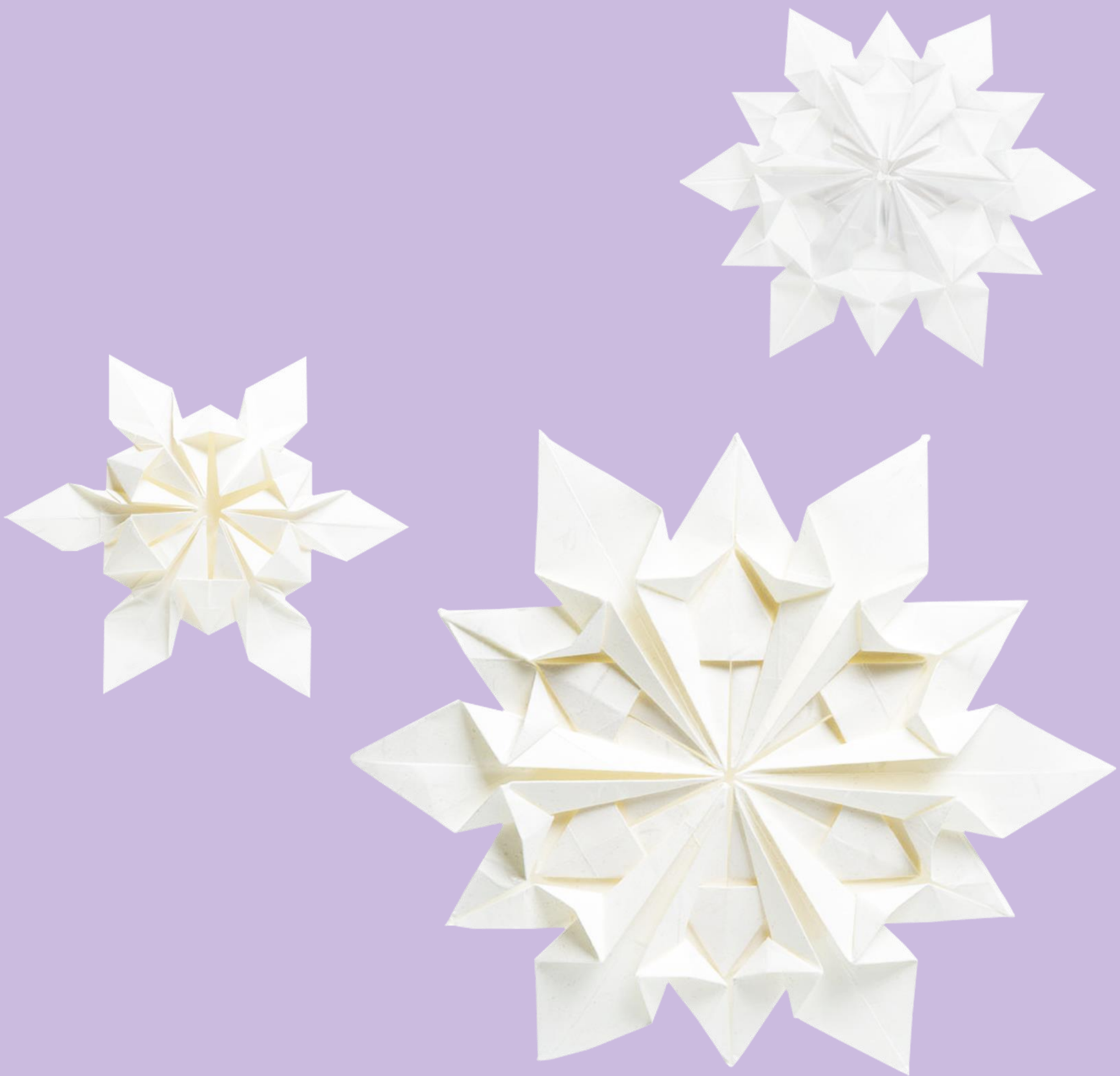
Pearson Edexcel

Level 3 Advanced
GCE in Further Mathematics (9FM0)



Enhanced Content Guidance for 9FM0

First teaching from September 2017

Contents

Content of this document is based on the accredited version of Pearson Edexcel Level 3 Advanced GCE in Further Mathematics (9FM0) and includes enhanced content guidance to accompany the specification document.

To support the co-teaching of this qualification with the AS Mathematics qualification, common content has been highlighted in bold.

Our free support for the AS and A level Mathematics and Further Mathematics specifications can be found on the Pearson Edexcel Mathematics website (quals.pearson.com/Alevelmaths2017) and on the Emporium (www.edexcelmaths.com).

Updates to this document since the previous version (version 1)	
Change(s) made	**Page no.**
Correction to inequality symbols in section 3.1 - changed to P($Y \leqslant n$) = P($X \geqslant r$)	31

Paper 1 and Paper 2: Core Pure Mathematics

Topic	What students need to learn:				Enhanced content guidance
	Content		Guidance	
1 **Proof**	1.1	**Construct proofs using mathematical induction.** **Contexts include sums of series, divisibility and powers of matrices.**	**To include induction proofs for** **(i) summation of series** **e.g. show** $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$ **or** **show** $\sum_{r=1}^{n} r(r+1) = \frac{n(n+1)(n+2)}{3}$ **(ii) divisibility e.g. show** $3^{2n}+11$ **is divisible by 4** **(iii) matrix products e.g. show** $\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^n = \begin{pmatrix} 2n+1 & -4n \\ n & 1-2n \end{pmatrix}$	**Students need to understand the concept of proof by induction and be able to apply it in a variety of contexts. Proof by induction in contexts other than those listed in the guidance may be set.** **For example:** **(i) Simple recurrence relations** **If** $u_{n+1} = 6u_n - 5$ **with** $u_1 = 6$, **prove that** $u_n = 5\times 6^{n-1}+1$ **(ii) Differentiation** **If** $y = \frac{1}{x}$ **prove that** $\frac{d^n y}{dx^n} = (-1)^n n!\, x^{-(n+1)}$
2 **Complex numbers**	2.1	**Solve any quadratic equation with real coefficients.** **Solve cubic or quartic equations with real coefficients.**	**Given sufficient information to deduce at least one root for cubics or at least one complex root or quadratic factor for quartics, for example:** (i) $f(z) = 2z^3 + 5z^2 + 7z + 10$ **Given that** $z + 2$ **is a factor of** $f(z)$**, use algebra to solve** $f(z) = 0$ **completely.** (ii) $g(x) = x^4 - x^3 + 6x^2 + 14x - 20$ **Given** $g(1) = 0$ **and** $g(-2) = 0$**, use algebra to solve** $g(x) = 0$ **completely.**	

Topic	What students need to learn:			
	Content		Guidance	Enhanced content guidance
2 **Complex numbers** *continued*	2.2	**Add, subtract, multiply and divide complex numbers in the form $x + \mathrm{i}y$ with x and y real.** **Understand and use the terms 'real part' and 'imaginary part'.**	**Students should know the meaning of the terms, 'modulus' and 'argument'.**	
	2.3	**Understand and use the complex conjugate.** **Know that non-real roots of polynomial equations with real coefficients occur in conjugate pairs.**	**Knowledge that if z_1 is a root of $\mathrm{f}(z) = 0$ then $z_1{}^*$ is also a root.**	
	2.4	**Use and interpret Argand diagrams.**	**Students should be able to represent the sum or difference of two complex numbers on an Argand diagram.**	
	2.5	**Convert between the Cartesian form and the modulus-argument form of a complex number.**	Knowledge of radians is assumed.	

Topic	What students need to learn:			
	Content		Guidance	Enhanced content guidance
2 **Complex numbers** *continued*	2.6	**Multiply and divide complex numbers in modulus argument form.**	**Knowledge of the results** $\lvert z_1 z_2 \rvert = \lvert z_1 \rvert \lvert z_2 \rvert$, $\left\lvert \frac{z_1}{z_2} \right\rvert = \frac{\lvert z_1 \rvert}{\lvert z_2 \rvert}$ $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$ Knowledge of radians and compound angle formulae is assumed.	
	2.7	**Construct and interpret simple loci in the argand diagram such as $\lvert z-a \rvert > r$ and $\arg(z-a) = \theta$.**	**To include loci such as $\lvert z-a \rvert = b$, $\lvert z-a \rvert = \lvert z-b \rvert$, $\arg(z-a) = \beta$, and regions such as $\lvert z-a \rvert \le \lvert z-b \rvert$, $\lvert z-a \rvert \le b$, $\alpha < \arg(z-a) < \beta$** Knowledge of radians is assumed.	

Topic	What students need to learn:			
	Content		Guidance	Enhanced content guidance
2 **Complex numbers** *continued*	2.8	Understand de Moivre's theorem and use it to find multiple angle formulae and sums of series.	To include using the results, $z + \frac{1}{z} = 2\cos\theta$ and $z - \frac{1}{z} = 2\text{i}\sin\theta$ to find $\cos p\theta$, $\sin q\theta$ and $\tan r\theta$ in terms of powers of $\sin\theta$, $\cos\theta$ and $\tan\theta$ and powers of $\sin\theta$, $\cos\theta$ and $\tan\theta$ in terms of multiple angles. For sums of series, students should be able to show that, for example, $1 + z + z^2 + \ldots + z^{n-1} = 1 + \text{i}\cot\left(\frac{\pi}{2n}\right)$ where $z = \cos\left(\frac{\pi}{n}\right) + \text{i}\sin\left(\frac{\pi}{n}\right)$ and n is a positive integer.	
	2.9	Know and use the definition $\text{e}^{\text{i}\theta} = \cos\theta + \text{i}\sin\theta$ and the form $z = r\text{e}^{\text{i}\theta}$	Students should be familiar with $\cos\theta = \frac{1}{2}(\text{e}^{\text{i}\theta} + \text{e}^{-\text{i}\theta})$ and $\sin\theta = \frac{1}{2\text{i}}(\text{e}^{\text{i}\theta} - \text{e}^{-\text{i}\theta})$	
	2.10	Find the n distinct nth roots of $r\text{e}^{\text{i}\theta}$ for $r \neq 0$ and know that they form the vertices of a regular n-gon in the Argand diagram.		
	2.11	Use complex roots of unity to solve geometric problems.		

Topic	What students need to learn:			
	Content		Guidance	Enhanced content guidance
3 **Matrices**	3.1	**Add, subtract and multiply conformable matrices.** **Multiply a matrix by a scalar.**		
	3.2	**Understand and use zero and identity matrices.**		
	3.3	**Use matrices to represent linear transformations in 2-D.** **Successive transformations.** **Single transformations in 3-D.**	**For 2-D, identification and use of the matrix representation of single and combined transformations from: reflection in coordinate axes and lines $y = \pm x$, rotation through any angle about $(0, 0)$, stretches parallel to the x-axis and y-axis, and enlargement about centre $(0, 0)$, with scale factor k, $(k \neq 0)$, where $k \in \mathbb{R}$.** **Knowledge that the transformation represented by $\mathbf{AB}$ is the transformation represented by $\mathbf{B}$ followed by the transformation represented by $\mathbf{A}$.** **3-D transformations confined to reflection in one of $x = 0$, $y = 0$, $z = 0$ or rotation about one of the coordinate axes.** **Knowledge of 3-D vectors is assumed.**	
	3.4	**Find invariant points and lines for a linear transformation.**	**For a given transformation, students should be able to find the coordinates of invariant points and the equations of invariant lines.**	

Topic	What students need to learn:			
	Content		Guidance	Enhanced content guidance
3 **Matrices** *continued*	3.5	**Calculate determinants of** **2 x 2 and 3 x 3 matrices and interpret as scale factors, including the effect on orientation.**	**Idea of the determinant as an area scale factor in transformations.**	**For 3 x 3 matrices the idea of the determinant as a volume scale factor in transformations.**
	3.6	**Understand and use singular and non-singular matrices.** **Properties of inverse matrices.** **Calculate and use the inverse of non-singular 2 x 2 matrices and 3 x 3 matrices.**	**Understanding the process of finding the inverse of a matrix is required.** **Students should be able to use a calculator to calculate the inverse of a matrix.**	**Candidates should use the most appropriate method of finding the inverse of a matrix.** **Questions will be phrased in such a way as to make clear if calculators should not be used.**
	3.7	**Solve three linear simultaneous equations in three variables by use of the inverse matrix.**		

Topic	What students need to learn:			
	Content		Guidance	Enhanced content guidance
3 **Matrices** *continued*	3.8	**Interpret geometrically the solution and failure of solution of three simultaneous linear equations.**	**Students should be aware of the different possible geometrical configurations of three planes, including cases where the planes,** **(i) meet in a point** **(ii) form a sheaf** **(iii) form a prism or are otherwise inconsistent**	
4 **Further algebra and functions**	4.1	**Understand and use the relationship between roots and coefficients of polynomial equations up to quartic equations.**	**For example, given a cubic polynomial equation with roots α, β and γ students should be able to evaluate expressions such as,** **(i)** $\alpha^2 + \beta^2 + \gamma^2$ **(ii)** $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **(iii)** $(3 + \alpha)(3 + \beta)(3 + \gamma)$ **(iv)** $\alpha^3 + \beta^3 + \gamma^3$	
	4.2	**Form a polynomial equation whose roots are a linear transformation of the roots of a given polynomial equation (of at least cubic degree).**		

Topic	What students need to learn:			
	Content		Guidance	Enhanced content guidance
4 **Further algebra and functions** *continued*	4.3	**Understand and use formulae for the sums of integers, squares and cubes and use these to sum other series.**	**For example, students should be able to sum series such as** $\sum_{r+1}^{n} r^2(r+2)$	
	4.4	Understand and use the method of differences for summation of series including use of partial fractions.	Students should be able to sum series such as $\sum_{r=1}^{n} \frac{1}{r(r+1)}$ by using partial fractions such as $\frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$	
	4.5	Find the Maclaurin series of a function including the general term.		
	4.6	Recognise and use the Maclaurin series for e^x, $\ln(1+x)$, $\sin x$, $\cos x$ and $(1+x)^n$, and be aware of the range of values of x for which they are valid (proof not required).	To include the derivation of the series expansions of compound functions.	

Topic	What students need to learn:			
	Content		Guidance	Enhanced content guidance
5 **Further calculus**	5.1	**Derive formulae for and calculate volumes of revolution.**	**Both** $\pi \int y^2 \, \mathrm{d}x$ **and** $\pi \int x^2 \, \mathrm{d}y$ **are required.** Students should be able to find a volume of revolution given either Cartesian equations or parametric equations.	
	5.2	Evaluate improper integrals where either the integrand is undefined at a value in the range of integration or the range of integration extends to infinity.	For example, $\int_0^\infty \mathrm{e}^{-x} \, \mathrm{d}x, \int_0^2 \frac{1}{\sqrt{x}} \, \mathrm{d}x$	
	5.3	Understand and evaluate the mean value of a function.	Students should be familiar with the mean value of a function f(x) as, $\frac{1}{b-a} \int_a^b \mathrm{f}(x) \, \mathrm{d}x$	
	5.4	Integrate using partial fractions.	Extend to quadratic factors $ax^2 + c$ in the denominator	
	5.5	Differentiate inverse trigonometric functions.	For example, students should be able to differentiate expressions such as, $\arcsin x + x\sqrt{(1 - x^2)}$ and $\frac{1}{2}\arctan x^2$	

Topic	What students need to learn:			
	Content		Guidance	Enhanced content guidance
5 **Further calculus** *continued*	5.6	Integrate functions of the form $(a^2 - x^2)^{-\frac{1}{2}}$ and $(a^2 - x^2)^{-1}$ and be able to choose trigonometric substitutions to integrate associated functions.		Associated functions to include expressions of the form $\left(px^2 + qx + r\right)^{-\frac{1}{2}}$ and $\left(px^2 + qx + r\right)^{-1}$
6 **Further vectors**	6.1	**Understand and use the vector and Cartesian forms of an equation of a straight line in 3-D.**	**The forms, $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and** $\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}$ **Find the point of intersection of two straight lines given in vector form.** **Students should be familiar with the concept of skew lines and parallel lines.**	
	6.2	**Understand and use the vector and Cartesian forms of the equation of a plane.**	**The forms** $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$ **and** $ax + by + cz = d$	

Topic	What students need to learn:			
	Content		Guidance	Enhanced content guidance
6 **Further vectors** *continued*	6.3	**Calculate the scalar product and use it to express the equation of a plane, and to calculate the angle between two lines, the angle between two planes and the angle between a line and a plane.**	$\mathbf{a.b} = \lvert\mathbf{a}\rvert\,\lvert\mathbf{b}\rvert\cos\theta$ **The form $\mathbf{r.n} = k$ for a plane.**	
	6.4	**Check whether vectors are perpendicular by using the scalar product.**	**Knowledge of the property that $\mathbf{a.b} = 0$ if the vectors a and b are perpendicular.**	
	6.5	**Find the intersection of a line and a plane.** **Calculate the perpendicular distance between two lines, from a point to a line and from a point to a plane.**	**The perpendicular distance from (α, β, γ) to $n_1x + n_2y + n_3z + d = 0$ is** $\dfrac{\lvert n_1\alpha + n_2\beta + n_3\gamma + d\rvert}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$	**Students may be asked to find the line of intersection of two planes.**

Topic	What students need to learn:			
	Content		Guidance	Enhanced content guidance
7 **Polar coordinates**	7.1	Understand and use polar coordinates and be able to convert between polar and Cartesian coordinates.		
	7.2	Sketch curves with r given as a function of θ, including use of trigonometric functions.	The sketching of curves such as $r = p\sec(\alpha - \theta)$, $r = a$, $r = 2a\cos\theta$, $r = k\theta$, $r = a(1 \pm \cos\theta)$, $r = a(3 + 2\cos\theta)$, $r = a\cos 2\theta$ and $r^2 = a^2\cos 2\theta$ may be set.	Questions will only be set where $r > 0$
	7.3	Find the area enclosed by a polar curve.	Use of the formula $\frac{1}{2}\int_{\alpha}^{\beta} r^2\,\mathrm{d}\theta$ for area. The ability to find tangents parallel to, or at right angles to, the initial line is expected.	
8 **Hyperbolic functions**	8.1	Understand the definitions of hyperbolic functions $\sinh x$, $\cosh x$ and $\tanh x$, including their domains and ranges, and be able to sketch their graphs.	For example, $\cosh x = \frac{1}{2}\left(\mathrm{e}^{x} + \mathrm{e}^{-x}\right)$	
	8.2	Differentiate and integrate hyperbolic functions.	For example, differentiate $\tanh 3x$, $x\sinh^2 x$, $\frac{\cosh 2x}{\sqrt{(x+1)}}$	

Topic	What students need to learn:			
	Content		Guidance	Enhanced content guidance
8 **Hyperbolic functions** *continued*	8.3	Understand and be able to use the definitions of the inverse hyperbolic functions and their domains and ranges.	$\text{arsinh}x = \ln\left[x+\sqrt{x^2+1}\right]$ $\text{arcosh}x = \ln\left[x+\sqrt{x^2-1}\right],\ x \geqslant 1$ $\text{artanh}x = \frac{1}{2}\ln\left[\frac{1+x}{1-x}\right],\ -1 < x < 1$	
	8.4	Derive and use the logarithmic forms of the inverse hyperbolic functions.		
	8.5	Integrate functions of the form $\left(x^2+a^2\right)^{-\frac{1}{2}}$ and $\left(x^2-a^2\right)^{-\frac{1}{2}}$ and be able to choose substitutions to integrate associated functions.		
9 **Differential equations**	9.1	Find and use an integrating factor to solve differential equations of form $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$ and recognise when it is appropriate to do so.	The integrating factor $\mathrm{e}^{\int P(x)\mathrm{d}x}$ may be quoted without proof.	

Topic	What students need to learn:			
	Content		Guidance	Enhanced content guidance
9 Differential equations *continued*	9.2	Find both general and particular solutions to differential equations.	Students will be expected to sketch members of the family of solution curves.	
	9.3	Use differential equations in modelling in kinematics and in other contexts.		
	9.4	Solve differential equations of form $y'' + ay' + by = 0$ where a and b are constants by using the auxiliary equation.		
	9.5	Solve differential equations of form $y'' + a y' + b y = \mathrm{f}(x)$ where a and b are constants by solving the homogeneous case and adding a particular integral to the complementary function (in cases where $\mathrm{f}(x)$ is a polynomial, exponential or trigonometric function).	$\mathrm{f}(x)$ will have one of the forms $k\,\mathrm{e}^{px}$, $A + Bx$, $p + qx + cx^2$ or $m \cos \omega x + n \sin \omega x$	

Topic	What students need to learn:			
	Content		Guidance	Enhanced content guidance
9 Differential equations *continued*	9.6	Understand and use the relationship between the cases when the discriminant of the auxiliary equation is positive, zero and negative and the form of solution of the differential equation.		
	9.7	Solve the equation for simple harmonic motion $\ddot{x} = -\omega^2 x$ and relate the solution to the motion.		
	9.8	Model damped oscillations using second order differential equations and interpret their solutions.	Damped harmonic motion, with resistance varying as the derivative of the displacement, is expected. Problems may be set on forced vibration.	

Topic	What students need to learn:			
	Content		Guidance	Enhanced content guidance
9 **Differential equations** *continued*	9.9	Analyse and interpret models of situations with one independent variable and two dependent variables as a pair of coupled first order simultaneous equations and be able to solve them, for example predator-prey models.	Restricted to coupled first order linear equations of the form, $\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by + \mathrm{f}(t)$ $\frac{\mathrm{d}y}{\mathrm{d}t} = cx + dy + \mathrm{g}(t)$	

Paper 3 and Paper 4: Further Mathematics Options

Paper 3A: Further Pure Mathematics 1

Topics	What students need to learn:			
	Content		Guidance	Enhanced content guidance
1 **Further Trigonometry**	1.1	**The *t*-formulae**	**The derivation and use of** $\sin\theta \equiv \frac{2t}{1+t^2}$, $\cos\theta \equiv \frac{1-t^2}{1+t^2}$, $\tan\theta \equiv \frac{2t}{1-t^2}$, **where** $t = \tan\frac{\theta}{2}$	
	1.2	**Applications of *t*-formulae to trigonometric identities**	**E.g. show that** $\frac{1+\operatorname{cosec}\theta}{\cot\theta} \equiv \frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}$	
	1.3	**Applications of *t*-formulae to solve trigonometric equations**	**E.g. the solution of equations of the form** $a\cos x + b\sin x = c$	
2 **Further calculus**	2.1	Derivation and use of Taylor series.	The derivation, for example, of the expansion of $\sin x$ in ascending powers of $(x - \pi)$ up to and including the term in $(x-\pi)^3$.	The general form of the Taylor series expansion of $f(x)$ about $x = k$ will be given in the question.
	2.2	Use of series expansions to find limits.	E.g. $\lim_{x\to 0}\frac{x-\arctan x}{x^3}$, $\lim_{x\to 0}\frac{e^{2x^2}-1}{x^2}$	
	2.3	Leibnitz's theorem.	Leibnitz's theorem for differentiating products.	

Topics	What students need to learn:			
	Content		Guidance	Enhanced content guidance
2 **Further calculus** *continued*	2.4	L'Hospital's Rule.	The use of derivatives to evaluate limits of indeterminate forms. Repeated applications and/or substitutions may be required. E.g. $\lim_{x\to 0}\frac{2\sin x-\sin 2x}{x-\sin x}$, $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x$	
	2.5	The Weierstrass substitution for integration.	The use of tangent half angle substitutions to find definite and indefinite integrals. E.g. $\int \operatorname{cosec} x\,\mathrm{d}x$, $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{1+\sin x-\cos x}\,\mathrm{d}x$ using $t=\tan\frac{x}{2}$.	
3 **Further differential equations**	3.1	Use of Taylor series method for series solution of differential equations.	For example, derivation of the series solution in powers of x, as far as the term in x^4, of the differential equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\frac{\mathrm{d}y}{\mathrm{d}x}+y=0$, where $y=1$, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ at $x=0$	The general form of the Taylor series expansion of $\mathrm{f}(x)$ about $x=k$ will be given in the question.
	3.2	Differential equations reducible by means of a given substitution.	Differential equations reducible to the types as specified in section 9 of the A level Further Pure Mathematics content for papers 1 and 2.	

Topics	What students need to learn:			
	Content		Guidance	Enhanced content guidance
4 **Coordinate systems**	4.1	**Cartesian and parametric equations for the parabola and rectangular hyperbola**, ellipse and hyperbola.	Students should be familiar with the equations: $y^2 = 4ax$; $x = at^2$, $y = 2at$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; $x = a\,x\cos t$, $y = b \sin t$. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; $x = a \sec t$, $y = b \tan t$, $x = a \cosh t$, $y = b \sinh t$	
	4.2	**The focus-directrix properties of the parabola**, ellipse and hyperbola, including the eccentricity.	For example, students should know that, for the ellipse $b^2 = a^2 (1 - e^2)$, the foci are $(ae, 0)$ and $(- ae, 0)$ and the equations of the directrices are $x = +\frac{a}{e}$ and $x = -\frac{a}{e}$	
	4.3	**Tangents and normals to these curves.**	**The condition for $y = mx + c$ to be a tangent to these curves is expected to be known.** Students are expected to be able to use algebraic differentiation to find $\frac{dy}{dx}$ for these curves.	
	4.4	**Loci problems.**		

Topics	What students need to learn:			
	Content		Guidance	Enhanced content guidance
5 **Further vectors**	5.1	**The vector product $\mathbf{a} \times \mathbf{b}$ of two vectors.**	To include the interpretation of $\lvert \mathbf{a} \times \mathbf{b} \rvert$ as an area.	
	5.2	**The scalar triple product $\mathbf{a}.\mathbf{b} \times \mathbf{c}$**	**Students should be able to use the scalar triple product to find the volume of a tetrahedron and a parallelepiped.**	
	5.3	Applications of vectors to three dimensional geometry involving points, lines and planes.	To include the equation of a line in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$. Direction ratios and direction cosines of a line.	
6 **Further numerical methods**	6.1	**Numerical solution of first order and second order differential equations.**	**The approximations** $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n \approx \frac{(y_{n+1} - y_n)}{h}$ $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n \approx \frac{(y_{n+1} - y_{n-1})}{2h}$ $\left(\frac{\mathrm{d}^2y}{dx^2}\right)_n \approx \frac{(y_{n+1} - 2y_n + y_{n-1})}{h^2}$	**The approximation formulae will be given in the question.**
	6.2	Simpson's rule.		Students are expected to know Simpson's rule and how to apply it for a given number of intervals.

Topics	What students need to learn:			
	Content		Guidance	Enhanced content guidance
7 Inequalities	7.1	**The manipulation and solution of algebraic inequalities and inequations**, including those involving the modulus sign.	**The solution of inequalities such as** $\dfrac{1}{x-a} > \dfrac{x}{x-b}$, $\lvert x^2 - 1 \rvert > 2(x+1)$	

Paper 4A: Further Pure Mathematics 2

Topics	What students need to learn:			
	Content		Guidance	Enhanced content guidance
1 **Groups**	1.1	**The Axioms of a group.**	**The terms 'binary operation, closure, associativity, identity and inverse'.**	
	1.2	**Examples of groups.** **Cayley tables.** **Cyclic groups.**	**For example, symmetries of geometrical figures, non-singular matrices, integers modulo *n* with operation addition, and/or multiplication permutation groups.**	
	1.3	**The order of a group and the order of an element.** **Subgroups.**		
	1.4	**Lagrange's theorem.**		
	1.5	Isomorphism.	Isomorphisms will be restricted to groups that have a maximum order of 8.	
2 **Further calculus**	2.1	Further Integration — Reduction formulae.	Students should be able to derive formulae such as: $nI_n = (n-1)I_{n-2},\ n \geq 2$, for $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$, $I_{n+2} = \frac{2\sin(n+1)x}{n+1} + I_n$ for $I_n = \int \frac{\sin nx}{\sin x} dx$	
	2.2	The calculation of arc length and the area of a surface of revolution.	The equation of the curve may be given in Cartesian, parametric or polar form.	

Topics	What students need to learn:			
	Content		Guidance	Enhanced content guidance
3 **Further matrix algebra**	3.1	**Eigenvalues and eigenvectors of 2 × 2** and 3 × 3 **matrices.**	**Understand the term *characteristic equation* for a 2 × 2 matrix.** **Repeated eigenvalues and complex eigenvalues.** **Normalised vectors may be required.**	
	3.2	**Reduction of matrices to diagonal form.**	**Students should be able to find a matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP}$ is diagonal** **Symmetric matrices and orthogonal diagonalisation.**	
	3.3	**The use of the Cayley-Hamilton theorem.**	**Students should understand and be able to use the fact that, every** 2×2 or 3×3 **matrix satisfies its own characteristic equation.**	
4 **Further complex numbers**	4.1	**Further loci and regions in the Argand diagram.**	**To include loci such as** $\|z - a\| = k\|z - b\|$, $\arg \frac{z-a}{z-b} = \beta$ **and** **regions such as** $\alpha \leq \arg(z - z_1) \leq \beta$ **and** $p \leq \text{Re}(z) \leq q$	
	4.2	Elementary transformations from the z-plane to the w-plane.	Transformations such as $w = z^2$ and $w = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ may be set.	

Topics	What students need to learn:			
	Content		Guidance	Enhanced content guidance
5 Number theory	5.1	**An understanding of the division theorem and its application to the Euclidean Algorithm and congruences.**	**Students should be able to apply the algorithm to find the highest common factor of two numbers.**	
	5.2	**Bezout's identity.**	**Students should be able to use back substitution to identity the Bezout's identity for two numbers.**	
	5.3	**Modular arithmetic. Understanding what is meant by two integers a and b to be congruent modulo an integer n. Properties of congruences.**	**The notation $a \equiv b \pmod{n}$ is expected.** **Knowledge of the following properties:** $a \equiv a \pmod{n}$ **if** $a \equiv b \pmod{n}$ **then** $b \equiv a \pmod{n}$ **if** $a \equiv b \pmod{n}$ **and** $b \equiv c \pmod{n}$ **then** $a \equiv c \pmod{n}$ **Addition and subtraction laws for congruences.** **Multiplication and power laws.**	
	5.4	Fermat's Little Theorem.	For example, students should be able to find the least positive residue of 4^{20} modulo 7 Proof is not required.	
	5.5	**Divisibility Tests.**	**For divisibility by $2, 3, 4, 5, 6, 9,$ 10 and 11.**	
	5.6	Solution of congruence equations.	Conditions under which solutions exist should be known. Use of Bezout's identity to find multiplicative inverses.	

Topics	What students need to learn:			
	Content		Guidance	Enhanced content guidance
5 **Number theory** *continued*	5.7	Combinatorics: counting problems, permutations and combinations.	The multiplicative principle of counting. Set notation is expected; the number of subsets of a set should be known. Addition and subtraction principles. Students should, for example, be able to determine the number of positive integers less than 1000 containing the digit 3 (at least once). Understanding both ${}^n\mathrm{P}_r$ and ${}^n\mathrm{C}_r$ and when to use them. For example, to determine how many ways there are to select a team of 11 players from a squad of 21 (a) regardless of playing positions and (b) if positions matter	
6 **Further sequences and series**	6.1	**First** and second **order recurrence relations.**	**Equations of the form** $\boldsymbol{u_{n+1} + \mathrm{f}(n)u_n = \mathrm{g}(n)}$ and $u_{n+2} + \mathrm{f}(n)u_{n+1} + \mathrm{g}(n)u_n = \mathrm{h}(n)$	
	6.2	**The solution of recurrence relations to obtain closed forms.**	**Students should be able to solve relations such as:** $\boldsymbol{u_{n+1} - 5u_n = 8,\ u_1 = 1,}$ $2u_{n+2} + 7u_{n+1} - 15u_n = 6,\ u_1 = 10,\ u_2 = -17$ **The terms, particular solution, complementary function and auxiliary equation should be known.** **Use of recurrence relations to model applications, e.g. population growth.**	

Topics	What students need to learn:			
	Content		Guidance	Enhanced content guidance
6 **Further sequences and series** *continued*	6.3	**Proof by induction of closed forms.**	**For example** **if $u_{n+1} - 3u_n = 4$ with $u_1 = 1$ prove by mathematical induction that $u_n = 3^n - 2$**	

Paper 3B: Further Statistics 1*

Topic	What students need to learn:			
	Content		Guidance	Enhanced content guidance
1 **Discrete probability distributions**	1.1	**Calculation of the mean and variance of discrete probability distributions.** **Extension of expected value function to include E(g(*X*))**	**Use of** $E(X) = \mu = \sum xP(X = x)$ **and** $Var(X) = \sigma^2 = \sum x^2P(X = x) - \mu^2$ **The formulae used to define** $g(x)$ **will be consistent with the level required in A level Mathematics and A level Further Mathematics.** **Questions may require candidates to use these calculations to assess the suitability of models.**	
2 **Poisson & binomial distributions**	2.1	**The Poisson distribution** **The additive property of Poisson distributions**	**Students will be expected to use this distribution to model a real-world situation and to comment critically on the appropriateness.** **Students will be expected to use their calculators to calculate probabilities including cumulative probabilities.** **Students will be expected to use the additive property of the Poisson distribution. E.g. if** X **= the number of events per minute and** $X \sim Po(\lambda)$**, then the number of events per** 5 **minutes ~** $Po(5\lambda)$**.** **If** X **and** Y **are independent random variables with** $X \sim Po(\lambda)$ **and** $Y \sim Po(\mu)$**, then** $X + Y \sim Po(\lambda + \mu)$ **No proofs are required.**	

Topic	What students need to learn:			
	Content		Guidance	Enhanced content guidance
2 **Poisson & binomial distributions** *continued*	2.2	**The mean and variance of the binomial distribution and the Poisson distribution.**	**Knowledge and use of :** **If $X \sim B(n, p)$, then $E(X) = np$ and $Var(X) = np(1 - p)$** **If $Y \sim Po(\lambda)$, then $E(Y) = \lambda$ and $Var(Y) = \lambda$** **Derivations are not required.**	
	2.3	**The use of the Poisson distribution as an approximation to the binomial distribution.**	**When n is large and p is small the distribution $B(n, p)$ can be approximated by $Po(np)$.** **Derivations are not required.**	The normal approximation to a Poisson distribution is not included.
3 **Geometric and negative binomial distributions**	3.1	Geometric and negative binomial distributions.	Models leading to the distributions $p(x) = p(1 - p)^{x-1}, x = 1, 2 \ldots$ and $p(x) = \binom{x-1}{r-1} p^r (1 - p)^{x-r}$ $x = r, r + 1, r + 2, \ldots$	Students could be asked to find cumulative probabilities using a negative binomial distribution using the relationship between the negative binomial and binomial distributions. For example if $Y \sim \text{NegBin}(r, p)$ and $X \sim B(n, p)$ then $P(Y \leqslant n) = P(X \geqslant r)$
	3.2	Mean and variance of a geometric distribution with parameter *p*.	Use of $\mu = \frac{1}{p}$ and $\sigma^2 = \frac{1-p}{p^2}$	
	3.3	Mean and variance of negative binomial distribution with $P(X = x)$ $= \binom{x-1}{r-1} p^r (1-p)^{(x-r)}$	Use of $\mu = \frac{r}{p}$ and $\sigma^2 = \frac{r(1-p)}{p^2}$	

Topic	What students need to learn:			
	Content		Guidance	Enhanced content guidance
4 **Hypothesis Testing**	4.1	**Extend ideas of hypothesis tests to test for the mean of a Poisson distribution**	**Hypotheses should be stated in terms of a population parameter μ or λ**	
	4.2	Extend hypothesis testing to test for the parameter p of a geometric distribution.	Hypotheses should be stated in terms of p	Hypothesis tests for the negative binomial distribution are not required.
5 **Central Limit Theorem**	5.1	Applications of the Central Limit Theorem to other distributions.	For a population with mean μ and variance σ^2, for large n $\bar{X} \approx\sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$ Applications may involve any of the distributions in A level Mathematics or A level Further Statistics 1 No proofs required.	When using the Central Limit Theorem as an approximation to a discrete distribution a continuity correction is not required.

Topic	What students need to learn:			
	Content		Guidance	Enhanced content guidance
6 **Chi Squared Tests**	6.1	**Goodness of fit tests and Contingency Tables** **The null and alternative hypotheses.** **The use of** $\sum_{i=1}^{n}\frac{(O_i - E_i)^2}{E_i}$ **as an approximate** χ^2 **statistic.** **Degrees of freedom.**	**Applications to include the discrete uniform, binomial, Poisson** and geometric **distributions.** **Lengthy calculations will not be required.** **Students will be expected to determine the degrees of freedom when one or more parameters are estimated from the data. Cells should be combined when** $E_i < 5$ **Students will be expected to obtain p-values from their calculator or use tables to find critical values.**	Applications involving the negative binomial distribution are not required. Situations where $E_i = 5$ will generally be avoided. There will not be any demand for p-values in connection with Chi-squared tests though, unless the question specifically says otherwise, they will be acceptable as an alternative to using the tables.
7 **Probability generating functions**	7.1	Definitions, derivations and applications. Use of the probability generating function for the negative binomial, geometric, binomial and Poisson distributions.		
	7.2	Use to find the mean and variance.	Proofs of standard results may be required.	e.g. the use of the probability generating functions for a standard distribution to find the mean or variance.
	7.3	Probability generating function of the sum of	$G_{X+Y}(t) = G_X(t) \times G_Y(t)$ Derivation is not required.	

		independent random variables.		
8 **Quality of tests**	8.1	Type I and Type II errors. Size and Power of Test. The power function.	Calculation of the probability of a Type I or Type II error. Use of Type I and Type II errors and power function to indicate effectiveness of statistical tests. Questions will use any of the distributions in A level Mathematics or A level Further Statistics 1	

*This paper is also the **Paper 4 option 4B** paper and will have the title '*Paper 4, Option 4B: Further Statistics 1'.*

Paper 4B: Further Statistics 2

Topics	What students need to learn:			Enhanced content guidance
	Content		Guidance	
1 **Linear Regression**	**1.1**	**Least squares linear regression. The concept of residuals and minimising the sum of squares of residuals.**	**Students should have an understanding of the process involved in linear regression.** **They should be able to calculate the regression coefficients for the equation y on x using standard formulae.**	
	1.2	**Residuals.** **The residual sum of squares (RSS)**	**An intuitive use of residuals to check the reasonableness of linear fit and to find possible outliers.** **Use in refinement of mathematical models.** **The formula RSS = $S_{yy} - \frac{(S_{xy})^2}{S_{xx}}$** **Derivations are not required**	

Topics	What students need to learn:			
	Content		Guidance	Enhanced content guidance
2 **Continuous probability distributions**	**2.1**	**The concept of a continuous random variable.** **The probability density function and the cumulative distribution function for a continuous random variable.**	**Students will be expected to link with their knowledge of histograms and frequency polygons.** **Use of the probability density function f(*x*), where** $\mathrm{P}(a < X \leqslant b) = \int_a^b \mathrm{f}(x)\,\mathrm{d}x$ **Use of the cumulative distribution function** $\mathrm{F}(x_0) = \mathrm{P}(X \leqslant x_0) = \int_{-\infty}^{x_0} \mathrm{f}(x)\,\mathrm{d}x$. The formulae used in defining f(*x*) and the calculus required will be consistent with the level expected in A level Mathematics and A level Further Mathematics.	
	2.2	**Relationship between probability density and cumulative distribution functions.**	$\mathrm{f}(x) = \dfrac{\mathrm{dF}(x)}{\mathrm{d}x}$.	

Topics	What students need to learn:			Enhanced content guidance
	Content		Guidance	
2 **Continuous probability distributions** *continued*	**2.3**	**Mean and variance of continuous random variables.** **Extension of expected value function to include $E(g(X))$** **Mode, median and percentiles of continuous random variables.** **Idea of skewness**	The formulae used to define $g(x)$ will be consistent with the level required in A level Mathematics and A level Further Mathematics **Questions may require candidates to use these calculations to assess the suitability of models.** **Candidates will be expected to describe the skewness as positive, negative or zero and give a suitable justification.**	
	2.4	**The continuous uniform (rectangular) distribution.**	**Including the derivation of the mean, variance and cumulative distribution function.**	
3 **Correlation**	**3.1**	**Use of formulae to calculate the product moment correlation coefficient.** **Knowledge of the conditions for the use of the product moment correlation coefficient.** **A knowledge of the effects of coding will be expected.**	**Students will be expected to be able to use the formula to calculate the value of a coefficient given summary statistics.**	

Topics	What students need to learn:			
	Content		Guidance	Enhanced content guidance
3 Correlation *continued*	**3.2**	**Spearman's rank correlation coefficient, its use, interpretation.**	**Use of** $r_s = 1 - \frac{6\sum d^2}{n(n^2-1)}$ **Numerical questions involving ties may be set. An understanding of how to deal with ties will be expected.** **Students will be expected to calculate the resulting correlation coefficient on their calculator or using the formula.**	**i.e. sharing of ranks and use of product moment correlation coefficient on these values.**
	3.3	**Testing the hypothesis that a correlation is zero using either Spearman's rank correlation or the product moment correlation coefficient.**	**Hypotheses should be in terms of** ρ **or** ρ_s **and test a null hypothesis that** r **or** $\rho_s = 0$**.** **Use of tables for critical values of Spearman's and product moment correlation coefficients.** **Students will be expected to know that the critical values for the product moment correlation coefficient require that the data comes from a population having a bivariate normal distribution. Formal verification of this condition is not required.**	
4 Combinations of random variables	4.1	Distribution of linear combinations of independent Normal random variables.	If $X \sim \text{N}(\mu_x, \sigma_x^2)$ and $Y \sim \text{N}(\mu_y, \sigma_y^2)$ independently, then $aX \pm bY \sim \text{N}(a\mu_x \pm b\mu_y, a^2\sigma_x^2 + b^2\sigma_y^2)$ No proofs required.	

Topics	What students need to learn:			
	Content		Guidance	Enhanced content guidance
5 **Estimation, confidence intervals and tests using a normal distribution**	5.1	Concepts of standard error, estimator, bias. Quality of estimators	The sample mean, $\bar{x}$, and the sample variance, $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$, as unbiased estimates of the corresponding population parameters. Candidates will be expected to compare estimators and look for features such as unbiasedness and small variance.	
	5.2	Concept of a confidence interval and its interpretation.	Link with hypothesis tests.	
	5.3	Confidence limits for a Normal mean, with variance known.	Candidates will be expected to know how to apply the Normal distribution and use the standard error and obtain confidence intervals for the mean, rather than be concerned with any theoretical derivations.	
	5.4	Hypothesis test for the difference between the means of two Normal distributions with variances known.	Use of $\frac{(\bar{X}-\bar{Y})-(\mu_x-\mu_y)}{\sqrt{\frac{\sigma_x^2}{n_x}+\frac{\sigma_y^2}{n_y}}} \sim \mathrm{N}(0,1)$	
	5.5	Use of large sample results to extend to the case in which the population variances are unknown.	Use of Central Limit Theorem and use of $\frac{(\bar{X}-\bar{Y})-(\mu_x-\mu_y)}{\sqrt{\frac{S_x^2}{n_x}+\frac{S_y^2}{n_y}}} \approx\sim \mathrm{N}(0,1)$	

Topics	What students need to learn:			
	Content		Guidance	Enhanced content guidance
6 **Other Hypothesis Tests and confidence intervals**	6.1	Hypothesis test and confidence interval for the variance of a Normal distribution.	Use of $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$ Candidates may use χ^2 –tables or their calculators to find critical values or p-values.	
	6.2	Hypothesis test that two independent random samples are from Normal populations with equal variances.	Use of $\frac{S_1^{\ 2}}{S_2^{\ 2}} \sim F_{n_1-1,\, n_2-1}$ under H_0. Candidates may use tables of the *F*-distribution or their calculators to find critical values or p-values.	
7 **Confidence intervals and tests using the *t* – distribution**	7.1	Hypothesis test and confidence interval for the mean of a Normal distribution with unknown variance.	Use of $\frac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}} \sim t_{n-1}$ Candidates may use *t*-tables or their calculators to calculate critical or p-values.	
	7.2	Paired *t*-test.		The calculation of a confidence interval is expected.
	7.3	Hypothesis test and confidence interval for the difference between two means from independent Normal distributions when the variances are equal but unknown. Use of the pooled estimate of variance.	Use of *t*-distribution. $\frac{\bar{X}-\bar{Y}-(\mu_x-\mu_y)}{S\sqrt{\frac{1}{n_x}+\frac{1}{n_y}}} \sim t_{n_x+n_y-2}$, under H_0. $s^2 = \frac{(n_1-1)s_1^{\ 2}+(n_2-1)s_2^{\ 2}}{n_1+n_2-2}$	

Paper 3C: Further Mechanics 1*

Topics	What students need to learn:			
		Content	Guidance	Enhanced content guidance
1 **Momentum and impulse**	1.1	**Momentum and impulse. The impulse-momentum principle. The principle of conservation of momentum applied to two spheres colliding directly.**	**The spheres may be modelled as particles.**	
	1.2	Momentum as a vector. The impulse-momentum principle in vector form.		Questions may involve the angle of deflection caused by an impulse.
2 **Work, energy and power**	2.1	**Kinetic and potential energy, work and power. The work-energy principle. The principle of conservation of mechanical energy.**	**Problems involving motion under a variable resistance and/or up and down an inclined plane may be set.**	**Questions will not involve the work done by a variable force.**
3 **Elastic strings and springs and elastic energy**	3.1	Elastic strings and springs. Hooke's law.		Strings and springs will be modelled as being light.
	3.2	Energy stored in an elastic string or spring.	Problems using the work-energy principle involving kinetic energy, potential energy and elastic energy may be set.	Students may be required to derive the formula for the elastic potential energy stored in an elastic string or spring.

Topics	What students need to learn:			
	Content		Guidance	Enhanced content guidance
4 **Elastic collisions in one dimension**	4.1	**Direct impact of elastic spheres. Newton's law of restitution. Loss of kinetic energy due to impact.**	**Students will be expected to know and use the inequalities $0 \leq e \leq 1$ (where e is the coefficient of restitution).** The spheres may be modelled as particles.	**Students will be expected to be able to appreciate the significance of the two special cases when $e = 0$ or when $e = 1$.**
	4.2	**Successive direct impacts of spheres and/or a sphere with a smooth plane surface.**	**The spheres may be modelled as particles.**	**Questions may be set in a variety of contexts and could involve, for example, finding the condition(s) for further impacts.**
5 **Elastic collisions in two dimensions**	5.1	Oblique impact of smooth elastic spheres and a smooth sphere with a fixed surface. Loss of kinetic energy due to impact.	Problems will only involve spheres with the same radius. Problems may be set in vector form. The spheres may be modelled as particles.	Questions may be set where the line of centres or the fixed surface is not parallel to either coordinate axis.
	5.2	Successive oblique impacts of a sphere with smooth plane surfaces.	The sphere may be modelled as particle.	Questions may be set where the fixed surfaces are not parallel to either coordinate axis.

*This paper is also the **Paper 4 option 4C** paper and will have the title '*Paper 4, Option 4C: Further Mechanics 1*'.

Paper 4C: Further Mechanics 2

Topics	What students need to learn:			Enhanced content guidance
	Content		Guidance	
1 **Motion in a circle**	1.1	**Angular speed.** $v = r\omega$ **.** **Uniform motion of a particle moving in a horizontal circle.** **Radial acceleration in circular motion.** **The forms** $r\omega^2$ **and** $\frac{v^2}{r}$ **are required.**	**Problems involving the 'conical pendulum', an elastic string, motion on a banked surface, as well as other contexts, may be set.**	Questions may be set in a variety of contexts e.g. the particle is attached to two strings and is moving in a horizontal circle, the particle is moving in a horizontal circle on a rough horizontal plane, the particle is moving in a horizontal circle on the smooth inner surface of a cone or a hemisphere etc.

Topics	What students need to learn:			
	Content		Guidance	Enhanced content guidance
1 **Motion in a circle** *continued*	1.2	Motion of a particle in a vertical circle. Radial and tangential acceleration in circular motion. Kinetic and potential energy and the conservation of energy principle applied to motion in a vertical circle.	Questions may be set which involve complete or incomplete circles.	Questions may be set in a variety of contexts where the particle *can* move off the circle e.g. the particle is attached to a string or *cannot* move off the circle e.g. the particle is attached to the end of a light rod. Students may be required to derive the conditions for the particle to make complete circles. Questions may involve incomplete circles followed by projectile motion. Questions may involve finding the magnitude and direction of the acceleration of a particle moving in a vertical circle.
2 **Centres of mass of plane figures**	2.1	**Moment of a force. Centre of mass of a discrete mass distribution in one and two dimensions.**		

Topics	What students need to learn:			
	Content		Guidance	Enhanced content guidance
2 **Centres of mass of plane figures** *continued*	2.2	**Centre of mass of uniform plane figures, and of composite plane figures.** **Centre of mass of frameworks.** **Equilibrium of a plane lamina or framework under the action of coplanar forces.**	**The use of an axis of symmetry will be acceptable where appropriate. Use of integration is not required.** **Questions may involve non-uniform composite plane figures/frameworks.** **Figures may include the shapes referred to in the formulae book. Unless otherwise stated in the question, results given in the formulae book may be quoted without proof.**	**Questions could involve a variety of contexts e.g. plane figures that are folded.**
3 **Further centres of mass**	3.1	Centre of mass of uniform and non-uniform rigid bodies and composite bodies.	Problems which involve the use of integration and/or symmetry to determine the centre of mass of a uniform or non-uniform body may be set. The level of calculus will be consistent with that required in A level Mathematics and Further Mathematics. Unless otherwise stated in the question, results given in the formulae book may be quoted without proof.	Students may be asked to derive, e.g using integration, results given in the formulae book.
	3.2	Equilibrium of rigid bodies under the action of coplanar forces.	To include (i) suspension of a body from a fixed point, (ii) a rigid body placed on a horizontal or inclined plane.	Questions will be set in a variety of contexts e.g. a composite body held in equilibrium by one or more forces.

Topics	What students need to learn:			
	Content		Guidance	Enhanced content guidance
3 **Further centres of mass** *continued*	3.3	Toppling and sliding of a rigid body on a rough plane.		Questions will be set in a variety of contexts e.g. toppling and sliding of a composite body placed on a rough inclined plane.
4 **Further dynamics**	4.1	Newton's laws of motion, for a particle moving in one dimension, when the applied force is variable.	The solution of the resulting equations will be consistent with the level of calculus required in A level Mathematics and Further Mathematics. Problems may involve the law of gravitation, i.e. the inverse square law.	
	4.2	Simple harmonic motion. Oscillations of a particle attached to the end of elastic string(s) or spring(s). Kinetic, potential and elastic energy in the context of SHM.	Proof that a particle moves with simple harmonic motion in a given situation may be required (i.e. showing that $\ddot{x} = -\omega^2 x$). Students will be expected to be familiar with standard formulae, which may be quoted without proof. e.g. $x = a\sin\omega t,\ x = a\cos\omega t,$ $v^2 = \omega^2\left(a^2 - x^2\right),\ T = \dfrac{2\pi}{\omega}$	Questions could involve one or more elastic strings or springs. Standard formulae can only be quoted if it is given in the question that the motion is simple harmonic. If this not the case, a candidate would first need to prove SHM before quoting these formulae. Questions could involve incomplete oscillations.

Topics	What students need to learn:			
	Content		Guidance	Enhanced content guidance
5 **Further kinematics**	5.1	**Kinematics of a particle moving in a straight line when the acceleration is a function of the** displacement (x), or **time (t) or velocity (v).**	**The setting up and solution of equations where** $\frac{dv}{dt} = f(t)$ **or** $\frac{dv}{dt} = f(v)$, $v\frac{dv}{dx} = f(v)$ or $v\frac{dv}{dx} = f(x)$, $\frac{dx}{dt} = f(x)$ **or** $\frac{dx}{dt} = f(t)$ **will be consistent with the level of calculus required in A level Mathematics** and Further Mathematics.	**Questions could be set in a variety of contexts e.g. a particle moving under gravity which is varying.**

Paper 3D: Decision Mathematics 1*

Topics	What students need to learn:			
	Content		Guidance	Enhanced content guidance
1 **Algorithms and graph theory**	1.1	**The general ideas of algorithms and the implementation of an algorithm given by a flow chart or text.**	**The meaning of the order of an algorithm is expected. Students will be expected to determine the order of a given algorithm and the order of standard network problems.**	**Students are not expected to learn or memorise the order of any standard network problems. Students will be given sufficient information in the question to determine the order of a given algorithm, or the order of a standard network problem. If the order of a standard network is required, then the order will be given in the question.**
	1.2	**Bin packing, bubble sort and quick sort.**	**When using the quick sort algorithm, the pivot should be chosen as the middle item of the list.**	**For 'Bin packing' students should be familiar with both the first-fit and first-fit decreasing algorithms and the full bin method.** **Unless the question states otherwise, when applying the bubble sort algorithm students should start at the left-hand end of the list and work from left to right. Furthermore, for the bubble sort algorithm, unless the question states otherwise, only the list at the end of each pass needs to be shown (and not the individual comparisons and swaps within each pass).**

Topics	What students need to learn:			
	Content		Guidance	Enhanced content guidance
1 **Algorithms and graph theory** *continued*	1.3	**Use of the order of the nodes to determine whether a graph is Eulerian, semi-Eulerian or neither.**	**Students will be expected to be familiar with the following types of graphs: complete (including K notation), planar and isomorphic.**	
	1.4	The planarity algorithm for planar graphs.	Students will be expected to be familiar with the term 'Hamiltonian cycle'.	Knowledge of Kuratowski's theorem for planar graphs is not required.
2 **Algorithms on graphs**	2.1	**The minimum spanning tree (minimum connector) problem. Prim's and Kruskal's algorithm.**	**Matrix representation for Prim's algorithm is expected. Drawing a network from a given matrix and writing down the matrix associated with a network may be required.**	

Topics	What students need to learn:			Enhanced content guidance
	Content		Guidance	
2 **Algorithms on graphs** *continued*	2.2	**Dijkstra's** and Floyd's **algorithm for finding the shortest path.**	When applying Floyd's algorithm, unless directed otherwise, students will be expected to complete the first iteration on the first row of the corresponding distance and route problems, the second iteration on the second row and so on until the algorithm is complete.	**For Dijkstra's algorithm students need only replace a working value if the new working value is smaller than the current working value. Students are reminded that when applying Dijkstra's algorithm, working values that have been replaced should not be crossed out.** **Questions on both Dijkstra's** and Floyd's **algorithm may contain directed arcs.** Candidates may be required, when applying Floyd's algorithm, to only complete one of the matrices (distance or route), and not necessarily both. For Floyd's algorithm in a directed network the convention in reading the initial (or any subsequent) distance matrix is that the entry in row X, column Y will correspond to the distance from X to Y at that stage of the algorithm.

Topics	What students need to learn:			
	Content		Guidance	Enhanced content guidance
3 **Algorithms on graphs II**	3.1	**Algorithm for finding the shortest route around a network, travelling along every edge at least once and ending at the start vertex (The Route Inspection Algorithm).**	**Also known as the 'Chinese postman' problem.** **Students will be expected to use inspection to consider all possible pairings of odd nodes. The network will contain at most four odd nodes.** If the network has more than four odd nodes then additional information will be provided that will restrict the number of pairings that will need to be considered.	The additional information required to restrict the number of pairings may be reflected in the nature of the network itself. For example, a network containing an arc XY such that node X has order 1 would require that X and Y would have to be paired together.
	3.2	The practical and classical Travelling Salesman problems. The classical problem for complete graphs satisfying the triangle inequality.	The use of short cuts to improve upper bound is included.	When using short cuts students are reminded that they must start from the initial upper bound (found by doubling the weight of the minimum spanning tree) and explicitly state which arcs they are either adding or removing from their initial upper bound.
		Determination of upper and lower bounds using minimum spanning tree methods.	The conversion of a network into a complete network of shortest 'distances' is included.	
		The nearest neighbour algorithm.		

Topics	What students need to learn:			
	Content		Guidance	Enhanced content guidance
4 **Critical path analysis**	4.1	**Modelling of a project by an activity network, from a precedence table.**	**Activity on arc will be used. The use of dummies is included.**	
	4.2	**Completion of the precedence table for a given activity network.**	**In a precedence network, precedence tables will only show immediate predecessors.**	
	4.3	**Algorithm for finding the critical path. Earliest and latest event times. Earliest and latest start and finish times for activities. Identification of critical activities and critical path(s).**	**Calculating the lower bound for the number of workers required to complete the project in the shortest possible time is required.**	
	4.4	**Calculation of the total float of an activity. Construction of Gantt (cascade) charts.**	**Each activity will require only one worker.**	**The calculation of the independent or interfering float for an activity is not required.**

Topics	What students need to learn:			
	Content		Guidance	Enhanced content guidance
4 **Critical path analysis** *continued*	4.5	Construct resource histograms (including resource levelling) based on the number of workers required to complete each activity.	The number of workers required to complete each activity of a project will be given and the number or workers required may not necessarily be one.	
	4.6	Scheduling the activities using the least number of workers required to complete the project.		
5 **Linear programming**	5.1	**Formulation of problems as linear programs** including the meaning and use of slack, surplus and artificial variables.	For example, $3x+2y \leqslant 20 \Rightarrow 3x+2y+s_1=20$ $2x+5y \leqslant 35 \Rightarrow 2x+5y+s_2=35$ $x+y \geqslant 5 \Rightarrow x+y-s_3+t_1=5$ where s_1, s_2 are slack variables, s_3 is a surplus variable and t_1 is an artificial variable.	

Topics	What students need to learn:			
	Content		Guidance	Enhanced content guidance
5 **Linear programming** *continued*	5.2	**Graphical solution of two variable problems using objective line and vertex methods including cases where integer solutions are required.**		**The use of branch-and-bound methods for finding the optimal integer solution is not required.**
	5.3	The Simplex algorithm and tableau for maximising and minimising problems with ≤ constraints.	Problems will be restricted to those with a maximum of four variables (excluding slack variables) and four constraints, in addition to non-negativity conditions.	
	5.4	The two-stage Simplex and big-M methods for maximising and minimising problems which may include both ≤ and ≥ constraints.	Problems will be restricted to those with a maximum of four variables (excluding slack, surplus and artificial variables) and four constraints, in addition to any non-negativity conditions.	Examples of setting up the equation constraints and objectives using both methods: Example 1 – two-stage maximisation Maximise $P = 2x + 4y - 3z$ subject to: $3x + 2y + z \leqslant 35$ $2x + y \geqslant 5$ $-3x + y + 5z \geqslant 15$ $x \geqslant 0, y \geqslant 0, z \geqslant 0$ Becomes: $3x + 2y + z + s_1 = 35$ $2x + y - s_2 + t_1 = 5$ $-3x + y + 5z - s_3 + t_2 = 15$ $P - 2x - 4y + 3z = 0$

Topics	What students need to learn:		Enhanced content guidance
	Content	Guidance	
			$I + t_1 + t_2 = 0$ $\Rightarrow I + x - 2y - 5z + s_2 + s_3 + 20 = 0$ <u>Example 2</u> – big-M minimisation Minimise $P = 2x + 4y - 3z$ subject to: $3x + 2y + z \leqslant 35$ $2x + y \geqslant 5$ $-3x + y + 5z \geqslant 15$ $x \geqslant 0, y \geqslant 0, z \geqslant 0$ Becomes: $3x + 2y + z + s_1 = 35$ $2x + y - s_2 + t_1 = 5$ $-3x + y + 5z - s_3 + t_2 = 15$ $Q = -2x - 4y + 3z - M(t_1 + t_2)$ leading to $Q + (2+M)x + 2(2-M)y - (3+5M)z + Ms_2 + Ms_3$ $= -20M$

*This paper is also the **Paper 4 option 4D** paper and will have the title '*Paper 4, Option 4D: Decision Mathematics 1'.*

Paper 4D: Decision Mathematics 2

Topics	What students need to learn:			
	Content		Guidance	Enhanced content guidance
1 **Transportation problems**	1.1	The north-west corner method for finding an initial basic feasible solution.	Problems will be restricted to a maximum of four sources and four destinations.	
	1.2	Use of the stepping-stone method for obtaining an improved solution. Improvement indices.	The ideas of dummy locations and degeneracy are required. Students should identify a specific entering cell and a specific exiting cell.	
	1.3	Formulation of the transportation problem as a linear programming problem.		Unbalanced problems: for the supply constraints use $\leqslant$ and for the demand constraints use $\geqslant$. For balanced problems the constraints may also be written as equalities. The constraints can also be written using sigma notation.

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2 **Allocation (assignment) problems**	2.1	**Cost matrix reduction.**	**Students should reduce rows first.**	**Questions may ask students to reduce columns first.**
		Use of the Hungarian algorithm to find a least cost allocation.	**Ideas of a dummy location is required.** **The adaption of the algorithm to manage incomplete data is required.**	**When dealing with incomplete data a <u>finite</u> value that is greater than the largest value currently in the (incomplete) table should be used. For example, if the largest value in the table is 48 then a value of at least 49 (but not ∞) should be used in all the blank cell(s).**
		Modification of the Hungarian algorithm to deal with a maximum profit allocation.	**Students should subtract all the values (in the original matrix) from the largest value (in the original matrix).**	
	2.2	Formulation of the Hungarian algorithm as a linear programming problem.		

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3 Flows in networks	3.1	**Cuts and their capacity.**	**Only networks with directed arcs will be considered.**	
	3.2	**Use of the labelling procedure to augment a flow to determine the maximum flow in a network.**	**The arrow in the same direction as the arc will be used to identify the amount by which the flow along that arc can be increased. The arrow in the opposite direction will be used to identify the amount by which the flow in the arc could be reduced.**	
	3.3	**Use of the max–flow min–cut theorem to prove that a flow is a maximum flow.**		
	3.4	Multiple sources and sinks. Vertices with restricted capacity.	Problems may include vertices with restricted capacity.	
	3.5	Determine the optimal flow rate in a network, subject to given constraints.	Problems may include both upper and lower capacities.	If there is a maximum and minimum capacity associated with an arc this will be written on each arc as (a, b) where a is the lower capacity and b the upper capacity. A single value on an arc will represent the upper capacity only (and it can therefore be assumed that the corresponding lower capacity is zero).

Topics	What students need to learn:			
	Content		Guidance	Enhanced content guidance
4 **Dynamic programming**	4.1	Principles of dynamic programming. Bellman's principle of optimality.	Students should be aware that any part of the shortest/longest path from source to sink is itself a shortest/longest path, that is, any part of an optimal path is itself optimal.	
		Stage variables and State variables. Use of tabulation to solve maximum, minimum, minimax or maximin problems.	Both network and table formats are required.	
5 **Game theory**	5.1	**Two person zero-sum games and the pay-off matrix.**	**A pay-off matrix will always be written from the row player's point of view unless directed otherwise.**	
	5.2	**Identification of play safe strategies and stable solutions (saddle points).**	**Students should be aware that in a zero-sum game there will be a stable solution if and only if the row maximin = the column minimax** **The proof of the stable solution theorem is not required.**	
	5.3	Reduction of pay-off matrices using dominance arguments.		Problems involving weak dominance may be set. Students need to understand that if an option is removed that is never better than another option it is still possible to find an optimal strategy. However, it is possible that an alternative optimal strategy may have been removed.

Topics	What students need to learn:			
	Content		Guidance	Enhanced content guidance
5 **Game theory** *continued*	5.4	**Optimal mixed strategies for a game with no stable solution by use of graphical methods for $2 \times n$ or $n \times 2$ games where $n = 1, 2, 3$ or 4**		
	5.5	Optimal mixed strategies for a game with no stable solution by converting games to linear programming problems that can be solved by the Simplex algorithm.		Students may be asked to set up an initial Simplex tableau and students need to understand the process of carrying out the Simplex algorithm in this context. However, students will not be asked to perform iterations of the Simplex algorithm. They may, however, be required to interpret a tableau that is given to them at different stages/iterations of the algorithm.
6 **Recurrence relations**	6.1	**Use of recurrence relations to model appropriate problems.**		Students may be asked to set up a recurrence relation to model a given problem.
	6.2	**Solution of first** and second **order linear homogeneous and non-homogeneous recurrence relations.**	**Students should be able to solve relations such as,** $u_{n+1} - 5u_n = 8,\ u_1 = 1$ $2u_{n+2} + 7u_{n+1} - 15u_n = 6,\ u_1 = 10,\ u_2 = -17$ **The terms, particular solution, complementary function and auxiliary equation should be known.** **Use of recurrence relations to model applications e.g. population growth**	The auxiliary equation for a second order recurrence relation may have (i) distinct real roots, (ii) a repeated root or (iii) complex roots. **First order recurrence relations are restricted to those that can be written in the form:** $u_{n+1} + au_n = g(n)$ **where** a **is a constant and** $g(n)$ **is either a polynomial function of** n**, constant or of the form** kp^n **where** p

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				can either be equal or not equal to a (including sums and differences). Similarly, second order recurrence relations are restricted to those that can be written in the form: $u_{n+2} + au_{n+1} + bu_n = \mathrm{g}(n)$ where a and b are constants and $\mathrm{g}(n)$ is either a polynomial function of n, constant or of the form kp^n where p can either be equal or not equal to the roots of the auxiliary equation (including sums and differences) . **In both cases, if $\mathrm{g}(n)$ takes any other form than those stated above, then the form of the particular solution will be given to the students.** **Students may be required to use a given substitution to reduce a recurrence relation to one of the forms stated above.** **The general solution of the recurrence relation should be given in the form $u_n = \text{C.F.+P.S.}$ where C.F. is the complementary function (the solution of the homogeneous recurrence relation) and where P.S. is a particular solution.** **Questions may be asked that require the finding of (i) the general solution for u_n (so this will include unknown arbitrary constant(s)) or (ii) the solution for u_n (in**

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	Content		Guidance	
				which initial (or other) conditions will be required to find the value(s) of the constant(s) in the corresponding general solution).
7 **Decision analysis**	7.1	Use, construct and interpret decision trees.	Students should be familiar with the terms: *decision nodes*, *chance nodes* and *pay-offs*.	
	7.2	Use of expected monetary values (EMVs) and utility to compare alternative courses of action.		